Math 251 — Midterm 2 — NAME: SOLUTIONS

F 22 Nov. 2002 — Student ID: $\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$

Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Monday, this means complaints must be received by Tuesday 26 Nov.

1. (1 point) True or false: L'Hospital's rule was included in the course for the purpose of confusing students who are having trouble remembering the quotient rule.

Solution: Don't say that to L'Hospital!

2. (9 points) Let f be a function such that $f'(x) = -8f(x)$. Find the derivative of the function $g(x) = \cos(f(x) + x)$. (Your answer might involve the function f.)

Solution: Use the chain rule:

$$\begin{aligned} g'(x) &= -\sin(f(x)+x)\cdot \tfrac{d}{dx}(f(x)+x) = -\sin(f(x)+x)(f'(x)+1) \\ &= -\sin(f(x)+x)(-8f(x)+1) = \sin(f(x)+x)(8f(x)-1). \end{aligned}$$

3. (15 points) Use the methods of calculus to find the exact values of x at which the function $f(x) = (3x^2 - 5x - 5)\, e^{-x-7}$ takes its absolute minimum and maximum values on the interval $[2, 10]$.

Hint: $f'(x) = (-3x^2 + 11x)\, e^{-x-7}$.

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate f at all critical numbers and at the endpoints, and compare values.

To find the critical numbers, we solve the equation $f'(x) = 0$. We are already given $f'(x)$, and to solve this equation we factor it:

$$f'(x) = \left(-3x^2 + 11x\right) e^{-x-7} = -x\,(3x - 11)\, e^{-x-7}.$$

This last expression is zero when $x = 0$ and when $x = \frac{11}{3}$. (The factor e^{-x-7} is never zero.)

We now have two critical numbers, namely 0 and $\frac{11}{3}$. Of these, 0 is not in the interval under consideration, so we ignore it. (I must see you reject this value.) So we must compare the values of f at $\frac{11}{3}$ and at the endpoints 2 and 10. We evaluate:

$$f(2) = \left(3\cdot 2^2 - 5\cdot 2 - 5\right) e^{-9} = -3e^{-9} \approx -0.000370229,$$

$$f\left(\tfrac{11}{3}\right) = \left(3\left(\tfrac{11}{3}\right)^2 - 5\left(\tfrac{11}{3}\right) - 5\right) e^{-\frac{11}{3}-7} = \left(\tfrac{121}{3} - \tfrac{55}{3} - 5\right) e^{-32/3} = 17e^{-32/3} \approx 0.000396255,$$

and

$$f(10) = \left(3\cdot 10^2 - 5\cdot 10 - 5\right) e^{-10-7} = 245e^{-17} \approx 0.0000101428.$$

(You will probably use your calculator to get the decimal approximations.) The smallest of these is $f(2)$ and the largest is $f\left(\frac{11}{3}\right)$, so the absolute minimum on the interval $[2, 10]$ occurs at $x = 2$ and the absolute maximum on the interval $[2, 10]$ occurs at $x = \frac{11}{3}$.

There is another way to find the maximum and minimum, which does not require the calculator at all. From the formula for $f'(x)$, we see that $f'(x) > 0$ for x in the interval $\left[2, \frac{11}{3}\right)$, and $f'(x) < 0$ for x in the interval $\left(\frac{11}{3}, 10\right]$. Therefore f is increasing on the interval $\left[2, \frac{11}{3}\right)$, and f is decreasing on the interval $\left(\frac{11}{3}, 10\right]$. It follows that the absolute maximum on the interval $[2, 10]$ occurs at $x = \frac{11}{3}$. The absolute minimum must be at one of the endpoints, and no calculator is needed to show that

$$f(2) = \left(3 \cdot 2^2 - 5 \cdot 2 - 5\right) e^{-9} = -3e^{-9} < 0$$

and

$$f(10) = \left(3 \cdot 10^2 - 5 \cdot 10 - 5\right) e^{-10-7} = 245e^{-17} > 0.$$

Therefore the absolute minimum on the interval $[2, 10]$ occurs at $x = 2$.

Note that $x = 0$ is not correct for the minimum, even though $f(0) = -5e^{-7} \approx -0.00455941$ is less than $f(2)$, because 0 is not in the interval $[2, 10]$.

4. (10 points/part) Evaluate the following limits. (Give exact values.)

(a) $\displaystyle\lim_{x\to 0} \frac{e^{k\sin(x)} - 1}{\sin(x)}$, where k is a nonzero constant.

Solution: Trying to substitute $x = 0$ gives the undefined expression $\frac{0}{0}$, so more work is needed. In this case, L'Hospital's Rule applies. Remember to use the chain rule when differentiating the numerator:

$$\lim_{x\to 0} \frac{e^{k\sin(x)} - 1}{\sin(x)} = \lim_{x\to 0} \frac{e^{k\sin(x)} \cdot k\cos(x)}{\cos(x)},$$

provided the last limit exists. This limit can be evaluated by substituting $x = 0$, and the result is k.

(b) $\displaystyle\lim_{x\to 2} \frac{x}{e^{3x}}$

Solution: Both the numerator and denominator are defined and continuous at $x = 2$, and the denominator is not zero there. Therefore

$$\lim_{x\to 2} \frac{x}{e^{3x}} = \frac{2}{e^{3\cdot 2}} = \frac{2}{e^6}.$$

Note that L'Hospital's rule does not apply, because the limit does not have an indeterminate form. If you try to apply it anyway, you get

$$\lim_{x\to 2} \frac{1}{3e^{3x}} = \frac{1}{3e^6},$$

which is not the correct answer.

5. (10 points) If $x^7 y = x + \cos(ky) + \pi^3$, where k is a constant, find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

Solution: Write the equation as

$$x^7 y(x) = x + \cos(ky(x)) + \pi^3.$$

Use the product rule on the left hand side and the chain rule on the second term on the right hand side. The derivative of the third term on the right hand side is zero because π^3 is a constant. Thus:

$$7x^6y(x) + x^7y'(x) = 1 - \sin(ky(x)) \cdot ky'(x).$$

Now solve for $y'(x)$:

$$7x^6y(x) - 1 = -x^7y'(x) - \sin(ky(x)) \cdot ky'(x) = -\big(x^7 + k\sin(ky(x))\big)\, y'(x)$$

$$y'(x) = -\frac{7x^6y(x) - 1}{x^7 + k\sin(ky(x))}.$$

This result can't be further simplified.

For those who prefer the other notation, here it is written that way. We use the product rule on the left hand side and the chain rule on the second term on the right hand side. The derivative of the third term on the right hand side is zero because π^3 is a constant. Thus:

$$7x^6y + x^7\frac{dy}{dx} = 1 - \sin(ky) \cdot k\frac{dy}{dx}.$$

Now solve for $\frac{dy}{dx}$:

$$7x^6y - 1 = -x^7\frac{dy}{dx} - \sin(ky) \cdot k\frac{dy}{dx} = -\big(x^7 + k\sin(k)\big)\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{7x^6y - 1}{x^7 + k\sin(ky)}.$$

This result can't be further simplified.

6. (25 points) A 13 foot ladder leans against a vertical wall in a room with a high ceiling and level floor. Because the floor is slippery, the foot of the ladder is sliding away from the wall. When the foot is 5 feet from the wall, it is sliding away at 3 feet per hour. At this time, how fast is the top of the ladder sliding down the wall?

Solution 1: Note: There is no picture in this file.

Let $x(t)$ be the distance of the foot of the ladder from the base of the wall, in feet. Let $h(t)$ be the height above the floor of the top of the ladder, also in feet. Recall that the ladder is 13 feet long, and this length does not change. The missing picture should therefore show a right triangle, with horizontal side labelled $x(t)$, vertical side labelled $h(t)$, and hypotenuse (diagonal) labelled 13.

Let t_0 be the time at which we are interested. We are given the following:

$$x(t_0) = 5 \quad \text{and} \quad x'(t_0) = 3.$$

(The derivative $x'(t_0)$ is positive because the distance from the foot of the ladder to the wall is increasing.)

Our quantities are related by the equation $[x(t)]^2 + [h(t)]^2 = 13^2$, since we have a right triangle. Differentiating this equation with respect to t gives

$$2x(t)x'(t) + 2h(t)h'(t) = 0.$$

(Don't forget to use the chain rule!) Dividing through by 2 and substituting t_0 for t gives

$$x(t_0)x'(t_0) + h(t_0)h'(t_0) = 0.$$

Substituting for the known quantities gives

$$5 \cdot 3 + h(t_0)h'(t_0) = 0.$$

We want to find $h'(t_0)$, so we still need $h(t_0)$. Substituting for the known quantities in the equation $[x(t_0)]^2 + [h(t_0)]^2 = 13^2$ gives $5^2 + [h(t_0)]^2 = 13^2$, which can be solved to give $h(t_0) = 12$. (We reject the solution $h(t_0) = -12$, because $h(t)$ is positive.) Therefore $5 \cdot 3 + 12h'(t_0) = 0$, whence

$$h'(t_0) = -\frac{5 \cdot 3}{12} = -\frac{5}{4}.$$

So the top of the ladder is sliding down the wall at $\frac{5}{4}$ feet per hour. (The units are required.)

Solution 2: Draw the same picture as in Solution 1, and label it the same way. Then obtain the equation $[x(t)]^2 + [h(t)]^2 = 13^2$ as before. Rewrite this as

$$h(t) = \sqrt{13^2 - [x(t)]^2}.$$

(We use the positive square root, because $h(t)$ is positive.) Differentiating this equation with respect to t gives

$$h'(t) = \tfrac{1}{2}\left(13^2 - [x(t)]^2\right)^{-1/2} \cdot (-2x(t)x'(t)) = -\frac{x(t)x'(t)}{\sqrt{13^2 - [x(t)]^2}}.$$

(Don't forget to use the chain rule!) Setting $t = t_0$ and substituting for the known quantities gives

$$h'(t_0) = -\frac{5 \cdot 3}{\sqrt{13^2 - 5^2}} = -\frac{5 \cdot 3}{12} = -\frac{5}{4}.$$

So the top of the ladder is sliding down the wall at $\frac{5}{4}$ feet per hour. (The units are required.)

Solution 3: Here, for reference, is what the first solution looks like in physicists' notation. We have $x^2 + h^2 = 13^2$. Differentiate with respect to t:

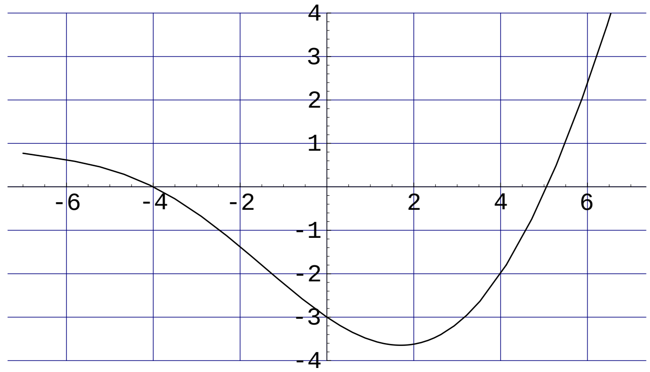
$$2x\frac{dx}{dt} + 2h\frac{dx}{dt} = 0.$$

(Don't forget the factors $\frac{dx}{dt}$ and $\frac{dh}{dt}$!) Use the equation $x^2 + h^2 = 13^2$ and the fact that $x = 5$ at the time of interest to find that $h = 12$ at the time of interest. Then substitute $x = 5$, $h = 12$, and $\frac{dx}{dt} = 5$, getting

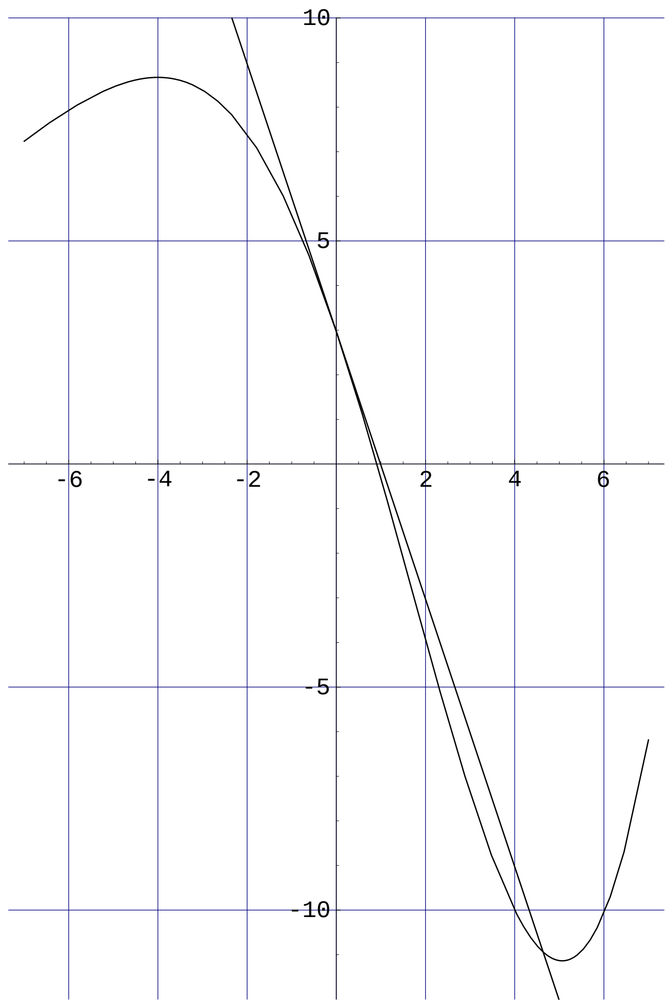
$$5 \cdot 3 + 12\frac{dh}{dt} = 0,$$

and solve for $\frac{dh}{dt}$ the same way we solved for $h'(t_0)$ above.

7. (5 points/part) The picture below is the graph of the *DERIVATIVE* $y = w'(x)$ for a certain function w. **CAUTION:** You are given the graph of the *derivative* $w'(x)$, *not* the graph of $w(x)$, but you are asked questions about $w(x)$.



For reference in the solutions below, here is the graph of the function f whose derivative is shown above and which satisfies $f(4) = -10$. Also shown on the graph is the tangent line at the point in part (c).



Any other function with the given derivative differs from this one by a constant, as we saw in Section 4.2 of the book. You can check, by drawing tangent lines, that the graph of the derivative of this function does indeed have the shape shown in the original graph.

For reference, this function has the formula

$$w(x) = -\frac{14}{3} + \frac{x}{24}\left(x^2 + 6x - 72\right) + \frac{24}{\pi}\cos(\pi x/8).$$

(a) Find all numbers c in $(-6, 6)$ such that w has a local maximum at c. Give reasons for your choices, or explain why there are none.

Solution: Local maximums occur at critical numbers, that is, numbers c such that $w'(c) = 0$ or $w'(c)$ does not exist. There are no numbers c in $(-6, 6)$ such that $w'(c)$ does not exist. We can read off the graph of w' that $w'(c) = 0$ for $c = -4$ and $c \approx 5.1$. We can test which of these are local maximums in either of two ways: the first derivative test or the second derivative test.

Here is the first derivative test. (This explanation is much more detailed than was required on the exam.) At $c = -4$, the values shown for $w'(x)$ on the graph of w', for $x < -4$ but x close to -4, are positive, while values shown for $w'(x)$, for $x > -4$ but x close to -4, are negative. So w is increasing before we get to -4, and decreasing afterwards; therefore, w has a local *maximum* at $c = -4$.

At $c \approx 5.1$, the values shown for $w'(x)$ on the graph of w', for $x < c$ but x close to c, are negative, while values shown for $w'(x)$, for $x > c$ but x close to c, are positive. So w is decreasing before we get to c, and increasing afterwards; therefore, w has a local *minimum* at $c \approx 5.1$.

Thus, the answer to the problem is $c = -4$ only.

Here is the second derivative test. (Again, this explanation is much more detailed than was required on the exam.) Since the graph shows w', the second derivative $w''(x)$, which is the derivative of w' at x, is the slope of the tangent line to the graph shown at x. The tangent line at $c = -4$ clearly has negative slope, while the one at $c \approx 5.1$ clearly has positive slope. Thus, the second derivative test tells us that w has a local maximum at $c = -4$ (the critical number where the second derivative is positive). (Also, there is a local minimum at $c \approx 5.1$, since this is a critical number where the second derivative is negative.)

You can see from the graph of w provided above that w does indeed have a local maximum at $c = -4$ and a local maximum at $c \approx 5.1$.

(b) Suppose we are given that $w(4) = -10$. Use the linear approximation to estimate $w(3.8)$.

Solution: Let $L(x)$ be the linear approximation at 4. Thus

$$w(x) \approx L(x) = w(4) + w'(4)(x - 4).$$

We read from the graph of w' that $w'(4) = -2$. Thus

$$w(3.8) \approx L(3.8) = w(4) + w'(4)(3.8 - 4) = -10 - 2(-0.2) = -9.6.$$

(For reference, $w(3.8) \approx -9.57095$, correct to five decimal places.)

(c) Is w concave up or concave down at $x = 0$, or does w (nearly) have an inflection point at $x = 0$, or is there not enough information provided to determine this? Why?

Solution: The graph shown is of the derivative w' of w. Therefore $w''(0)$ is the slope (first derivative) of the graph shown at 0. Clearly it is negative. Thus $w''(0) < 0$. Therefore w is concave down at $x = 0$. The tangent line to the graph of w at $x = 0$ is shown on the graph above, and you can see that it is above the graph of w.

(d) Let $f(x) = w(9 - x^2)$. Find $f'(-3)$, or explain why there is not enough information to do so.

Solution: Use the chain rule:

$$f'(x) = w'(9 - x^2) \cdot (-2x) = -2xw'(9 - x^2).$$

Therefore $f'(-3) = (-2)(-3)w'(9 - (-3)^2) = 6w'(0)$. From the graph of w' we can read off $w'(0) = -3$, so the answer is $f'(-3) = 6(-3) = -18$.

Extra credit. (Remember that these problems will only be counted if you get a grade of B or better on the main part of this exam.)

EC1. (7 extra credit points) Find a function f such that $f'(x) = x\sin(x^2)$ for all x. Check your function to be sure its derivative really is what you think it is.

Solution: The function $f(x) = -\frac{1}{2}\cos(x^2)$ will work. Differentiate it (with the chain rule):

$$\frac{d}{dx}\left(-\tfrac{1}{2}\cos(x^2)\right) = -\tfrac{1}{2}\left(-\sin(x^2)\right) \cdot \frac{d}{dx}\left(x^2\right) = -\tfrac{1}{2}\left(-\sin(x^2)\right) \cdot 2x = x\sin(x^2).$$

EC2. (15 extra credit points) Let g be a continuous function on $(-1, 1)$ such that $g(0) = 0$ and $g'(0) = 1$. Suppose that g'' exists and is continuous on $(-1, 1)$. Define

$$f(x) = \begin{cases} g(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Show that $f'(0)$ exists, and prove a formula for it in terms of a suitable derivative of g.

Solution: By definition,

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h},$$

if this limit exists. Putting in the formula for f, we get

$$f'(0) = \lim_{h \to 0} \frac{g(h)/h - 1}{h} = \lim_{h \to 0} \frac{g(h) - h}{h^2}.$$

This limit has the indeterminate form $\frac{0}{0}$, so work is needed. We we may use L'Hospital's Rule. Thus

$$\lim_{h \to 0} \frac{g(h) - h}{h^2} = \lim_{x \to \infty} \frac{g'(h) - 1}{2h},$$

if the second limit exists. We rewrite this limit as

$$\frac{1}{2} \lim_{x \to \infty} \frac{g'(h) - g'(0)}{h} = \tfrac{1}{2}g''(0).$$

Therefore

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \tfrac{1}{2}g''(0).$$

Alternate solution: We begin by using L'Hospital's Rule, as in the first solution. We must find

$$\lim_{x\to\infty} \frac{g'(h)-1}{2h}.$$

By assumption, $g'(0) = 1$. Also, g' is differentiable, and therefore continuous. So this limit also has the indeterminate form $\frac{0}{0}$. We apply L'Hospital's Rule again:

$$\lim_{x\to\infty} \frac{g'(h)-1}{2h} = \lim_{x\to\infty} \frac{g''(h)}{2}.$$

if the limit on the right hand side exists. Since g'' is continuous, this limit is $\frac{1}{2}g''(0)$. Therefore

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \tfrac{1}{2}g''(0).$$