Math 251 | Midterm 1 | NAME: <u>SOLUTIONS</u>

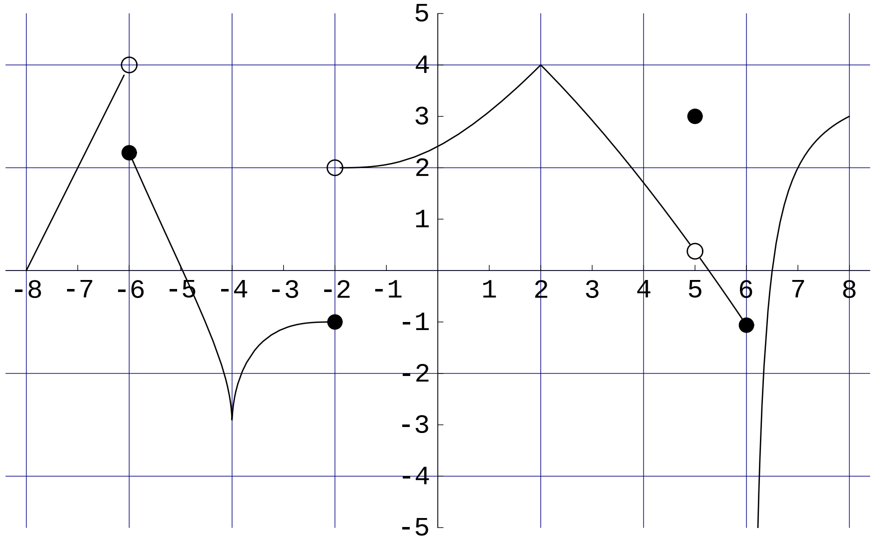
F 25 Oct. 2002 | | Student ID: $\underline{\pi\pi\pi\text{-}\pi\pi\text{-}\pi\pi\pi\pi}$

Reminder: Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned. If, as planned, I return the exam Monday, this means complaints must be received by Tuesday 29 Oct.

1. (1 point) True of false: I hate limits which contain square roots.

Solution: If you think those are bad, try limits which contain cube roots.

2. (5 points/part; total 20 points.) For the function $y = w(x)$ graphed below, answer the following questions:



(a) List all numbers a in $(-8, 8)$ such that $\lim_{x\to a} w(x)$ does not exist.

Solution: The answer is $a = -6$, $a = -2$, and $a = 6$. At all three of these numbers, the one sided limits $\lim_{x\to a^+} w(x)$ and $\lim_{x\to a^-} w(x)$ are different. (Also, at $a = 6$, one of the one sided limits is $-\infty$.)

Note that $\lim_{x\to 5} w(x)$ does exist, even though it is not equal to $w(5)$.

(b) Find the largest interval containing -3 on which w is continuous.

Solution: The largest interval containing -3 on which w is continuous is $(-6, -2)$. (Note that w is not continuous at -6 and at -2, because $\lim_{x\to -6} w(x)$ and $\lim_{x\to -2} w(x)$ do not exist. Therefore w is not continuous on any interval containing those points. On the other hand, w is continuous at -4, even though w is not differentiable there.)

(c) List all numbers a in $(-8, 8)$ such that w is not differentiable at a. Give reasons.

The answer is $a = -6$, $a = -4$, $a = -2$, $a = 2$, $a = 5$, and $a = 6$.

For $a = -6$, $a = -2$, and $a = 6$, the limits $\lim_{x\to a} w(x)$ do not exist, so w is not continuous at these numbers, hence not differentiable. For $a = 5$, the limit $\lim_{x\to 5} w(x)$ does exist, but is not equal to $w(5)$. So again w is not continuous there, hence not differentiable. Finally, at $a = -4$ and $a = 2$, the function is continuous but still not differentiable: there is a cusp at -4 and a corner at 2.

(d) Which of the following best describes $w'(4)$?
(1) $w'(4)$ does not exist.
(2) $w'(4)$ is close to 0.
(3) $w'(4)$ is positive and not close to 0.
(4) $w'(4)$ is negative and not close to 0.

Solution: $w'(4)$ is the slope of the tangent line to the graph of $y = w(x)$ at $x = 4$. You can tell from inspection that this slope is negative and not close to 0 (choice (4) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around $-4/3$. (In fact, it is clear that $w'(x) < 0$ for $2 < x < 5$.)

3. (10 points.) Let f be a function such that $f'(x) = 2e^{-x^2} + \sqrt{7}$. Find the derivative of the function $g(x) = \dfrac{x}{f(x)}$. (Your answer might involve the function f. You need not do this directly from the definition.)

Solution: Use the quotient rule:

$$g'(x) = \frac{f(x) - xf'(x)}{[f(x)]^2} = \frac{f(x) - x\big(2e^{-x^2} + \sqrt{7}\big)}{[f(x)]^2} = \frac{f(x) - 2xe^{-x^2} - \sqrt{7}\,x}{[f(x)]^2}.$$

4. (7 points) Let f and g be functions which are differentiable at -2 and which satisfy

$$f(-2) = -5, \quad f'(-2) = -3, \quad g(-2) = 4, \quad \text{and} \quad g'(-2) = 2.$$

Let $w(x) = x - f(x)g(x)$ for all x. Find $w'(-2)$.

Solution: Using the product rule on the second part, we get:

$$w'(x) = 1 - [f'(x)g(x) + f(x)g'(x)] = 1 - f'(x)g(x) - f(x)g'(x).$$

This gives

$$w'(-2) = 1 - f'(-2)g(-2) - f(-2)g'(-2) = 1 - (-3)(4) - (-5)(2) = 23.$$

5. (a) (7 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

(b) (13 points.) If $f(x) = \dfrac{1}{x+4}$, compute the derivative $f'(3)$ *directly from the definition.* (You should check your answer using the differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient. To handle the expression that appears in the difference quotient, we subtract the fractions in the numerator and then cancel common factors in the numerator and denominator:

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{1}{3+h+4} - \frac{1}{3+4}}{h} = \lim_{h \to 0} \frac{\frac{1}{7+h} - \frac{1}{7}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{7-(7+h)}{7(7+h)}\right)}{h} = \lim_{h \to 0} \frac{\left(\frac{-h}{7(7+h)}\right)}{h} = \lim_{h \to 0} \frac{-1}{7(7+h)} = -\frac{1}{49}. \end{aligned}$$

We can check using the differentiation formulas. It is easiest to use the chain rule from Section 3.5: $f(x) = (x+4)^{-1}$, so $f'(x) = -(x+4)^{-2} \cdot 1 = -(x+4)^{-2}$, whence $f'(3) = -\frac{1}{49}$. It can also be done using the quotient rule. (However, you get no credit if this is the only thing you do.)

6. (10 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them.

(a) $\displaystyle\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$.

Solution: This has the indeterminate form $\frac{0}{0}$, so work is needed. We rationalize the numerator, and then cancel common factors:

$$\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} = \lim_{x \to 3} \frac{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})}{(x-3)(\sqrt{x} + \sqrt{3})} = \lim_{x \to 3} \frac{x - 3}{(x-3)(\sqrt{x} + \sqrt{3})} = \lim_{x \to 3} \frac{1}{\sqrt{x} + \sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

(b) $\displaystyle\lim_{x \to \infty} \frac{4x^2 + 6070x + 193}{17x^2 - 9x + 21}$.

Solution: This has the indeterminate form $\frac{\infty}{\infty}$, so work is needed. We factor out x^2 from both the numerator and denominator, and then use the limit laws:

$$\begin{aligned} \lim_{x \to \infty} \frac{4x^2 + 6070x + 193}{17x^2 - 9x + 21} &= \lim_{x \to \infty} \frac{x^2\left(4 + \frac{6070}{x} + \frac{193}{x^2}\right)}{x^2\left(17 - \frac{9}{x} + \frac{21}{x^2}\right)} = \lim_{x \to \infty} \frac{4 + \frac{6070}{x} + \frac{193}{x^2}}{17 - \frac{9}{x} + \frac{21}{x^2}} \\ &= \frac{4 + \lim_{x \to \infty} \frac{6070}{x} + \lim_{x \to \infty} \frac{193}{x^2}}{17 - \lim_{x \to \infty} \frac{9}{x} + \lim_{x \to \infty} \frac{21}{x^2}} = \frac{4 + 0 + 0}{17 - 0 + 0} = \frac{4}{17}. \end{aligned}$$

Here is a different way to arrange essentially the same calculation:

$$\lim_{x\to\infty} \frac{4x^2+6070x+193}{17x^2-9x+21} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)(4x^2+6070x+193)}{\left(\frac{1}{x^2}\right)(17x^2-9x+21)} = \lim_{x\to\infty} \frac{4+\frac{6070}{x}+\frac{193}{x^2}}{17-\frac{9}{x}+\frac{21}{x^2}}$$
$$= \frac{4+\lim_{x\to\infty}\frac{6070}{x}+\lim_{x\to\infty}\frac{193}{x^2}}{17-\lim_{x\to\infty}\frac{9}{x}+\lim_{x\to\infty}\frac{21}{x^2}} = \frac{4+0+0}{17-0+0} = \frac{4}{17}.$$

(c) $\displaystyle\lim_{x\to 4^-} \frac{f(x)}{x-4}$, given that $\displaystyle\lim_{x\to 4} f(x) = 9$.

Solution: For $x < 4$ but very close to 4, we have $f(x)$ close to 9 and $x-4$ negative and very close to zero. Therefore $\dfrac{f(x)}{x-4}$ is negative and very far from zero, that is, very small. So $\displaystyle\lim_{x\to 4^-} \frac{f(x)}{x-4} = -\infty$.

7. (12 points) Let c be a positive constant. Find the equation of the tangent line to the graph of $f(x) = e^x + cx$ at $x = 3$. (Use exact values in your answer—no calculator approximations.) You need not calculate the derivative directly from the definition.

Solution: We need the slope, which is $f'(3)$, and a point on the line, such as

$$(3,\ f(3)) = (3,\ e^3 + 3c).$$

Differentiating, we get $f'(x) = e^x + c$, so $f'(3) = e^3 + c$. Therefore the equation is

$$y - (e^3 + 3c) = (e^3 + c)(x - 3),$$

which can be rearranged to give

$$y = (e^3 + c)(x - 3) + e^3 + 3c = (e^3 + c)x - 2e^3.$$

The simplification is necessary.

Note that we want the slope at the *particular* value $x = 3$. Therefore we must substitute $x = -2$ in the formula for the derivative $f'(x)$ *before* using it as the slope of a line. The equation

$$y - (e^3 + 3c) = (e^x + c)(x - 3)$$

is wrong—it is not even the equation of a line.

Extra credit. (Remember that these problems will only be counted if you get a grade of B or better on the main part of this exam.)

EC1. (8 extra credit points) Let $f(x) = e^{2x}$. Using only the differentiation rules in Sections 3.1 and 3.2 of the book (the ones we have done so far), find $f'(x)$. (You might want to check your answer using the chain rule, if you have read that far in the book. However, no credit will be given for a calculation that uses any form of the chain rule in any step.)

Solution: We write

$$f(x) = e^{2x} = (e^x)^2 = e^x \cdot e^x.$$

In this form, we can differentiate $f(x)$ using the product rule. (The simplification at the end is necessary.)

$$f'(x) = \left(\frac{d}{dx}(e^x)\right) \cdot e^x + e^x \cdot \frac{d}{dx}(e^x) = e^x \cdot e^x + e^x \cdot e^x = 2(e^x)^2 = 2e^{2x}.$$

Note 1: No differentiation rule in Section 3.1 or 3.2 applies to the function $f(x) = e^{2x}$ in this form. In particular, the rule for differentiating x^n does not apply, since the variable in e^{2x} is in the exponent rather than the base. The rule for differentiating e^x does not apply, since we have here e^{2x} rather than e^x.

Note 2: The "general power rule",

$$\frac{d}{dx}(g(x)^n) = ng(x)^{n-1}g'(x),$$

is nothing more than a special case of the chain rule, and is therefore also not allowed.

EC2. (10 extra credit points) Give a convincing argument to show that the function $f(x) = e^{4x} + x + 2$ has an inverse whose domain is the entire real line. (Don't use calculator graphs.)

Solution: We first observe that f is one to one. Suppose $x_1 < x_2$. Then also $e^{4x_1} < e^{4x_2}$, so $e^{4x_1} + x_1 < e^{4x_2} + x_2$. Thus $f(x_1) < f(x_2)$. Similarly, if $x_1 > x_2$ then $f(x_1) > f(x_2)$. So f is in fact one to one. Therefore f has an inverse.

It remains to show that the domain of the inverse of f is the entire real line. This is the same as showing that if b is a real number, then there is some a with $f(a) = b$. Now $\lim_{x\to-\infty}(x+2) = -\infty$ and $\lim_{x\to-\infty} e^{4x} = 0$, so $\lim_{x\to-\infty} f(x) = -\infty$. In particular, there must be some x_1 with $f(x_1) < b$. Also, $\lim_{x\to\infty} e^{4x} = \infty$ and $\lim_{x\to\infty}(x+2) = \infty$, so $\lim_{x\to\infty} f(x) = \infty$. Therefore there must be some x_2 with $f(x_2) > b$. Since f is continuous, the Intermediate Value Theorem implies that there is some a between x_1 and x_2 such that $f(a) = b$.