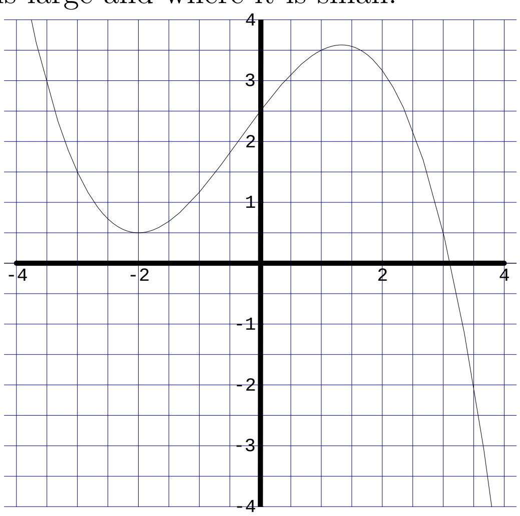
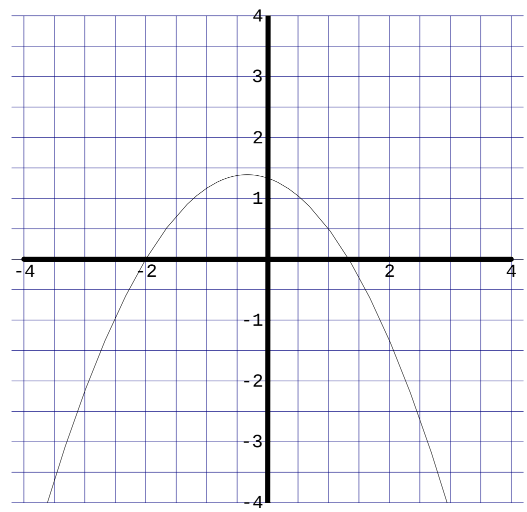
At least 80% of the points on the real exam will be modifications of problems from Midterm 1 from the last time I taught the course, the problems below, homework problems (particularly written homework), and problems from the sample and real Midterms 0. Note, though, that the exact form of the functions to be differentiated and of the limits to be computed could vary substantially, and the methods required to do them might occur in different combinations.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen in the book and on the blackboard what the correct notation for limits is; *use it*. The right notation will help you get the mathematics right, and I will complain about incorrect notation.

---

1. (15 points) For the function $y = f(x)$ graphed below (left), sketch the graph of the derivative $y = f'(x)$ on the grid provided. Do not worry about great accuracy; your graph should, however, make it clear where the derivative is positive and where it is negative, and where the derivative is large and where it is small.





Solution: The graph above (right) shows $y = f'(x)$ with great accuracy (much more than required).

When grading such a problem, I will pay careful attention to where your graph crosses the x-axis, and other easily distinguishable features.

2. (10 points/part) Find the exact values of the following limits, or explain why they do not exist:

(a) $\displaystyle\lim_{x\to-2} \frac{x+2}{x^2-3x-10}$.

Solution: This has the indeterminate form $\frac{0}{0}$, so work is needed. We factor the denominator and cancel common factors:

$$\lim_{x\to-2} \frac{x+2}{x^2-3x-10} = \lim_{x\to-2} \frac{x+2}{(x+2)(x-5)} = \lim_{x\to-2} \frac{1}{x-5} = \frac{1}{-2-5} = -\frac{1}{7}.$$

(b) $\displaystyle\lim_{x\to0^+} \left(\frac{1}{2x} - \frac{1}{2x+\sqrt{x}} \right)$.

Solution: This has the indeterminate form $\infty - \infty$, so work is needed. Subtract the fractions and cancel common factors:

$$\lim_{x\to 0^+}\left(\frac{1}{2x} - \frac{1}{2x+\sqrt{x}}\right) = \lim_{x\to 0^+}\frac{(2x+\sqrt{x})-2x}{(2x)(2x+\sqrt{x})} = \lim_{x\to 0^+}\frac{\sqrt{x}}{(2x)(2x+\sqrt{x})} = \lim_{x\to 0^+}\frac{1}{(2\sqrt{x})(2x+\sqrt{x})}.$$

For $x > 0$ and close to 0, the numerator is 1 and the denominator is positive and close to 0. So the function becomes arbitrarily large:

$$\lim_{x\to 0^+}\left(\frac{1}{2x} - \frac{1}{2x+\sqrt{x}}\right) = \infty.$$

(c) $\displaystyle\lim_{x\to -\infty}\frac{x^2 - x + 17}{7x^2 + 9x + 19}$.

Solution: This has the indeterminate form $\frac{\infty}{\infty}$, so work is needed. We factor out x^2 from both the numerator and denominator, and then use the limit laws:

$$\begin{aligned}\lim_{x\to -\infty}\frac{x^2 - x + 17}{7x^2 + 9x + 19} &= \lim_{x\to -\infty}\frac{x^2\left(1 - \frac{1}{x} + \frac{17}{x^2}\right)}{x^2\left(7 + \frac{9}{x} + \frac{19}{x^2}\right)} = \lim_{x\to -\infty}\frac{1 - \frac{1}{x} + \frac{17}{x^2}}{7 + \frac{9}{x} + \frac{19}{x^2}} \\ &= \frac{1 - \lim_{x\to -\infty}\frac{1}{x} + \lim_{x\to -\infty}\frac{17}{x^2}}{7 + \lim_{x\to -\infty}\frac{9}{x} + \lim_{x\to -\infty}\frac{19}{x^2}} = \frac{1-0+0}{7+0+0} = \frac{1}{7}.\end{aligned}$$

Alternate solution: Here is a different way to arrange essentially the same calculation:

$$\begin{aligned}\lim_{x\to -\infty}\frac{x^2 - x + 17}{7x^2 + 9x + 19} &= \lim_{x\to -\infty}\frac{\left(\frac{1}{x^2}\right)(x^2 - x + 17)}{\left(\frac{1}{x^2}\right)(7x^2 + 9x + 19)} = \lim_{x\to -\infty}\frac{1 - \frac{1}{x} + \frac{17}{x^2}}{7 + \frac{9}{x} + \frac{19}{x^2}} \\ &= \frac{1 - \lim_{x\to -\infty}\frac{1}{x} + \lim_{x\to -\infty}\frac{17}{x^2}}{7 + \lim_{x\to -\infty}\frac{9}{x} + \lim_{x\to -\infty}\frac{19}{x^2}} = \frac{1-0+0}{7+0+0} = \frac{1}{7}.\end{aligned}$$

(d) $\displaystyle\lim_{x\to -\infty}\frac{x}{\sqrt{5x^2+9}}$.

Solution: This has the indeterminate form $\frac{-\infty}{\infty}$, so work is needed. We multiply the numerator and denominator by $\frac{1}{x}$, since we should consider the highest degree part of the denominator to be $\sqrt{5x^2} = \pm\sqrt{5}\,x$ (the sign depending on whether $x > 0$ or $x < 0$). When we do the calculation, we are only interested in negative values of x, since what happens for $x > 0$ has no effect on a limit as $x \to -\infty$. Therefore when we put $\frac{1}{x}$ under the square root in the denominator, we will have to calculate as follows: $\frac{1}{x} = -\sqrt{1/x^2}$, *not* $\sqrt{1/x^2}$, for $x < 0$. Accordingly,

$$\begin{aligned}\lim_{x\to -\infty}\frac{x}{\sqrt{5x^2+9}} &= \lim_{x\to -\infty}\frac{\left(\frac{1}{x}\right)x}{\left(\frac{1}{x}\right)\sqrt{5x^2+9}} = \lim_{x\to -\infty}\frac{1}{-\sqrt{\frac{1}{x^2}}\sqrt{5x^2+9}} = \lim_{x\to -\infty}\frac{1}{-\sqrt{\left(\frac{1}{x^2}\right)(5x^2+9)}} \\ &= \lim_{x\to -\infty}\frac{1}{-\sqrt{5+\frac{9}{x^2}}} = \frac{1}{-\sqrt{5+\lim_{x\to -\infty}\frac{9}{x^2}}} = \frac{1}{-\sqrt{5+0}} = -\frac{1}{\sqrt{5}}.\end{aligned}$$

(e) $\displaystyle\lim_{x\to 0}\frac{\sin(ax)}{x}$, where a is a constant.

Solution: If $a = 0$, then $\sin(ax) = 0$ for all x, so clearly the limit is 0. (A small amount of credit will be lost for not treating this case separately, since the argument below does not make sense for $a = 0$.)

If $a \neq 0$, then the limit has the indeterminate form $\frac{0}{0}$, so work is needed. From

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

we get

$$\lim_{x \to 0} \frac{\sin(ax)}{ax},$$

so

$$\lim_{x \to 0} \frac{\sin(ax)}{x} = a.$$

Thus

$$\lim_{x \to 0} \frac{\sin(ax)}{x} = a$$

in all cases.

(f) $\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)}$, where a and b are nonzero constants.

Solution: This has the indeterminate form $\frac{0}{0}$, so work is needed. We write

$$\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x \to 0} \frac{\frac{\sin(ax)}{x}}{\frac{\sin(bx)}{x}}.$$

Since $a \neq 0$, we have

$$\lim_{x \to 0} \frac{\sin(ax)}{x} = a \lim_{x \to 0} \frac{\sin(ax)}{ax} = a \lim_{x \to 0} \frac{\sin(x)}{x} = a \cdot 1 = a.$$

Similarly

$$\lim_{x \to 0} \frac{\sin(bx)}{x} = b.$$

Therefore

$$\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x \to 0} \frac{\frac{\sin(ax)}{x}}{\frac{\sin(bx)}{x}} = \frac{\lim_{x \to 0} \frac{\sin(ax)}{x}}{\lim_{x \to 0} \frac{\sin(bx)}{x}} = \frac{a}{b}.$$

(g) $\lim_{x \to 0} \frac{\sin(ax)}{\cos(bx)}$, where a and b are nonzero constants.

Solution: The function $f(x) = \frac{\sin(ax)}{\cos(bx)}$ is defined and continuous at $x = 0$, so

$$\lim_{x \to 0} \frac{\sin(ax)}{\cos(bx)} = \frac{\sin(a \cdot 0)}{\cos(b \cdot 0)} = \frac{\sin(0)}{\cos(0)} = 0.$$

3. (10 points/part) Let f be a function such that $\lim_{x \to 2} f(x) = 7$. Calculate the following expressions, or explain why there is not enough information to do so:

(a) $\lim_{x \to 2} \frac{f(x)}{\sqrt{x}}$.

Solution:
$$\lim_{x\to 2}\frac{f(x)}{\sqrt{x}} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2}\sqrt{x}} = \frac{7}{\sqrt{2}}.$$

(b) $\displaystyle\lim_{x\to 2}\frac{f(x)}{x-2}$.

Solution: This limit does not exist. The solution to part (c) below shows that $\displaystyle\lim_{x\to 2^-}\frac{f(x)}{x-2} = -\infty$. Similar reasoning shows that $\displaystyle\lim_{x\to 2^+}\frac{f(x)}{x-2} = +\infty$. Therefore one can't even correctly say that $\displaystyle\lim_{x\to 2}\frac{f(x)}{x-2}$ is $-\infty$ or that it is $+\infty$.

(c) $\displaystyle\lim_{x\to 2^-}\frac{f(x)}{x-2}$.

Solution: For $x < 2$ but very close to 2, we have $f(x)$ close to 7 and $x-2$ negative and very close to zero. Therefore $\dfrac{f(x)}{x-2}$ is negative and very far from zero. So $\displaystyle\lim_{x\to 2^-}\frac{f(x)}{x-2} = -\infty$.

(d) $\displaystyle\lim_{x\to 7} f(x)$.

Solution: Nothing can be said about $\lim_{x\to 7} f(x)$. The statement $\lim_{x\to 2} f(x) = 7$ says something about the behavior of $f(x)$ for x close to 2, but nothing about the behavior of $f(x)$ for x close to 7.

(e) $f(2)$.

Solution: Nothing can be said about $f(2)$. The statement $\lim_{x\to 2} f(x) = 7$ says nothing about $f(2)$. It only tells about the values of $f(x)$ for x close to, but not equal to, 2.
(Note: If we also knew that f is continuous at 2, then we could say that $f(2) = 7$.)

(f) $\displaystyle\lim_{x\to 2}\frac{f(x)-7}{x-2}$.

Solution: Nothing can be said about $\displaystyle\lim_{x\to 2}\frac{f(x)-7}{x-2}$. It has the indeterminate form $\frac{0}{0}$, so requires more work. But without knowing anything else about f, there is no way to do any more work.
Examples:
If $f(x) = x+5$, then $\displaystyle\lim_{x\to 2}\frac{f(x)-7}{x-2} = 1$.
If $f(x) = 2x+3$, then $\displaystyle\lim_{x\to 2}\frac{f(x)-7}{x-2} = 2$.
If $f(x) = (x-2)^2+7$, then $\displaystyle\lim_{x\to 2}\frac{f(x)-7}{x-2} = 0$.
If $f(x) = (x-2)^{1/3}+7$, then $\displaystyle\lim_{x\to 2}\frac{f(x)-7}{x-2} = \infty$.
All the given choices of f satisfy $\lim_{x\to 2} f(x) = 7$.

(g) $\lim_{x\to 2} \sqrt{f(x)}$.

Solution:

$$\lim_{x\to 2} \sqrt{f(x)} = \sqrt{\lim_{x\to 2} f(x)} = \sqrt{7}.$$

4. (9 points/part) Differentiate the following functions:

(a) $f(y) = \left(7y^3 + \frac{1}{2}\right)\left(\frac{1}{8}y^7 - 16\sqrt{y} + \frac{2}{3}\right)$.

Solution: We use the product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Before doing so, we rewrite the factors to make then easy to differentiate:

$$f(y) = \left(7y^3 + \tfrac{1}{2}\right)\left(\tfrac{1}{8}y^7 - 16y^{1/2} + \tfrac{2}{3}\right).$$

Therefore

$$f'(y) = 21y^2\left(\tfrac{1}{8}y^7 - 16y^{1/2} + \tfrac{2}{3}\right) + \left(7y^3 + \tfrac{1}{2}\right)\left(\tfrac{7}{8}y^6 - 8y^{-1/2}\right).$$

(b) $h(t) = \dfrac{\sqrt[3]{t} - 2\pi}{\sqrt[3]{t} + 2\pi}$.

Solution: Use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Before doing so, we rewrite the numerator and denominator to make them easy to differentiate:

$$h(t) = \frac{t^{1/3} - 2\pi}{t^{1/3} + 2\pi}.$$

Therefore

$$h'(t) = \frac{\frac{1}{3}t^{-2/3}(t^{1/3} + 2\pi) - (t^{1/3} - 2\pi)\frac{1}{3}t^{-2/3}}{(t^{1/3} + 2\pi)^2} = \frac{\frac{1}{3}t^{-2/3} \cdot 4\pi}{(t^{1/3} + 2\pi)^2} = \frac{4\pi}{3t^{2/3}(t^{1/3} + 2\pi)^2}.$$

(The simplification is necessary.) Note that 2π is a *constant*, so its derivative is zero.

(c) Given that $h'(x) = 3h(x)$, find $\dfrac{d}{dx}\left(\dfrac{x}{h(x)}\right)$. (Your answer might involve the function h.)

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$\frac{d}{dx}\left(\frac{x}{h(x)}\right) = \frac{1 \cdot h(x) - xh'(x)}{h(x)^2} = \frac{h(x) - x \cdot 3h(x)}{h(x)^2} = \frac{1 - 3x}{h(x)}.$$

(The simplification is necessary.)

(d) $g(t) = \dfrac{\cos(t)}{t^2 + 1} - 2$.

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$g'(t) = \frac{-\sin(t)(t^2+1) - \cos(t)\cdot 2t}{(t^2+1)^2} = \frac{-(t^2+1)\sin(t) - 2t\cos(t)}{(t^2+1)^2}.$$

(e) $g(t) = \dfrac{t^2 - 17}{k + \tan(t)} - \sqrt{2}$, where k is a constant.

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$g'(t) = \frac{2t(k+\tan(t)) - (t^2-17)\sec^2(t)}{\big(k+\tan(t)\big)^2} = \frac{2kt + 2t\tan(t) - t^2\sec^2(t) + 17\sec^2(t)}{\big(k+\tan(t)\big)^2}.$$

The "simplification" at the second step isn't really any simpler, so is not required. Note that $\sqrt{2}$ is a *constant*, so its derivative is zero.

5. (15 points) Let g be a continuous function on $(-1, 1)$ such that $g(0) = 0$ and $g'(0) = 1$. Define

$$f(x) = \begin{cases} g(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Is f continuous at $x = 0$? Why? (Show your work!)

Solution: We compare $\lim_{x\to 0} f(x)$ with $f(0)$. When computing the limit, we use the formula which applies for $x \neq 0$, ignoring the value at 0. The limit has the indeterminate form $\frac{0}{0}$, so work is needed. We proceed as follows:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = g'(0) = 1 = f(0).$$

So f is continuous at 0.