MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 8.

This sheet is part of the homework for Week 8, and is due in class on Wednesday 27 February 2008.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WebAssign), give exact answers (not decimal approximations; again, unlike WebAssign), and use correct notation. Some of the grade will be based on correctness of notation in the work shown.

1. (Section 4.5, Problem 18). Use the guidelines of Section 4.5 to sketch the graph of

$$f(x) = \frac{x}{x^3 - 1}.$$

Reminder of the guidelines: Find the domain, intercepts, symmetry, asymptotes, intervals of increase and decrease, local minimums and maximums, intervals of concavity up and down, and inflection points.

Solution: The domain clearly consists of all real numbers except 1.

The y-intercept is $f(0) = 0$.

To find the x-intercepts, we solve $f(x) = 0$:

$$\frac{x}{x^3 - 1} = 0$$

$$x = 0.$$

(A fraction is zero if and only if the numerator is zero.)

There is no symmetry.

There is a vertical asymptote at $x = 1$, because $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = -\infty$. (To check these, note the numerator is positive for x close to 1, while the denominator is positive for $x > 1$ and negative for $x < 1$.)

For horizontal asymptotes, we check

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^3 - 1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^3}\right)x}{\left(\frac{1}{x^3}\right)(x^3 - 1)} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)}{1 - \left(\frac{1}{x^3}\right)} = \frac{\lim_{x\to\infty} \frac{1}{x^2}}{1 - \lim_{x\to\infty} \frac{1}{x^3}} = \frac{0}{1 - 0} = 0.$$

Similarly, $\lim_{x\to -\infty} f(x) = 0$. This, the x-axis is a horizontal asymptote in both directions.

Next, we need the derivative:

$$f'(x) = \frac{x^3 - 1 - x(3x^2)}{(x^3 - 1)^2} = -\frac{2x^3 + 1}{(x^3 - 1)^2}.$$

(Important: *simplify* your answer before trying to do anything with it!) This is zero when $x = -1/\sqrt[3]{2}$. It is of course not defined at $x = -1$, since f is not defined there.

On the interval $\left(-\infty, -1/\sqrt[3]{2}\right)$, the factor $2x^3 + 1$ above is negative and $(x^3 - 1)^2$ is positive. Because of the minus sign in front of the whole expression, f' is positive on this interval. Therefore f is increasing on this interval.

On the interval $\left(-1/\sqrt[3]{2}, 1\right)$, the factor $2x^3 + 1$ above is positive and $(x^3 - 1)^2$ is positive. Because of the minus sign in front of the whole expression, f' is negative on this interval. Therefore f is decreasing on this interval.

It now follows that f has a local maximum at $x = -\sqrt[3]{2}$. For use in graphing, we compute decimal approximations for the values of x and $f(x)$, getting $-1/\sqrt[3]{2} \approx -0.793701$ and $f\left(-1/\sqrt[3]{2}\right) \approx 0.529134$.

On the interval $(1, \infty)$, the factor $2x^3 + 1$ above is positive and $(x^3 - 1)^2$ is positive. Because of the minus sign in front of the whole expression, f' is negative on this interval. Therefore f is decreasing on this interval.

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Now, we need the second derivative:

$$f''(x) = -\frac{6x^2(x^3-1)^2 - (2x^3+1)\cdot 2(x^3-1)\cdot 3x^2}{(x^3-1)^4} = -\frac{(x^3-1)\big(6x^2(x^3-1) - 6x^2(2x^3+1)\big)}{(x^3-1)^4}$$
$$= -\frac{6x^2(x^3-1) - 6x^2(2x^3+1)}{(x^3-1)^3} = \frac{6x^2(x^3+2)}{(x^3-1)^3}.$$

(Important: *simplify* your answer before trying to do anything with it!) This is zero when $x = -\sqrt[3]{2}$ and $x = 0$. It is of course not defined at $x = -1$, since f is not defined there.

On the interval $\big(-\infty, -\sqrt[3]{2}\big)$, the factor $6x^2$ above is positive, $x^3 + 2$ is negative, and $(x^3 - 1)^3$ is negative. So f'' is positive on this interval. Therefore f is concave up on this interval.

On the interval $\big(-\sqrt[3]{2}, 0\big)$, the factor $6x^2$ above is positive, $x^3 + 2$ is positive, and $(x^3 - 1)^3$ is negative. So f'' is negative on this interval. Therefore f is concave down on this interval.

It follows that there is an inflection point at $x = -\sqrt[3]{2}$. For use in graphing, we compute decimal approximations for the values of x and $f(x)$, getting $-\sqrt[3]{2} \approx -1.25992$ and $f\big(-\sqrt[3]{2}\big) \approx 0.419974$.

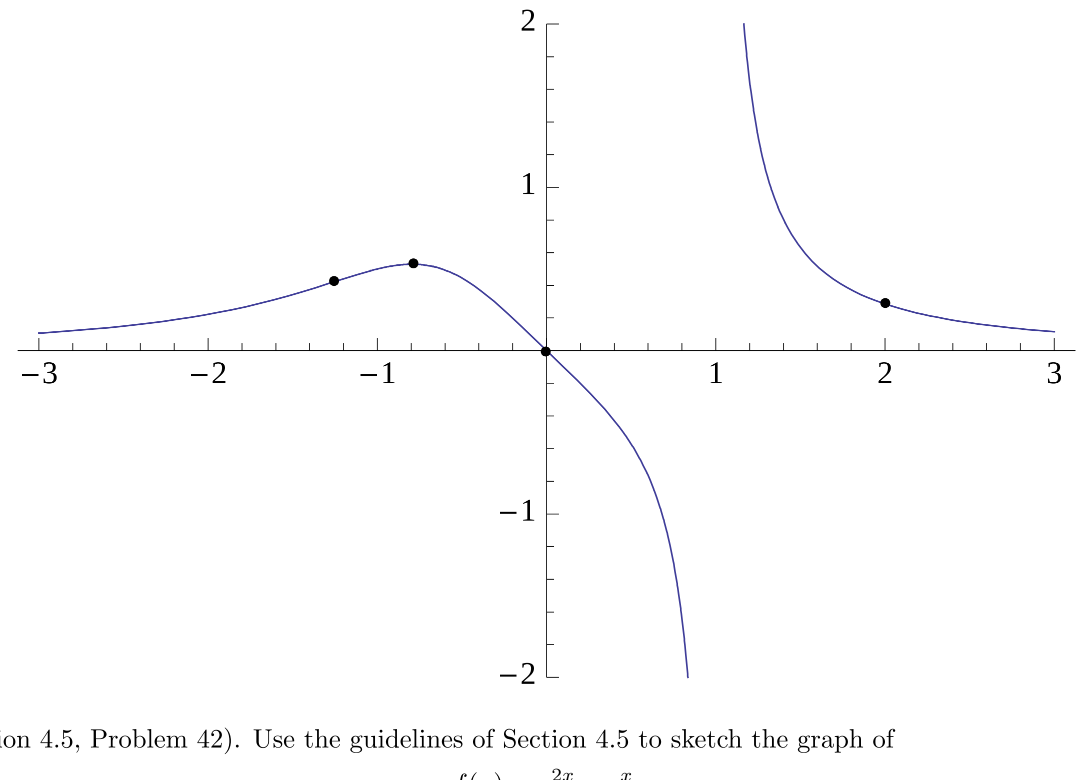
On the interval $(0, 1)$, the factor $6x^2$ above is positive, $x^3 + 2$ is positive, and $(x^3 - 1)^3$ is negative. So f'' is negative on this interval. Therefore f is concave down on this interval.

Since the concavity does not change at $x = 0$, there is *no* inflection point at $x = 0$.

On the interval $(1, \infty)$, the factor $6x^2$ above is positive, $x^3 + 2$ is positive, and $(x^3 - 1)^3$ is positive. So f'' is positive on this interval. Therefore f is concave up on this interval.

Now we are almost ready to graph the function. But we should plot at least one point to the right of the vertical asymptote. I chose $x = 2$, and computed the approximation $f(2) \approx 0.285714$.

Here is the graph, showing as black dots the inflection point, the point on the graph corresponding to the local maximum, the point on the graph corresponding to the x-intercept, and the point on the graph corresponding to $x = 2$.



2. (Section 4.5, Problem 42). Use the guidelines of Section 4.5 to sketch the graph of

$$f(x) = e^{2x} - e^x.$$

Reminder of the guidelines: Find the domain, intercepts, symmetry, asymptotes, intervals of increase and decrease, local minimums and maximums, intervals of concavity up and down, and inflection points.

Solution: The domain clearly consists of all real numbers.

The y-intercept is $f(0) = 0$.

To find the x-intercepts, we solve $f(x) = 0$:

$$e^{2x} - e^x = 0$$

$$(e^x)^2 - e^x = 0$$

$$e^x(e^x - 1) = 0$$

$$e^x = 0 \quad \text{or} \quad e^x - 1 = 0.$$

Now e^x is never zero, and $e^x = 1$ exactly when $x = 0$, so the only x-intercept is at $x = 0$.

There is no symmetry.

There are no vertical asymptotes, since f is defined and continuous on $(-\infty, \infty)$.

For horizontal asymptotes, we check

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (e^{2x} - e^x) = \lim_{x\to\infty} e^x(e^x - 1).$$

Since $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty}(e^x - 1) = \infty$, we get $\lim_{x\to\infty} f(x) = \infty$. Therefore there is no horizontal asymptote in the positive direction. Also,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{2x} - \lim_{x\to-\infty} e^x = 0 - 0 = 0.$$

This, the x-axis is a horizontal asymptote in the negative direction.

Next, we need the derivative:

$$f'(x) = 2e^{2x} - e^x.$$

To find the critical numbers, we solve $f'(x) = 0$:

$$2e^{2x} - e^x = 0$$

$$2(e^x)^2 - e^x = 0$$

$$e^x(2e^x - 1) = 0$$

$$e^x = 0 \quad \text{or} \quad 2e^x - 1 = 0.$$

Now e^x is never zero, and $2e^x = 1$ exactly when $e^x = \frac{1}{2}$, so the only critical number is $x = \ln\left(\frac{1}{2}\right) = -\ln(2)$.

On the interval $(-\infty, -\ln(2))$, the factor e^x above is positive and $2e^x - 1$ is negative. So f' is negative on this interval. Therefore f is decreasing on this interval.

On the interval $(-\ln(2), \infty)$, the factor e^x above is positive and $2e^x - 1$ is positive. So f' is positive on this interval. Therefore f is increasing on this interval.

It now follows that f has a local minimum at $x = -\ln(2)$. For use in graphing, we compute decimal approximations for the values of x and $f(x)$, getting $-\ln(2) \approx -0.693147$ and $f(-\ln(2)) = -\frac{1}{4} = -0.25$.

Now, we need the second derivative:

$$f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1).$$

By a similar calculation as above, this is zero exactly when $x = -\ln(4)$.

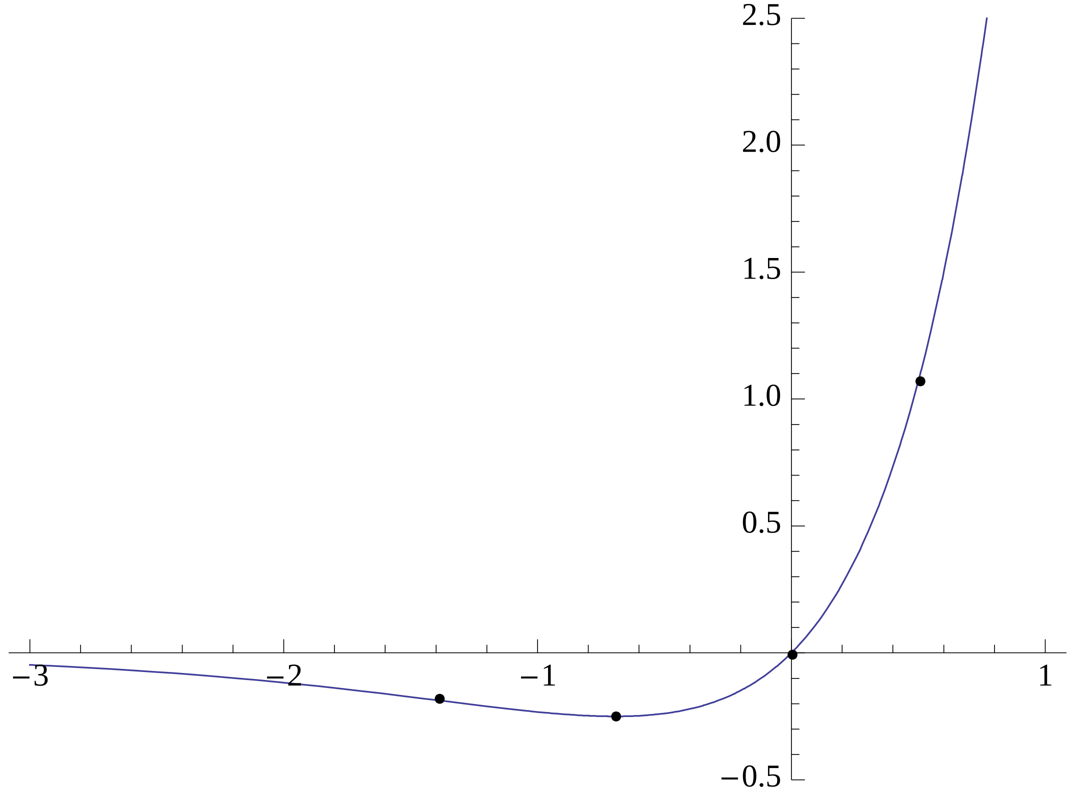
On the interval $(-\infty, -\ln(4))$, the factor e^x above is positive and $2e^x - 1$ is negative. So f'' is negative on this interval. Therefore f is concave down on this interval.

On the interval $(-\ln(4), \infty)$, the factor e^x above is positive and $2e^x - 1$ is positive. So f'' is positive on this interval. Therefore f is concave up on this interval.

It follows that there is an inflection point at $x = -\ln(4)$. For use in graphing, we compute decimal approximations for the values of x and $f(x)$, getting $-\ln(4) \approx -1.38629$ and $f(-\ln(4)) = -\frac{3}{16} = -0.1875$.

Now we are almost ready to graph the function. But we should plot at least one point to the right of the y-axis. I chose $x = \frac{1}{2}$, and computed the approximation $f\left(\frac{1}{2}\right) \approx 1.06956$.

Here is the graph, showing as black dots the inflection point, the point on the graph corresponding to the local minimum, the point on the graph corresponding to the x-intercept, and the point on the graph corresponding to $x = \frac{1}{2}$.


2.5
2.0
1.5
1.0
0.5
−0.5
−3
−2
−1
1