MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 7.

This sheet is part of the homework for Week 7, and is due in class on Friday 22 February 2008.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WebAssign), give exact answers (not decimal approximations; again, unlike WebAssign), and use correct notation. Some of the grade will be based on correctness of notation in the work shown.

1. (Section 4.2, Problem 6) Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$, but that there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Explain why this outcome does not contradict Rolle's Theorem.

Solution: For the first part, we have $\tan(0) = 0$ and $\tan(\pi) = 0$.

For the second part, we have $f'(x) = \sec^2(x)$. Since $\sec(x)$ is never zero, $\sec^2(x)$ is also never zero.

For the third part, we observe that the hypotheses of Rolle's theorem do not hold, because f is not even defined at the number $\frac{1}{2}\pi$ and this number is in $(0, \pi)$.

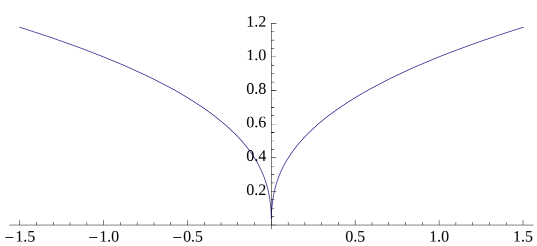
2. Let $f(x) = \sqrt[5]{x^2}$. Show that $f(-1) = f(1)$, and explain why there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Explain why this outcome does not contradict Rolle's Theorem.

Solution: For the first part, we have $f(-1) = 1$ and $f(1) = 1$.

For the second part, we rewrite $f(x) = \left(x^2\right)^{1/5}$. (We must be careful here since x can be negative.) Then $f'(x) = \frac{2}{5}\left(x^2\right)^{-4/5} \cdot 2x$ for $x \neq 0$. This expression is not zero for $x \neq 0$, and is not even defined for $x = 0$.

It only remains to show that $f'(0) \neq 0$. In fact, $f'(0)$ does not exist, because the graph of f has a cusp at $x = 0$.

The statement about the cusp is sufficient explanation, but here are more details. First, here is the graph of f, showing the cusp:



Second, we can actually calculate directly from the definition of the derivative:

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^{2/5} - 0}{h} = \lim_{h \to 0^+} h^{-3/5} = \infty,$$

so $f'(0)$ does not exist. (Similarly, and being careful about signs, one gets

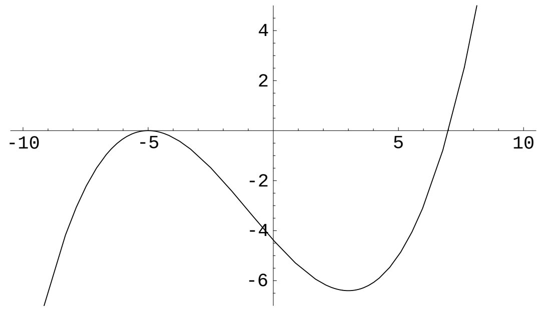
$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -\infty.$$

But already the first result shows that $f'(0)$ does not exist.)

For the third part, we observe that the hypotheses of Rolle's theorem do not hold, because f is not differentiable at the number 0, and this number is in $(-1, 1)$.

3. The picture below is the graph of the *DERIVATIVE* $y = f'(x)$ for a certain function f. **CAUTION:** You are given the graph of the *derivative* $f'(x)$, *not* the graph of $f(x)$, but you are asked questions about $f(x)$.

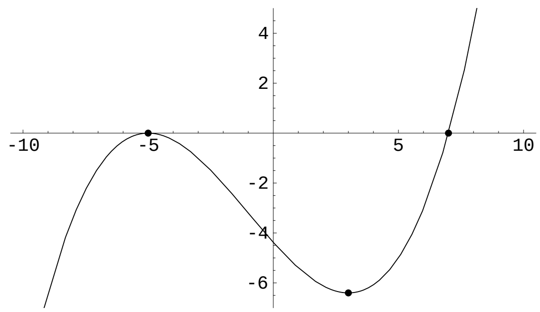
Date: 22 February 2008.



Find and identify all critical numbers, local minimums, local maximums, and inflection points of f in the interval $(-10, 8)$. Give reasons.

In case anyone wants to experiment, the function happens to have the formula $f'(x) = \frac{1}{40}(x-7)(x+5)^2$. The function f has the formula $f(x) = \frac{1}{160}x\left(x^3 + 4x^2 - 90x - 700\right) + C$ for some unknown constant C. For graphing purposes, you might as well take $C = 0$.

Solution: Here is a graph on which the relevant points are marked with dots.



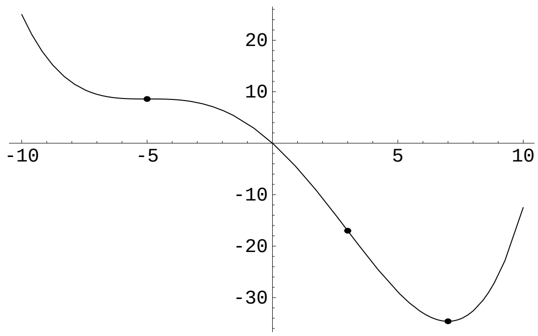
To solve the problem, we first read directly from the graph of f' all the numbers x with $f'(x) = 0$. These are $x = -5$ and $x = 7$. So the critical numbers of f are -5 and 7.

At $x = 7$, we can read directly from the graph of f' that f' changes from negative to positive there. So f changes from decreasing to increasing at 7, and f has a local minimum at 7. (Here is an alternate way to see that f has a local minimum at 7. We can see from the graph that the slope of f' is positive at 7. Therefore $f''(7) > 0$, so f is concave up there, and so has a local minimum at 7.)

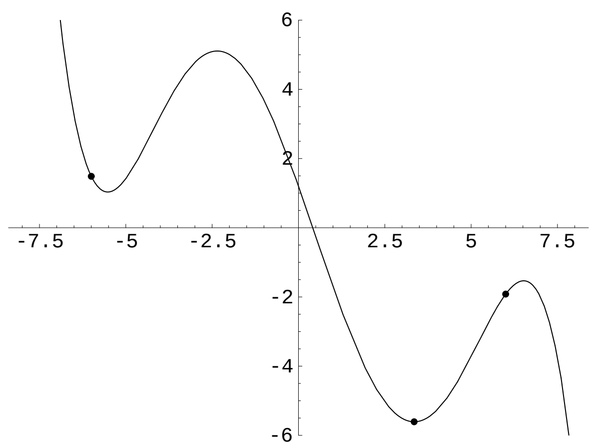
At $x = -5$, we can read directly from the graph of f' that f' does not change sign there. Indeed, for x close to but less than -5, we see that $f'(x) < 0$, and for x close to but greater than -5, again $f'(x) < 0$. So f has a critical number, but neither a local maximum nor a local minimum, at $x = -5$. In fact, the slope of f' at $x = -5$, that is, $f''(-5)$, is equal to zero. Since the slope of f' does change sign at -5 (from positive to negative), we know that the value of f'' changes sign at -5. Therefore f has an inflection point at $x = -5$.

The slope of f' is also zero at about $x = 3$. The slope of f' changes sign near 3, from negative to positive. So, as at $x = -5$, the function f also has an inflection point at about $x = 3$. (In fact, for the function shown, the inflection point is exactly at $x = 3$.)

Here is a graph showing one possibility for f, namely $f(x) = \frac{1}{160}x\left(x^3 + 4x^2 - 90x - 700\right)$. (You were not required to produce such a graph in the problem.) The points discussed above are marked with dots.

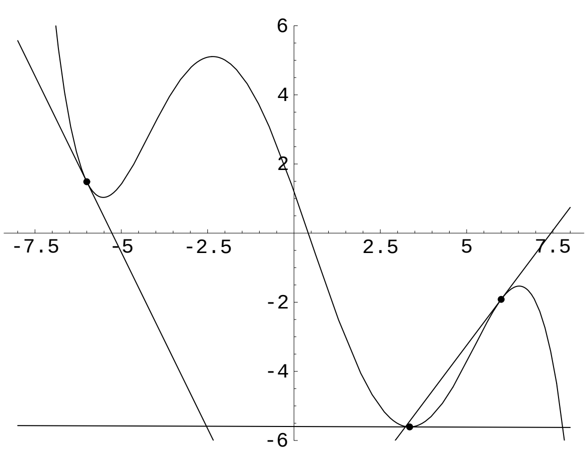


4. The picture below is the graph of $y = h(x)$ for a certain function h. (This is a graph of the function, *not* its derivative.)



For each value of x listed, answer the following three questions. Give reasons for your answers.
(1) Is $h(x)$ positive, negative, or near zero, or is there not enough information provided to determine this?
(2) Is $h'(x)$ positive, negative, or near zero, or is there not enough information provided to determine this?
(3) Is $h''(x)$ positive, negative, or near zero, or is there not enough information provided to determine this?
(The dots on the graph indicate the points on the graph corresponding to the given values of x.)

The graph below will help explain the solution. It shows the same function as above, but with the tangent lines at all the points added.



Also, in case anyone wants to experiment, the function happens to have the formula

$$\begin{aligned} h(x) &= -\frac{1}{16000}\left(19305 - 45434x - 3512x^2 + 2288x^3 + 80x^4 - 32x^5\right) \\ &= -\frac{1}{16000}(2x-15)(2x-11)(2x-1)(2x+9)(2x+13) - \tfrac{1}{2}x \end{aligned}$$

(a) $x = 6$.

Solution: (1) $h(6) < 0$ is clear from the graph.
(2) $h'(6) > 0$ since the slope of the tangent line is clearly positive.
(3) $h''(6) < 0$ because h is concave down at 6. Pictorially, the best way to see the concavity is to observe that the graph of h is below the tangent line at 6.

(b) $x = 3.35$.

Solution: (1) $h(3.35) < 0$ is clear from the graph.
(2) $h'(3.35) \approx 0$ since the graph is nearly flat at 3.35, so the slope of the tangent line is close to zero. (In fact, for the particular function shown, $h'(3.35) \approx -0.00336006$, which is certainly close to zero. The tangent line is not quite flat.)
(3) $h''(3.35) > 0$ because h is concave up at 3.35. Pictorially, the best way to see the concavity is to observe that the graph of h is above the tangent line at 3.35.

(c) $x = -6$.

Solution: (1) $h(-6) > 0$ is clear from the graph.
(2) $h'(-6) < 0$ since the slope of the tangent line is clearly negative.
(3) $h''(-6) > 0$ because h is concave up at -6. Pictorially, the best way to see the concavity is to observe that the graph of h is above the tangent line at -6.

Extra credit 1: Do the "applied project" on pages 279–280 of the book (on rainbows).

Solution: I do not expect to provide a solution to this problem. (Extra credit will be given to someone willing to type one, but you have to learn TEX to do it.)

Extra credit 2 (Section 4.2, Problem 33): Prove the trigonometric identity

$$\arcsin\left(\frac{x-1}{x+1}\right) = 2\arctan\left(\sqrt{x}\right) - \frac{\pi}{2}$$

for $x > 0$.

Solution: Let

$$f(x) = \arcsin\left(\frac{x-1}{x+1}\right) - 2\arctan\left(\sqrt{x}\right).$$

The method is to show that $f'(x) = 0$ for all $x > 0$. Therefore f must be constant. Evaluation of $f(x_0)$ for a suitable choice of x_0 then identifies the constant. We use the chain rule multiple times, and $\sqrt{(1+x)^2} = 1 + x$ (which is true because $1 + x \geq 0$):

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right) - \frac{2}{1+\left(\sqrt{x}\right)^2} \cdot \frac{d}{dx}\left(\sqrt{x}\right) \\ &= \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \left(\frac{x+1-(x-1)}{(x+1)^2}\right) - \frac{2}{1+x} \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{\sqrt{\frac{(x+1)^2-(x-1)^2}{(x+1)^2}}} \cdot \left(\frac{2}{(x+1)^2}\right) - \frac{1}{(1+x)\sqrt{x}} \\ &= \frac{1}{\left(\frac{\sqrt{4x}}{\sqrt{(x+1)^2}}\right)} \cdot \left(\frac{2}{(x+1)^2}\right) - \frac{1}{(1+x)\sqrt{x}} = \left(\frac{x+1}{\sqrt{4x}}\right)\left(\frac{2}{(x+1)^2}\right) - \frac{1}{(1+x)\sqrt{x}} = 0. \end{aligned}$$

This is true for all $x > 0$, so f is constant on $(0, \infty)$. To determine the constant, we calculate

$$f(1) = \arcsin(0) - 2\arctan(1) = 0 - 2\left(\tfrac{\pi}{4}\right) = -\tfrac{\pi}{2}.$$

So $f(x) = -\frac{\pi}{2}$ for all $x > 0$, which is the desired identity.