MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 2.

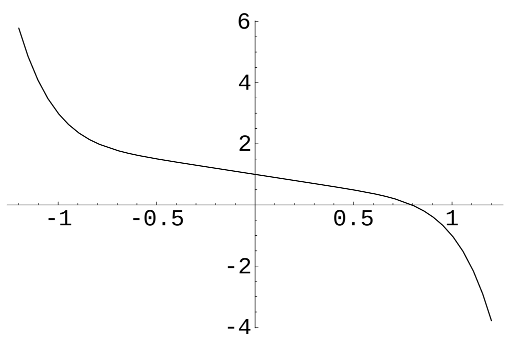
This sheet is part of the homework for Week 2, and is due in class on Friday 18 January 2008.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WebAssign), give exact answers (not decimal approximations; again, unlike WebAssign), and use correct notation. Some of the grade will be based on correctness of notation in the work shown.

1. Prove that there exists a real solution to the equation $-x^7 - x + 1 = 0$. Give a complete justification for any theorems that you use, in particular being sure to check that the hypotheses hold.

Solution: We will use the Intermediate Value Theorem. Set $f(x) = -x^7 - x + 1 = 0$. Then f is a polynomial function, so it is continuous at all real numbers. By substitution, you can check that $f(0) = 1$ and $f(1) = -1$. (These numbers were found by trying small integers at random. Note that $x = 0$ is a good choice because $f(0)$ is particularly easy to evaluate. Using $f(-1)$ and $f(1)$, for example, would also work fine.) Since $f(0) > 0 > f(1)$, the Intermediate Value Theorem therefore tells us that there is some c in the interval $(0, 1)$ such that $f(c) = 0$.

Here is the graph of f for x near zero (not required as part of the solution):



Note that the quadratic formula can't be used, since the original equation is not a quadratic equation.

2. Find the exact values of the following limits, or explain why they do not exist:

(a) $\displaystyle\lim_{x\to 9^+} \frac{f(x)}{x-9}$, given that $\displaystyle\lim_{x\to 9} f(x) = -4$.

Solution: For $x > 9$ but very close to 9, we have $f(x)$ close to -4 and $x - 9$ positive and very close to zero. Therefore $\dfrac{f(x)}{x-9}$ is negative and very far from zero, that is, very small. So $\displaystyle\lim_{x\to 9^+} \frac{f(x)}{x-9} = -\infty$.

(b) $\displaystyle\lim_{x\to 3} \frac{\sqrt{3x}-3}{x-3}$.

Solution: This has the indeterminate form $\frac{0}{0}$, so work is needed. We rationalize the numerator and then cancel common factors:

$$\lim_{x\to 3} \frac{\sqrt{3x}-3}{x-3} = \lim_{x\to 3} \frac{\left(\sqrt{3x}-3\right)\left(\sqrt{3x}+3\right)}{(x-3)\left(\sqrt{3x}+3\right)} = \lim_{x\to 3} \frac{3x-9}{(x-3)\left(\sqrt{3x}+3\right)}$$

$$= \lim_{x\to 3} \frac{3(x-3)}{(x-3)\left(\sqrt{3x}+3\right)} = \lim_{x\to 3} \frac{3}{\sqrt{3x}+3} = \frac{3}{\sqrt{9}+3} = \frac{1}{2}.$$

Alternate solution: Here is another approach. It is based on factoring the numerator and denominator, and cancelling a common factor. The key idea is to regard the denominator as a difference of squares, and factor

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accordingly:

$$x-3=(\sqrt{x})^2-\left(\sqrt{3}\right)^2=\left(\sqrt{x}+\sqrt{3}\right)\left(\sqrt{x}-\sqrt{3}\right).$$

Thus:

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{3x}-3}{x-3} &= \lim_{x\to 3}\frac{\sqrt{3}\left(\sqrt{x}-\sqrt{3}\right)}{\left(\sqrt{x}+\sqrt{3}\right)\left(\sqrt{x}-\sqrt{3}\right)} \\ &= \lim_{x\to 3}\frac{\sqrt{3}}{\sqrt{x}+\sqrt{3}}=\frac{\sqrt{3}}{\sqrt{3}+\sqrt{3}}=\frac{1}{2}.\end{aligned}$$

(c) $\lim_{x\to 0}\left[\frac{7}{12}+x^2\cos\left(\frac{1}{x^{1/3}}\right)\right]$.

Solution: We start by finding

$$\lim_{x\to 0}x^2\cos\left(\frac{1}{x^{1/3}}\right).$$

Even if you write $x^2\cos\left(\frac{1}{x^{1/3}}\right)$ as a fraction, no number of applications of L'Hospital's Rule gives any improvement; indeed, it makes things worse. Instead, use the Squeeze Theorem. For $x \neq 0$, we have

$$-1 \le \cos\left(\frac{1}{x^{1/3}}\right) \le 1,$$

so

$$-x^2 \le x^2\cos\left(\frac{1}{x^{1/3}}\right) \le x^2.$$

Since

$$\lim_{x\to 0}x^2=\lim_{x\to 0}(-x^2)=0,$$

the Squeeze Theorem implies that

$$\lim_{x\to 0}x^2\cos\left(\frac{1}{x^{1/3}}\right)=0.$$

Therefore

$$\lim_{x\to 0}\left[\frac{7}{12}+x^2\cos\left(\frac{1}{x^{1/3}}\right)\right]=\frac{7}{12}.$$

Alternate solution (sketch): By reasoning similar to the first solution, one sees that

$$\frac{7}{12}-x^2 \le \frac{7}{12}+x^2\cos\left(\frac{1}{x^{1/3}}\right) \le \frac{7}{12}+x^2$$

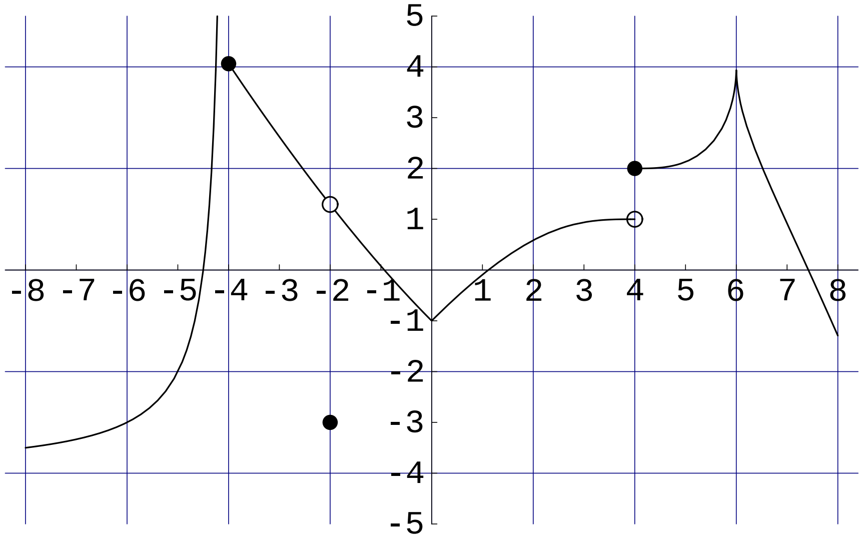
for all real $x \neq 0$. Also,

$$\lim_{x\to 0}\left(\frac{7}{12}-x^2\right)=\lim_{x\to 0}\left(\frac{7}{12}+x^2\right)=\frac{7}{12}.$$

Now use the Squeeze Theorem to get

$$\lim_{x\to 0}\left[\frac{7}{12}+x^2\cos\left(\frac{1}{x^{1/3}}\right)\right]=\frac{7}{12}.$$

3. For the function $y = q(x)$ graphed below, answer the following questions:



(a) List all numbers a in $(-8, 8)$ such that q is not continuous at a. Give reasons.

Solution: The answer is $a = -4$, $a = -2$, and $a = 4$. The function q is not continuous at -4 and at 4, because $\lim_{x\to-4} q(x)$ and $\lim_{x\to 4} q(x)$ do not exist. (We have $\lim_{x\to-4^-} q(x) = \infty$, and at 4 the two one sided limits exist but are not equal.) The function q is not continuous at -2 because, although $\lim_{x\to-2} q(x)$ exists, it is not equal to $q(-2)$. Note that q *is* continuous at 0 and at 6 (although q is not differentiable at those numbers).

(b) Find the largest interval containing -3 on which q is continuous.

Solution: The largest interval containing -3 on which q is continuous is $(-4, -2)$. The function q is not continuous at either -4 or -2, because $\lim_{x\to-4} q(x)$ does not exist and $\lim_{x\to-2} q(x) \neq q(-2)$.)