SOLUTIONS TO MATH 251 ADDITIONAL SAMPLE FINAL PROBLEMS

1. Helium is leaking from a spherical balloon. At a certain time, helium is being lost at the rate of 2 cubic centimeters per minute, and the radius of the balloon is 10 centimeters. At what rate is the radius decreasing at that time? (Be sure to include the correct units in your answer.)

Note: There is no picture in this file.

Solution: Let $r(t)$ be the radius at time t, measured in centimeters and with time measured in minutes, and let $V(t)$ be the volume at time t, measured in cubic centimeters. Note that both the volume and the radius vary with time, so must be treated as functions not constants.

Let t_0 be the time at which we are interested, so that $r(t_0) = 10$. Then $V'(t_0) = -2$. (Note that it is negative, since the volume is *decreasing*.) The functions $V(t)$ and $r(t)$ are related by the equation

$$V(t) = \tfrac{4}{3}\pi\,[r(t)]^3 .$$

(You are expected to know this formula.) Differentiate with respect to t:

$$V'(t) = \tfrac{4}{3}\pi \cdot 3\,[r(t)]^2\, r'(t) = 4\pi\,[r(t)]^2\, r'(t).$$

(Don't forget the factor $r'(t)$! That will spoil the whole thing!) Evaluate this at t_0, using $r(t_0) = 10$ and $V'(t_0) = -2$. This gives

$$\begin{aligned} V'(t_0) &= 4\pi\,[r(t_0)]^2\, r'(t_0) \\ -2 &= 4\pi \cdot 10^2 \cdot r'(t_0) \\ r'(t_0) &= -\frac{2}{400\pi} = -\frac{1}{200\pi}\ \text{cm/min} \approx -0.001592\ \text{cm/min}. \end{aligned}$$

(Don't forget the units!) So the radius is decreasing at the rate of $\frac{1}{200\pi}$ centimeters per minute.

Here, for reference, is what the solution looks like in physicists' notation:

$$V = \tfrac{4}{3}\pi r^3.$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \tfrac{4}{3}\pi \cdot 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

(Don't forget the factor $\dfrac{dr}{dt}$!) Now substitute $r = 10$ and $\dfrac{dV}{dt} = -2$, getting $-2 = 4\pi \cdot 10^2\dfrac{dr}{dt}$, and solve for $\dfrac{dr}{dt}$ the same way we solved for $r'(t_0)$ above.

2. A certain section of the San Andreas Fault runs straight north-south. On 29 February 1996, the west side was moving north (relative to the east side) at 3 cm/year (0.03 meters/year). At the same time, the town of Hicksville was 1 km (1000 meters) west of the fault, and the town of Quoggin was 2 km (2000 meters) east of the fault and 4 km (4000 meters) farther north than Hicksville. Were these two towns getting closer together or farther apart at this time? At what rate?

Solution: Note: There is no picture in this file. A picture may be provided separately.

The route from Hicksville to Quoggin at the present time, as described in the problem, is a zigzag line, first going east 1000 meters to the fault, then north 4000 meters along the fault, then east 2000 meters from the fault to Quoggin. Both the east-west distances are constant, but the north-south distance along the fault is actually changing. So let's call it $y(t)$, and say that $t = 0$ is the present time. Thus $y(0) = 4000$ (measuring all distances in meters). A more appropriate description of the route from Hicksville to Quoggin, valid at an arbitrary time, is therefore a zigzag line which first goes east 1000 meters to the fault, then north $y(t)$ meters along the fault, then east 2000 meters from the fault to Quoggin.

To find the distance, we are better off considering the route which starts at Hicksville, goes 3000 meters east, and then goes $y(t)$ meters north to Quoggin. These are two sides of a right triangle, so the distance $l(t)$ in meters from Hicksville to Quoggin at time t is

$$l(t) = \sqrt{3000^2 + y(t)^2} = \left(3000^2 + y(t)^2\right)^{1/2}.$$

We want to find $l'(0)$. Differentiating, we get

$$l'(t) = \tfrac{1}{2}\left(3000^2 + y(t)^2\right)^{-1/2} \cdot 2y(t)y'(t) = y(t)y'(t)\left(3000^2 + y(t)^2\right)^{-1/2}.$$

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(Don't forget to use the chain rule!) Put $t = 0$ and substitute values. (Note that this can only be done *after* differentiating!) We know $y(0) = 4000$. We need $y'(0)$, which we can see from the statement is -0.03. (It is negative because $y(t)$ is decreasing.) So

$$l'(0) = y(0)y'(0)\left(3000^2 + y(0)^2\right)^{-1/2} = 4000 \cdot (-0.03) \cdot \tfrac{1}{5000} = -0.024.$$

Therefore Hicksville and Quoggin are getting closer together at 0.024 meters per year, or 2.4 cm/year. (The units are necessary!)

Some people in Quoggin consider this to be bad news. (So do some people in Hicksville.)

Here are descriptions of some alternatives. First, you could differentiate the equation $l(t)^2 = 3000^2 + y(t)^2$, getting

$$2l(t)l'(t) = 2y(t)y'(t),$$

so that

$$l'(t) = \frac{y(t)y'(t)}{l(t)}.$$

Now put $t = 0$, and substitute $y(0) = 4000$, $y'(0) = -0.03$, and $l(0) = 5000$. (You still need to calculate $l(0)$ from the Pythagorean Theorem.)

You could also do everything in physicists' notation. I will only show the first version. The equation for l is

$$l = \sqrt{3000^2 + y^2} = \left(3000^2 + y^2\right)^{1/2}.$$

Differentiating (using the chain rule, because everything is a function of t!), we get

$$\frac{dl}{dt} = \tfrac{1}{2}\left(3000^2 + y^2\right)^{-1/2} \cdot 2y\frac{dy}{dt} = y\frac{dy}{dt}\left(3000^2 + y^2\right)^{-1/2}.$$

Substituting values (implicitly putting $t = 0$, and using $l = 5000$ at $t = 0$, as above):

$$\frac{dl}{dt} = y\frac{dy}{dt}\left(3000^2 + y^2\right)^{-1/2} = 4000 \cdot (-0.03) \cdot \tfrac{1}{5000} = -0.024,$$

as before.

3. The Wang Container Corporation plans to manufacture wooden boxes with square bases and hinged lids. The wood for the bottom and sides costs \$3 per square foot, and the wood for the lid costs \$1 per square foot. [Evidently the lid will be rather flimsy.] Furthermore, each box requires hinges and a latch costing a total of \$6. If the total cost of the materials is only allowed to be \$54, what are the dimensions of the largest volume box that can be manufactured?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Solution: Note: There is no picture in this file. A picture may be provided separately.

Let the linear dimensions of the box be x, y, and z, measured in feet, with z being the height. Since the base is square,

$$x = y.$$

We want to maximize the volume

$$V = xyz = x^2 z.$$

The constraint on the total cost allows the elimination of one more variable (necessary if we are to have a single variable problem). Let C be the total cost, measured in dollars. The bottom costs $3x^2$, the four sides cost $3xz$ each, the top costs x^2, and the hinges and latch cost 6. So the total cost is

$$C = 3x^2 + 4 \cdot 3xz + x^2 + 6 = 4x^2 + 12xz + 6.$$

There is clearly no benefit to using less than \$54 worth of material, so we set

$$54 = C = 4x^2 + 12xz + 6.$$

It is easier to solve for z than for x, so we get:

$$48 - 4x^2 = 12xz$$
$$\frac{48 - 4x^2}{12x} = z.$$

Since z is determined by x, we can write the volume V as a function $V(x)$ of x. That is,

$$V(x) = x^2 z = x^2\left(\frac{48 - 4x^2}{12x}\right) = \frac{48x - 4x^3}{12} = 4x - \tfrac{1}{3}x^3.$$

Note that we want to write $V(x)$ in the simplest possible form, to make further work easy.

We now determine the constraints. One is obvious: $x \geq 0$. (Allowing the "degenerate" case $x = 0$ will allow us to find the maximum over a closed bounded interval, which simplifies later steps.) The other constraint is $z \geq 0$, which is the same as $48 - 4x^2 \geq 0$. Rewriting this, we get $x^2 \leq 12$, which means $-\sqrt{12} \leq x \leq \sqrt{12}$. Since we already know $x \geq 0$, our final constraint is:

$$0 \leq x \leq \sqrt{12}.$$

(Note that $x = \sqrt{12}$ is also a degenerate case.)

We now must maximize $V(x) = 4x - \frac{1}{3}x^3$ for x in $[0, \sqrt{12}]$. Differentiate:

$$V'(x) = 4 - x^2.$$

Set the derivative equal to zero and solve:

$$0 = V'(x) = 4 - x^2$$
$$x = \pm 2.$$

We ignore the solution $x = -2$, since -2 is not in the interval $[0, \sqrt{12}]$. Since we are maximizing over a closed bounded interval, we need only compare the numbers $V(0)$, $V(2)$, and $V(\sqrt{12})$. These are

$$V(0) = 4 \cdot 0 - \tfrac{1}{3} \cdot 0^3 = 0, \quad V(2) = 4 \cdot 2 - \tfrac{1}{3} \cdot 2^3 = 8 - \tfrac{8}{3} = \tfrac{16}{3},$$

and

$$V(\sqrt{12}) = 4 \cdot \sqrt{12} - \tfrac{1}{3}\left(\sqrt{12}\right)^3 = 4 \cdot \sqrt{12} - \tfrac{12}{3} \cdot \sqrt{12} = 0.$$

Clearly the largest value is at $x = 2$. Therefore $y = 2$ and

$$z = \frac{48 - 4x^2}{12x} = \frac{48 - 16}{24} = \frac{4}{3}.$$

So the dimensions are 2 feet $\times$ 2 feet $\times$ $\frac{4}{3}$ feet. (Be sure to include the units!)

For reference, here are the other two approaches to test for a maximum. For both, we start at the point above where we found that $x = 2$ is the only number in our interval with $V'(x) = 0$. In both cases, one must still find the dimensions as above. Note that both of these also work over the *open* interval $(0, \sqrt{12})$.

First derivative method: We know $V'(x) = 4 - x^2$. So $V'(x) > 0$ for $0 \leq x < 2$, and $V'(x) < 0$ for $x > 2$. This shows that $V(x)$ is increasing for $0 \leq x < 2$ and decreasing for $x > 2$. So $V(2)$ must be the largest value of $V(x)$ for x in $[0, \sqrt{12}]$.

Second derivative method: We know $V'(x) = 4 - x^2$. So $V''(x) = -2x$. This is negative on the entire interval $(0, \sqrt{12})$, so $V(x)$ is concave down there, and any number x with $V'(x) = 0$ must give a global maximum.

4. James Tutt Snodgrass III has finished Math 251 and gone home for the holidays to his mother's farm, only to be asked to solve the following problem:

His mother wants to fence off a rectangular enclosure for llamas. The west side of the enclosure will be a wall which is already present. The north side will be an old fence which will cost \$10 per foot to adequately reinforce. The remaining two sides will consist of new fencing costing \$20 per foot. What are the dimensions of the largest enclosure that can be made for \$6000?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Solution: Note: There is no picture in this file. A picture may be provided separately.

The arrangement of fences and the wall will be a rectangle with sides running north-south and east-west. We are supposed to maximize the total area of the enclosure; call it A.

Let x be the length of the enclosure in the east-west direction, and let y be its length in the north-south direction, both measured in feet. Then the total area is $A = xy$. This expression has too many variables, and ignores the fact that x and y are not independent. We must eliminate one of the variables by using the restriction on the total cost. There is nothing to be gained by using less than the allowed amount of money, so we assume the total cost is exactly 6000 (measured in dollars). The cost of the west side is 0, regardless of its length (this is where the wall is). The east side has length y feet and costs \$20 per foot, for a cost of $20y$. The south side has length x feet and costs \$20 per foot, for a cost of $20x$. The north side also has length x feet, but costs only \$10 per foot, for a cost of $10x$. Adding these together give a total cost C of

$$6000 = C = 0 + 20y + 20x + 10x = 20y + 30x.$$

It doesn't matter much which variable you solve for. I decided to solve for y, giving

$$y = \frac{6000 - 30x}{20} = 300 - \tfrac{3}{2}x.$$

Substitute this for y in the formula $A = xy$, write it as a function of x, and multiply out so that it is convenient to differentiate:

$$A(x) = x\left(300 - \tfrac{3}{2}x\right) = 300x - \tfrac{3}{2}x^2.$$

The constraints are that x and y are both nonnegative. (Allowing the "degenerate" cases $x = 0$ and $y = 0$ will allow us to find the maximum over a closed bounded interval, which simplifies later steps.) The significance of the requirement $x \geq 0$ is clear, but we must express the requirement $y \geq 0$ in terms of x. Since $y = 300 - \frac{3}{2}x$, it says that $x \leq 200$.

(Here is another way to see this. The total cost of the north and south sides is $30x$, and this can be at most 6000.) Our problem is therefore to maximize the function $A(x) = 300x - \frac{3}{2}x^2$ for x in the closed bounded interval $[0, 200]$.

We search for critical numbers. Differentiate:

$$A'(x) = 300 - 3x.$$

Set the derivative equal to zero and solve:

$$0 = A'(x) = 300 - 3x$$
$$x = 100.$$

Since we are maximizing over a closed bounded interval, we need only compare the numbers $A(0)$, $A(100)$, and $A(200)$. These are

$$A(0) = 300 \cdot 0 - \tfrac{3}{2} \cdot 0^2 = 0, \quad A(100) = 300 \cdot 100 - \tfrac{3}{2} \cdot 100^2 = 30000 - 15000 = 15000,$$

and

$$A(200) = 300 \cdot 200 - \tfrac{3}{2} \cdot 200^2 = 60000 - 60000 = 0.$$

Clearly the largest value is at $x = 100$. Therefore the east-west length should be 100 feet, and the north-south length should be $300 - \frac{3}{2} \cdot 100 = 150$ feet. (Include the units!)

For reference, here are the other two approaches to test for a maximum. For both, we start at the point above where we found that $x = 100$ is the only number in our interval with $A'(x) = 0$. In both cases, one must still find the dimensions as above. Note that both of these also work over the *open* interval $(0, 200)$, and so could have been done without allowing the degenerate cases $x = 0$ and $y = 0$.

First derivative method: We know $A'(x) = 300 - 3x$. So $A'(x) > 0$ for $0 \leq x < 100$, and $A'(x) < 0$ for $x > 100$. This shows that $A(x)$ is increasing for $0 \leq x < 100$ and decreasing for $x > 100$. So $A(100)$ must be the largest value of $A(x)$ for x in $[0, 200]$.

Second derivative method: We know $A'(x) = 300 - 3x$. So $A''(x) = -3$. This is negative on the entire interval $(0, 200)$, so $A(x)$ is concave down there, and any number x with $A'(x) = 0$ must give a global maximum.

5. Let $g(x) = -\frac{1}{8}x^8 + \frac{1297}{96}x^6 - \frac{81}{64}x^4 - 4$. Produce graphs (more than one, if necessary) of $y = g(x)$ which reveal all the important features of the function. In particular, estimate the intervals of increase and decrease, critical numbers, extreme values, intervals of concavity, and inflection points, either using graphs of the first and second derivatives of the function, or directly from the formulas for these derivatives. (Your graphs must be shown on the test paper.)

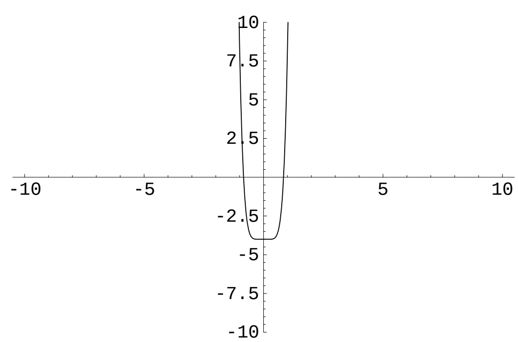
Hint: Here are the first and second derivatives in partially factored form:

$$g'(x) = x^3(81 - x^2)\left(x^2 - \tfrac{1}{16}\right) \quad \text{and} \quad g''(x) = -\tfrac{1}{16}x^2(112x^4 - 6485x^2 + 243).$$

The roots of $112x^4 - 6485x^2 + 243 = 0$ are approximately ± 0.193637 and approximately ± 7.60686.

Solution: This function is a polynomial of degree more than 1, so has no asymptotes and is defined everywhere. One very useful point is that the function is even: $g(-x) = g(x)$. This means that the graph is symmetric about the y-axis. In particular, this reduces the number of times we must evaluate g, which is helpful because of the somewhat messy coefficients.

For reference, here is the graph of g in the standard viewing rectangle on the TI-83 calculator, namely $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$:



We will see that this graph shows very little of the true shape of the graph of g.

We start the main part of our work by finding the critical numbers. From the factored form, this is easy: they are 0, ± 9, and $\pm \frac{1}{4} = \pm 0.25$. To get an idea of the shape of things, evaluate g at these numbers, using a calculator. Using the fact that g is even, we need only do three evaluations, of which one is very easy:

$$g(0) = -4, \quad g\left(\tfrac{1}{4}\right) = g\left(-\tfrac{1}{4}\right) \approx -4.00165, \quad \text{and} \quad g(9) = g(-9) \approx 1.79084 \cdot 10^6.$$

Since the derivative can't change sign on any of the intervals determined by the critical numbers, by simply comparing their sizes we see that:

g is decreasing on $\left(-9,-\frac{1}{4}\right)$.
g is increasing on $\left(-\frac{1}{4},0\right)$.
g is decreasing on $\left(0,\frac{1}{4}\right)$.
g is increasing on $\left(\frac{1}{4},9\right)$.

We could also find this by evaluating $g'(x)$ for one number x in each of these intervals, by examining the sign of the derivative directly from the formula, or by looking at suitable graphs.

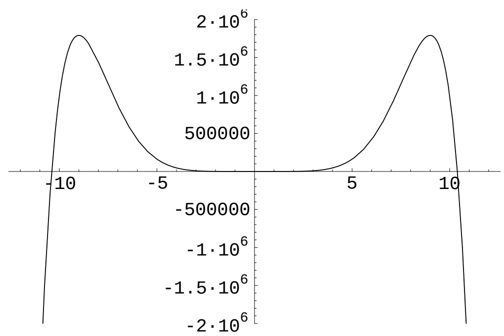
The derivative also can't change sign on either of the intervals $(-\infty,-9)$ or $(9,\infty)$. By considering the signs of $g'(-10)>0$ and $g'(10)<0$, we see that g' is positive on $(-\infty,-9)$ and negative on $(9,\infty)$. Therefore:
g is increasing on $(-\infty,-9)$.
g is decreasing on $(9,\infty)$.

We also see that g has local maximums at ± 9 and at 0, and local minimums at $\pm\frac{1}{4}$, with the values as calculated above.

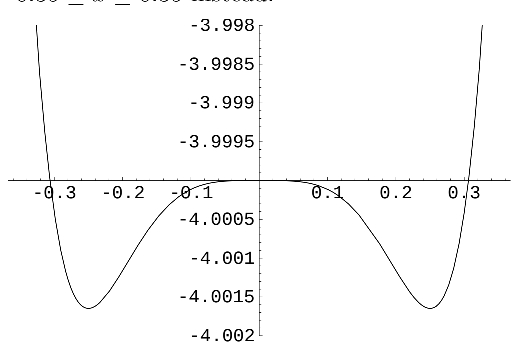
Since g is a polynomial with even degree and negative leading coefficient, we have

$$\lim_{x\to\infty} g(x)=-\infty \quad \text{and} \quad \lim_{x\to-\infty} g(x)=-\infty.$$

So a reasonable choice of a viewing rectangle for the largest scale features of the graph might be $-20\le x\le 20$ and $-2\cdot 10^6\le y\le 2\cdot 10^6$. The number 20 is chosen to be enough larger than 9 that we expect the graph to have gone back down below zero, and the number $2\cdot 10^6$ is chosen to be a convenient number not much larger than $g(9)$. The resulting plot shows that the function goes down very steeply indeed outside the interval $[-9,9]$, so I redid it with $-12\le x\le 12$ instead. Here is the result:



Our work above shows that, contrary to the appearance of this graph, there is a local maximum at $x=0$. To exhibit it and the neighboring local minimums properly, let's choose a viewing rectangle containing $x=\pm\frac{1}{4}$ and whose vertical direction includes the values of g at these places and at $x=0$. The viewing rectangle $-\frac{1}{2}\le x\le\frac{1}{2}$ and $-4.002\le y\le -3.998$ looks like a reasonable choice. After looking at it, I again decided to make the range of x a little narrower. Here is what I got using $-0.35\le x\le 0.35$ instead:



Finally, we consider inflection points and concavity. Using the hint, it is clear that $g''(x)=0$ for the following (approximate) values of x:

$$x=0, \quad x\approx\pm 0.193637, \quad \text{and} \quad x\approx\pm 7.60686.$$

There is no inflection point at $x=0$; that is a local maximum instead. The other four numbers are all inflection points. The points at $x\approx\pm 7.60686$ are clearly visible on the larger of the two graphs above, and the points at $x\approx\pm 0.193637$ are clearly visible on the smaller of the two graphs above. Therefore:
g is concave down approximately on $(-\infty,-7.60686)$.
g is concave up approximately on $(-7.60686,-0.193637)$.
g is concave down approximately on $(-0.193637,0.193637)$.
g is concave up approximately on $(0.193637,7.60686)$.

g is concave down approximately on $(7.60686, \infty)$.

Finally, it seems reasonable (although is not strictly necessary) to give a graph with a scale intermediate between the two scales above. After a bit of experimenting, I chose the viewing rectangle $-1.5 \leq x \leq 1.5$ and $-10 \leq y \leq 100$, getting the following: