# SOLUTIONS TO MATH 251 FINAL EXAM (WEDNESDAY)

1. (1 point) What do you think math professors do on spring break?

*Solution:* Well, some of them grade final exams.

2. (12 points/part) Find the exact values of the following limits (possibly including $\infty$ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\lim_{x \to 0} \dfrac{\cos(3x) - 1}{7x^2}$

*Solution:* The limit has the indeterminate form $\dfrac{0}{0}$. Therefore we may use L'Hospital's Rule. The chain rule gives $\dfrac{d}{dx}\big(\cos(3x) - 1\big) = -3\sin(3x)$, so

$$\lim_{x \to 0} \frac{\cos(3x) - 1}{7x^2} = \lim_{x \to 0} \frac{-3\sin(3x)}{14x},$$

if the second limit exists. The second limit also has the indeterminate form $\dfrac{0}{0}$. Therefore we may use L'Hospital's Rule again. A similar calculation shows that

$$\lim_{x \to 0} \frac{-3\sin(3x)}{14x} = \lim_{x \to 0} \frac{-9\cos(3x)}{14},$$

if the limit on the right exists. But the limit on the right is equal to $-9\cos(3 \cdot 0)/14 = -9/14$. Therefore

$$\lim_{x \to 0} \frac{\cos(3x) - 1}{7x^2} = -\frac{9}{14}.$$

(b) $\lim_{x \to 2} \dfrac{x^2 - 4}{x^2 - 5x - 6}$.

*Solution:* As always, the first thing we do is try to substitute $x = 2$. We get

$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 5x - 6} = \frac{2^2 - 4}{2^2 - 5 \cdot 2 - 6} = \frac{0}{-12} = 0.$$

Note that L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

$$\lim_{x \to 2} \frac{2x}{2x - 5} = -4,$$

which is the wrong answer.

(c) $\lim_{x \to \infty} \dfrac{7x + 2}{\sqrt{2x^2 + 7189}}$.

*Solution 1:* We have

$$\lim_{x \to \infty} (7x + 2) = \infty \quad \text{and} \quad \lim_{x \to \infty} \sqrt{2x^2 + 7189} = \infty,$$

so the limit has the indeterminate form $\dfrac{\infty}{\infty}$, and more work is needed. We multiply the numerator and denominator by $\frac{1}{x}$. When simplifying the resulting numerator, note that $\sqrt{x^2} = x$ for $x > 0$, and we need only consider $x > 0$ when dealing with a limit at $+\infty$. Thus:

$$\lim_{x \to \infty} \frac{7x + 2}{\sqrt{2x^2 + 7189}} = \lim_{x \to \infty} \frac{\frac{1}{x}\big(7x + 2\big)}{\frac{1}{x}\sqrt{2x^2 + 7189}} = \lim_{x \to \infty} \frac{7 + \frac{2}{x}}{\sqrt{\frac{1}{x^2}}\sqrt{2x^2 + 7189}} = \lim_{x \to \infty} \frac{7 + \frac{2}{x}}{\sqrt{\frac{1}{x^2}\left(2x^2 + 7189\right)}}$$

$$= \lim_{x \to \infty} \frac{7 + \frac{2}{x}}{\sqrt{2 + \frac{7189}{x^2}}} = \frac{\lim_{x \to \infty} \big(7 + \frac{2}{x}\big)}{\lim_{x \to \infty} \sqrt{2 + \frac{7189}{x^2}}} = \frac{7}{\sqrt{2}}.$$

*Date*: 21 March 2008.

*Solution 2:* (This solution uses L'Hospital's rule, but is not recommended. See Solution 3 instead.) As in Solution 1, we find that the limit has the indeterminate form $\frac{\infty}{\infty}$. Therefore we use L'Hospital's Rule. Replacing both the numerator and denominator by their derivatives, as called for in L'Hospital's Rule, we get

$$\lim_{x\to\infty} \frac{7}{\frac{1}{2}(2x^2+7189)^{-1/2}\cdot 2\cdot 2x} = \lim_{x\to\infty} \frac{7\sqrt{2x^2+7189}}{2x}.$$

It does not help very much to apply L'Hospital's Rule again, since that gives

$$\lim_{x\to\infty} \frac{\frac{1}{2}\left(2x^2+7189\right)^{-1/2} 2\cdot 2x}{2} = \lim_{x\to\infty} \frac{7x}{\sqrt{2x^2+7189}}.$$

Further applications of L'Hospital's Rule keep yielding the same two expressions over and over again. Instead, we use $x=\sqrt{x^2}$ (as in Solution 1, we need only consider positive values of $x$) to bring the denomenator under the square root:

$$\begin{aligned}\lim_{x\to\infty} \frac{7\sqrt{2x^2+7189}}{2x} &= \lim_{x\to\infty} \frac{7\sqrt{2x^2+7189}}{2\sqrt{x^2}} = \lim_{x\to\infty} \frac{7}{2}\sqrt{\frac{2x^2+7189}{x^2}} \\ &= \frac{7}{2}\sqrt{2+\lim_{x\to\infty}\frac{7189}{x^2}} = \frac{7}{2}\sqrt{2}.\end{aligned}$$

Since this limit exists, we also get, by L'Hospital's rule,

$$\lim_{x\to\infty} \frac{\sqrt{7x+2}}{2x^2+7189} = \lim_{x\to\infty} \frac{7\sqrt{2x^2+7189}}{2x} = \frac{7}{2}\sqrt{2}.$$

This can be shown to be the same answer as in Solution 1.

*Solution 3:* (This is better than Solution 2, but not as good as Solution 1.) For $x>0$, we have

$$\frac{7x+2}{\sqrt{2x^2+7189}} > 0.$$

Therefore we can write

$$\lim_{x\to\infty} \frac{7x+2}{\sqrt{2x^2+7189}} = \lim_{x\to\infty} \sqrt{\left(\frac{7x+2}{\sqrt{2x^2+7189}}\right)^2} = \lim_{x\to\infty} \sqrt{\frac{(7x+2)^2}{2x^2+7189}} = \sqrt{\lim_{x\to\infty} \frac{(7x+2)^2}{2x^2+7189}}.$$

So it suffices to find

$$\lim_{x\to\infty} \frac{(7x+2)^2}{2x^2+7189}.$$

One way is to write

$$\begin{aligned}\lim_{x\to\infty} \frac{(7x+2)^2}{2x^2+7189} &= \lim_{x\to\infty} \frac{7^2x^2+2\cdot 7\cdot 2x+2^2}{2x^2+7189} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x^2}\left(7^2x^2+2\cdot 7\cdot 2x+2^2\right)}{\frac{1}{x^2}\left(2x^2+7189\right)} \\ &= \lim_{x\to\infty} \frac{7^2+\frac{2\cdot 7\cdot 2}{x}+\frac{2^2}{x^2}}{2+\frac{1}{x^2}} = \frac{7^2+0+0}{2+0+0} = \frac{7^2}{2}.\end{aligned}$$

Another way is to use L'Hospital's rule twice. In condensed form:

$$\lim_{x\to\infty} \frac{(7x+2)^2}{2x^2+7189} = \lim_{x\to\infty} \frac{2\cdot 7(7x+2)}{2\cdot 2\cdot x} = \lim_{x\to\infty} \frac{2\cdot 7^2}{2\cdot 2} = \frac{7^2}{2}.$$

In either case, one gets

$$\lim_{x\to\infty} \frac{\sqrt{7x+2}}{2x^2+7189} = \sqrt{\frac{7^2}{2}} = \frac{7}{\sqrt{2}}.$$

3. (14 points) If $\sin(xy) = x+y+\ln(2)$, find $\frac{dy}{dx}$. (Use implicit differentiation. You must solve for $\frac{dy}{dx}$.)

*Solution:* We can write the equation as $\sin(xy(x)) = x+y(x)+\ln(2)$. Differentiate both sides with respect to $x$, using the product and and the chain rules on the left:

$$\cos(xy(x))\big(1\cdot y(x)+xy'(x)\big) = 1+y'(x).$$

(The derivative of $\ln(2)$ is immediately seen to be zero because $\ln(2)$ is a constant.) Now solve for $y'(x)$:

$$\cos(xy(x))y(x)+\cos(xy(x))xy'(x) = 1+y'(x)$$

$$\cos(xy(x))y(x)-1 = y'(x)-\cos(xy(x))xy'(x) = \big(1-x\cos(xy(x))\big)y'(x)$$

$$y'(x) = \frac{\cos(xy(x))y(x) - 1}{1 - x\cos(xy(x))}.$$

This expression can't be simplified.

For those who prefer the other notation, here it is written using $\frac{dy}{dx}$:

$$\cos(xy)\left(1 \cdot y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}.$$

(The derivative of $\ln(2)$ is immediately seen to be zero because $\ln(2)$ is a constant.) Now solve for $\frac{dy}{dx}$:

$$\cos(xy)y + \cos(xy)x\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\cos(xy)y - 1 = \frac{dy}{dx} - \cos(xy)x\frac{dy}{dx} = \big(1 - x\cos(xy)\big)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cos(xy)y - 1}{1 - x\cos(xy)}.$$

This expression can't be simplified.

4. (44 points) Professor Malvixx wants to build a walled rectangular enclosure. One side will be part of a long already existing wall which runs east-west, leaving three walls to build. The enclosure is to be divided in half by a fourth wall which is perpendicular to the existing wall. (She wants to put a Crumple-Horned Snorkack in one half, and a Spiral-Horned Snorkack in the other half.) If the total area is to be 1200 square meters, what is the shortest possible total length of wall needed to build such an enclosure?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

*Solution:* Note: There is no picture in this file. A picture may be provided separately.

The arrangement of walls will be a rectangle with sides running north-south and east-west. We are supposed to minimize the total length of the new walls; call it $L$.

Let $x$ be the length of the enclosure in the east-west direction, and let $y$ be its length in the north-south direction, both measured in meters. Then the total length of the walls is $L = x + 3y$. (There are three new north-south walls, each of length $y$, and there is one new east-west wall, of length $x$.) This expression has too many variables, and ignores the fact that $x$ and $y$ are not independent. We must eliminate one of the variables by using the restriction on the total area, which is $xy = 1200$. It doesn't matter much which variable you solve for. I decided to solve for $x$, giving

$$x = \frac{1200}{y}.$$

Substitute this for $x$ in the formula $L = x + 3y$, and write it as a function of $y$:

$$L(y) = \frac{1200}{y} + 3y = 1200\,y^{-1} + 3y.$$

The constraints are that $x$ and $y$ are both strictly positive. The significance of the requirement $y > 0$ is clear, but we must express the requirement $x > 0$ in terms of $y$. Since $y = 1200/x$, this constraint also says that $y > 0$. Our problem is therefore to minimize the function $L(y) = 1200\,y^{-1} + 3y$ for $y$ in the interval $(0, \infty)$.

We search for critical numbers. Differentiate:

$$L'(y) = -1200\,y^{-2} + 3.$$

Set the derivative equal to zero and solve:

$$0 = L'(y) = -1200\,y^{-2} + 3$$

$$1200\,y^{-2} = 3$$

$$1200 = 3y^2$$

$$400 = y^2$$

$$y = \pm 20$$

We reject $y = -20$ because it is not in $(0, \infty)$. (I must see you do this. Otherwise, I can't tell if you even knew that $y = -20$ is a solution.)

Is it a maximum or minimum? The easiest test to use here is the second derivative test. One easily checks that

$$L''(y) = 2400y^{-3},$$

which is positive for all $x$ in the interval $(0, \infty)$. Therefore $L$ is concave up everywhere on the interval $(0, \infty)$, and any critical number must be an absolute minimum.

The problem asked for the minimum total length of the walls. This is now

$$L(20) = \frac{1200}{20} + 3 \cdot 20 = 120$$

meters. (Don't forget the units!)

For those who tried it instead of the second derivative test as done above, here is how the first derivative test works. We factor the derivative as follows:

$$L'(x) = 3y^{-2}(y^2 - 400).$$

All factors except $y^2 - 400$ are always positive. For $0 < y < 20$, the factor $y^2 - 400$ is negative. Thus $L'(y) < 0$, and $L$ is decreasing on the whole interval $(0, 20)$. For $y > 20$, the factor $y^2 - 400$ is positive. Thus $L'(y) > 0$, and $L$ is increasing on the whole interval $(20, \infty)$. Clearly, then, there is an absolute minimum at $y = 20$.

Here is a different approach to the first derivative test. We know that $L'(y)$ is continuous on the whole interval $(0, \infty)$. It is zero only at $y = 20$. Therefore it must have the same sign throughout the interval $(0, 20)$, and we can find that sign by computing, say, $L'(1) = -1200 + 3 < 0$. The derivative must also have the same sign throughout the interval $(20, \infty)$, and we get it by computing, say, $L'(30) = 3 \cdot 30^{-2}(30^2 - 400) > 0$. As above, then, $L(y)$ is decreasing on the whole interval $(0, 20)$ and increasing on the whole interval $(20, \infty)$, so has an absolute minimum at $y = 20$.

Finally, here is the test using limits at the ends of the domain. We compare $L(20) = 600$ (computed above),

$$\lim_{y \to 0^+} L(y) = \lim_{y \to 0^+} \left( \frac{1200}{y} + 3y \right) = \infty,$$

and

$$\lim_{y \to \infty} L(y) = \lim_{y \to \infty} \left( \frac{1200}{y} + 3y \right) = \infty.$$

Obviously 600 is the smallest of these.

5. (10 points/part)

(a) Let $h$ be a function such that $h'(x) = \dfrac{\sin(x)}{x} + x$. Find the derivative of the function $f(x) = h(e^x) + \pi^2$. (Your answer might involve the function $h$.)

*Solution:* Use the chain rule on the first term, and note that the second term is a constant, so that its derivative is zero:

$$f'(x) = h'(e^x)e^x = \left( \frac{\sin(e^x)}{e^x} + e^x \right) e^x = \sin(e^x) + (e^x)^2 = \sin(e^x) + e^{2x}.$$

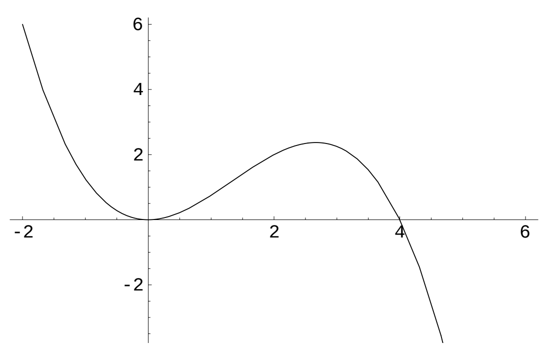
(The simplification is necessary.)

(b) Let $c$ be a constant. Let $g(x) = x \tan(cx^2 + x)$. Find $g'(x)$.

*Solution:* We use the product rule, remembering to use the chain rule on the second factor:

$$\begin{aligned} g'(x) &= 1 \cdot \tan(cx^2 + x) + x \sec^2(cx^2 + x) \frac{d}{dx}(cx^2 + x) \\ &= \tan(cx^2 + x) + x \sec^2(cx^2 + x)(2cx + 1) = \tan(cx^2 + x) + x(2cx + 1)\sec^2(cx^2 + x). \end{aligned}$$

6. (5 points/part) The picture below shows the graph of $y = f(x)$ for a particular function $f$. (This is a graph of the function, *not* its derivative.)




For each part, use this graph to find at least one number $x$ in the interval shown satisfying the stated conditions, or explain why no such number $x$ exists.

(a) $f(x) > 0$ and $f'(x) < 0$.

*Solution:* We are looking for values of $x$ at which the function is positive ($f(x) > 0$) and decreasing ($f'(x) < 0$). Any $x$ with $-2 < x < 0$ will do, as will values of $x$ near 3 and between 3 and 4. So $x = -1$ and $x = 3$ would be correct answers, and there are many others.

(b) $f'(x) < 0$ and $f''(x) > 0$.

*Solution:* We are looking for values of $x$ at which the function is decreasing ($f'(x) < 0$) and concave up ($f''(x) > 0$). Any $x$ with $-2 < x < 0$ will do, but there are no other such points shown. So $x = -1$ and $x = -\frac{3}{2}$ would be correct answers, and there are many others.

(c) $x < 0$ and $f''(x) < 0$.

*Solution:* We are looking for values of $x$ to the left of the $y$-axis at which the function is concave down ($f''(x) < 0$). We can see that the graph that is concave up for $-2 < x < 0$. Thus, no such values of $x$ are shown in the picture.

(d) $f(x) = 0$, $f'(x) = 0$, and $f''(x) \geq 0$.

*Solution:* We are looking for values of $x$ at which the function value is 0 ($f(x) = 0$), at which the function is flat ($f'(x) = 0$) (that is, critical points of the function), and concave up, or at least not concave down ($f''(x) \geq 0$). There are only two points shown at which $f(x) = 0$, namely $x = 0$ and $x = 4$. One of them, namely $x = 0$, matches this description, and the other doesn't ($f'(4) < 0$ and $f''(4) < 0$). So there is only one correct answer, namely $x = 0$.

7. (30 points) A spherical balloon was being slowly blown up, starting at noon one day. At 12:10 pm, its radius was 20 inches, and was increasing at 5 inches per minute. Was its surface area increasing or decreasing? At what rate? (Be sure to include the correct units in your answer.)

Note: There is no picture in this file.

*Solution:* Let $r(t)$ be the radius at time $t$, measured in inches and with time measured in minutes after noon, and let $S(t)$ be the surface area at time $t$, measured in square inches. Note that both the surface area and the radius vary with time, so must be treated as functions not constants.

The time we are interested in is $t = 10$. We are given $r(10) = 20$ and $r'(10) = 5$. The functions $S(t)$ and $r(t)$ are related by the equation

$$S(t) = 4\pi \left[r(t)\right]^2 .$$

(You are expected to know this formula.) Differentiate with respect to $t$:

$$S'(t) = 4\pi \cdot 2r(t)r'(t) = 8\pi r(t)r'(t).$$

(Don't forget the factor $r'(t)$! That will spoil the whole thing!) Evaluate this at $t = 10$, using $r(10) = 20$ and $r'(10) = 5$. This gives

$$S'(10) = 8\pi r(10)r'(10) = 8\pi \cdot 20 \cdot 5 = 800\pi.$$

So the surface area is increasing at the rate of $800\pi$ square inches per minute. (Don't forget the units!)

Here, for reference, is what the solution looks like in physicists' notation:

$$S = 4\pi r^2.$$

Differentiate with respect to $t$:

$$\frac{dS}{dt} = 4\pi \cdot 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}.$$

(Don't forget the factor $\frac{dr}{dt}$!) Now substitute $r = 10$ and $\frac{dr}{dt} = 5$, getting $\frac{dS}{dt} = 8\pi \cdot 20 \cdot 5 = 800\pi$.

8. (35 points) Let $g(x) = -\frac{1}{7}x^7 - \frac{17}{4}x^6 - \frac{433}{10}x^5 - 153x^4$. Use your calculator to produce graphs (more than one, if necessary) of $y = g(x)$ which reveal all the important features of the function. In particular, estimate the intervals of increase and decrease, critical numbers, extreme values, intervals of concavity, and inflection points, either using graphs of the first and second derivatives of the function, or directly from the formulas for these derivatives. (Your graphs must be shown on the test paper.)

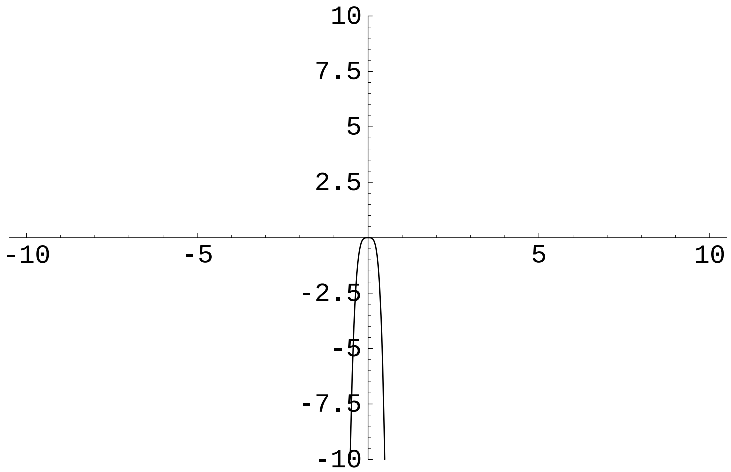
Hint: Here are the first and second derivatives in partially factored form:

$$g'(x) = -x^3(x+8)(x+9)\left(x + \tfrac{17}{2}\right) \quad \text{and} \quad g''(x) = -\tfrac{1}{2}x^2(12x^3 + 255x^2 + 1732x + 3672).$$

The roots of $12x^3 + 255x^2 + 1732x + 3672 = 0$ are approximately $-4.23039$, approximately $-8.22163$, and approximately $-8.79798$.

*Solution:* This function is a polynomial of degree more than 1, so has no asymptotes and is defined everywhere.

For reference, here is the graph of $g$ in the standard viewing rectangle on the TI-83 calculator, namely $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$ :




We will see that this graph shows rather little of the true shape of the graph of $g$.

The solution below describes how to find things; I don't expect every graph shown to be on your paper. I definitely want to see graphs that look like the ones referred to in the text in boldface as Graph 1 and Graph 2; the one referred to as Graph 3 would also be nice. Thus, I am asking for only two or three graphs.

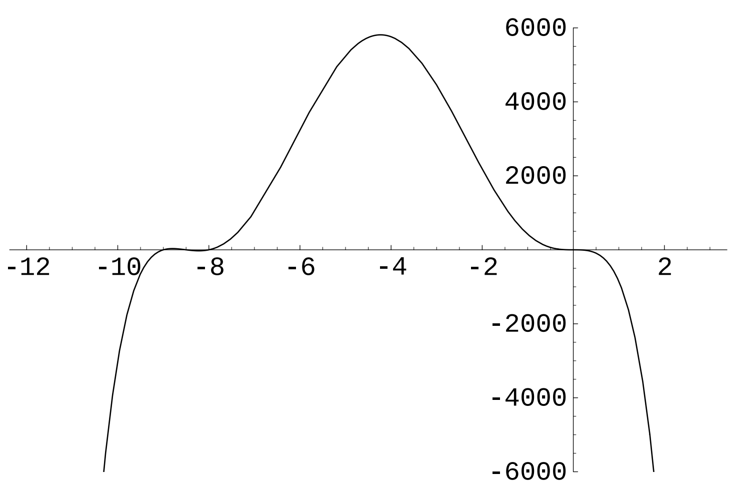
We start the main part of our work by finding the critical numbers. From the factored form, this is easy: they are 0, $-8$, $-\frac{17}{2} = -8.5$, and $-9$. To get an idea of the shape of the graph, evaluate $g$ at these numbers, using a calculator. (Probably the trace function is the easiest method.) On my computer, I get (to more accuracy than needed)

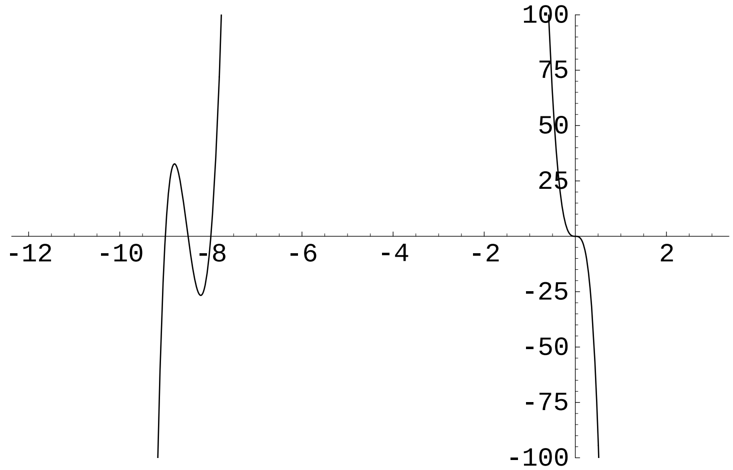
$$g(0) = 0, \quad g(-8) \approx -22352.5, \quad g\left(-\tfrac{17}{2}\right) \approx -22343.7, \quad \text{and} \quad g(-9) \approx -22354.3.$$

Since the derivative can't change sign on any of the intervals determined by the critical numbers, by simply comparing their sizes we see that:

- $g$ is increasing on $\left(-9, -\frac{17}{2}\right)$.
- $g$ is decreasing on $\left(-\frac{17}{2}, -8\right)$.
- $g$ is increasing on $(-8, 0)$.

We could also find this by evaluating $g(x)$ for one number $x$ in each of these intervals, by examining the sign of the derivative directly from the formula, or by looking at suitable graphs. Here, for reference, are two graphs of the derivative $y = g'(x)$, one in the viewing rectangle $-13 \leq x \leq 4$ and $-6000 \leq y \leq 6000$, and one in the viewing rectangle $-13 \leq x \leq 4$ and $-100 \leq y \leq 100$:






(I have not shown the work needed to choose a reasonable range for $y$ for these graphs, but trial and error will not do too badly, provided one uses the information about where $g'(x) = 0$. The range $-13 \leq x \leq 4$ was taken from choices made below.)

The derivative also can't change sign on either of the intervals $(-\infty, -9)$ or $(0, \infty)$. By considering the signs of

$$g'(-10) = -3000 < 0 \quad \text{and} \quad g'(1) = (-1)^3 \cdot 10 \cdot 9 \cdot \tfrac{19}{2} < 0,$$

we see that $g'$ is negative on $(-\infty, -9)$ and also negative on $(9, \infty)$. (This can also be easily done by examining the signs of the factors of the factored form of $g'$.) Therefore:

- $g$ is decreasing on $(-\infty, -9)$.
- $g$ is decreasing on $(0, \infty)$.

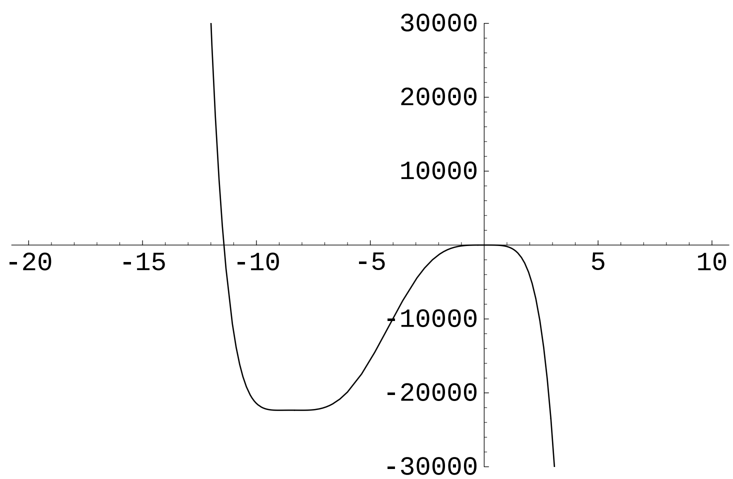
We have identified all the intervals of increase and decrease, so we can determine the nature of all the critical numbers. We get (using the values calculated above):

- $g$ has a local minimum at $-9$, with $g(-9) \approx -22354.3$.
- $g$ has a local maximum at $-\frac{17}{2}$, with $g\left(-\frac{17}{2}\right) \approx -22343.7$.
- $g$ has a local minimum at $-8$, with $g(-8) \approx -22352.5$.
- $g$ has a local maximum at 0, with $g(0) = 0$.

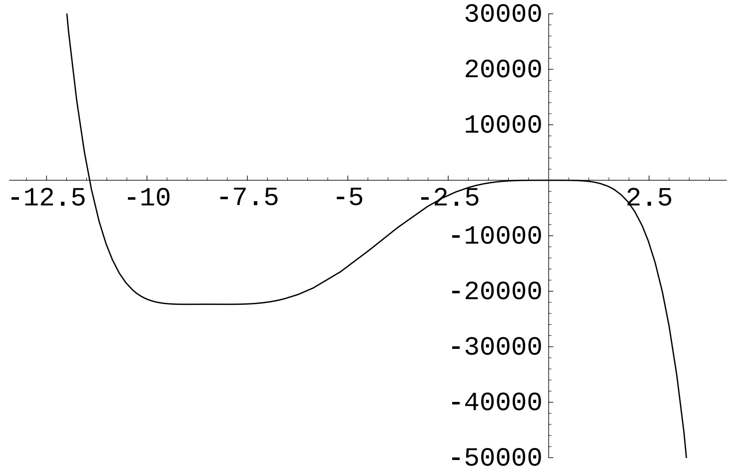
Since $g$ is a polynomial with odd degree and negative leading coefficient, we have

$$\lim_{x \to \infty} g(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} g(x) = \infty.$$

So a reasonable choice of a viewing rectangle for the largest scale features of the graph might be $-20 \leq x \leq 10$ and $-30000 \leq y \leq 30000$. The number $-20$ is chosen to be enough smaller than $-9$ that we expect the graph to have gone back up above zero, and the number 30000 is chosen to be a convenient number with $-30000$ not much smaller than $g(9)$. Here is what I got:



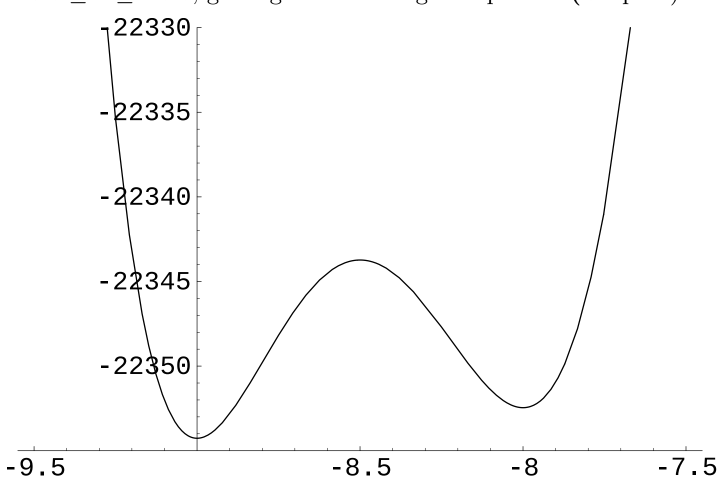
The resulting plot shows that the graph is very steep indeed outside the interval $[-11, 2]$, so I redid it with $-13 \le x \le 4$ instead. Also, since more of the interesting part of the graph is below the horizontal axis, I extended the vertical range in the negative direction to $-50000 \le y \le 30000$. Here is the result (Graph 1):



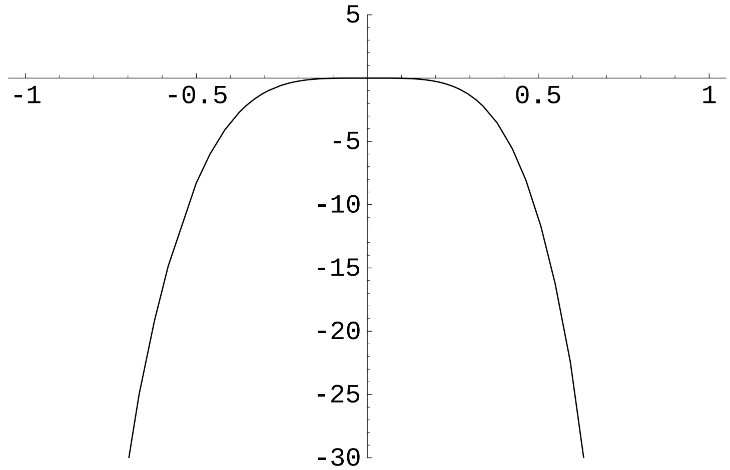

Our work above shows that there is indeed a local maximum at $x = 0$. However, contrary to the appearance of this graph, there is a local maximum at $x = -8.5$. To exhibit it and the neighboring local minimums properly, let's choose a viewing rectangle containing $x = -8$ and $x = -9$ and whose vertical direction includes the values of $g$ at these places and at $x = -8.5$. The viewing rectangle

$$-10 \le x \le -7 \quad \text{and} \quad -22355 \le y \le -22330$$

looks like a reasonable choice. The result is not shown, because the range of values of $x$ is again too wide. I therefore replaced $-10 \le x \le -7$ by $-9.5 \le x \le -7.5$, getting the following nice picture (Graph 2):




It is nice (although not really necessary) to examine a neighborhood of $x = 0$ at a similar scale to what we used above, to show how it differs from what we have just been looking at. I tried the viewing rectangle $-2 \le x \le 2$ and $-30 \le y \le 5$ (not shown). Once again, a narrower range of values of $x$ looked appropriate, so here is the viewing rectangle $-1 \le x \le 1$ and $-30 \le y \le 5$ (Graph 3):



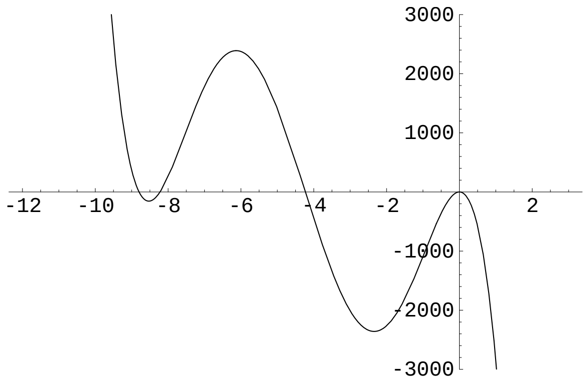

Finally, we consider inflection points and concavity. Using the hint, it is clear that $g''(x) = 0$ for the following (approximate) values of $x$ :

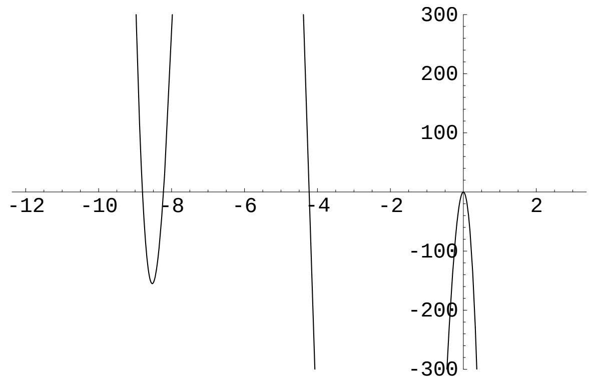
$$x = 0, \quad x \approx -4.23039, \quad x \approx -8.22163, \quad \text{and} \quad x \approx -8.79798.$$

There is no inflection point at $x = 0$; that is a local maximum instead, as we have already seen. The other three numbers are all inflection points. We can see this for $x \approx -4.23039$ by inspecting Graph 1 and for $x \approx -8.22163$ and $x \approx -8.79798$ inspecting Graph 2. By looking at these graphs, and also knowing where the inflection points are, we further find that:

- $g$ is concave up approximately on $(-\infty, -8.79798)$.
- $g$ is concave down approximately on $(-8.79798, -8.22163)$.
- $g$ is concave up approximately on $(-8.22163, -4.23039)$.
- $g$ is concave down approximately on $(-4.23039, \infty)$.

For reference (these were not required, but could help you determine the sign of $g''$ on various intervals), here are graphs of $y = g''(x)$, in the viewing rectangles $-13 \leq x \leq 4$ and $-3000 \leq y \leq 3000$, and $-13 \leq x \leq 4$ and $-300 \leq y \leq 300$ :




EC. Use the methods of calculus to prove the following statements:

(a) (2 extra credit points) $e^x > 1$ for all $x > 0$.

*Solution:* Let $f(x) = e^x$ for all real $x$. Then $f'(x) = e^x > 0$ for all real $x$. Therefore $f$ is (strictly) increasing. In particular, if $x > 0$ then $e^x = f(x) > f(0) = 1$.

(b) (5 extra credit points) $e^x > 1 + x$ for all $x > 0$.

*Solution:* Let $f(x) = e^x - 1 - x$ for all real $x$. Then $f'(x) = e^x - 1$, so Part (a) gives $f'(x) > 0$ for all $x > 0$. It follows that $f$ is (strictly) increasing on the interval $(0, \infty)$.

In fact, $f$ must be (strictly) increasing on the interval $[0, \infty)$. To see this, suppose that $f(x) = f(0)$ for some $x > 0$. Then $f\left(\frac{1}{2}x\right) < f(0)$ and $f\left(\frac{1}{2}x\right) < f(x)$. Choose $y_0$ strictly between $f(0)$ and $f\left(\frac{1}{2}x\right)$. Since $f$ is continuous, the intermediate value theorem gives $a$ in $\left(0, \frac{1}{2}x\right)$ and $b$ in $\left(\frac{1}{2}x, x\right)$ with $f(a) = f(b) = y_0$. But $0 < a < b$, so this contradicts the fact that $f$ is (strictly) increasing on $(0, \infty)$.

Since $f$ must be (strictly) increasing on the interval $[0, \infty)$, we have $e^x - 1 - x = f(x) > f(0) = 0$ for all $x > 0$. That is, $e^x > 1 + x$ for all $x > 0$.

(c) (8 extra credit points) $e^x > 1 + x + \frac{1}{2}x^2$ for all $x > 0$.

*Solution:* Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$ for all real $x$. Then $f'(x) = e^x - 1 - x$, so Part (b) gives $f'(x) > 0$ for all $x > 0$. It follows, as in Part (b), that $f$ is (strictly) increasing on the interval $[0, \infty)$. Thus, $e^x - 1 - x - \frac{1}{2}x^2 = f(x) > f(0) = 0$ for all $x > 0$. That is, $e^x > 1 + x + \frac{1}{2}x^2$ for all $x > 0$.

(d) (25 extra credit points) For every positive integer $n$,

$$e^x > 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

for all $x > 0$.

*Solution:* We prove this by induction on $n$. The case $n = 1$ is Part (b) (and, in fact, the case $n = 2$ is Part (c)). Assume therefore that the result is known for some $n$. We prove the statement for $n + 1$. Let

$$f(x) = e^x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{(n+1)!}\right) = e^x - 1 - x - \frac{x^2}{2} - \cdots - \frac{x^{n+1}}{(n+1)!}.$$

Then

$$f'(x) = e^x - 1 - x - \frac{x^2}{2} - \cdots - \frac{x^n}{n!},$$

as can be seen by differentiating term by term. The induction assumption implies that $f'(x) > 0$ for all $x > 0$. Therefore $f$ is (strictly) increasing on the interval $(0, \infty)$. Since $f$ is continuous, it follows (as in Part (b)) that $f$ is (strictly) increasing on the interval $[0, \infty)$. So $f(x) > f(0) = 0$ for all $x > 0$. Therefore

$$e^x > 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{(n+1)!}$$

for all $x > 0$. This completes the induction step, and hence the proof.