

MATH 251 EXTRA DIFFERENTIATION WORKSHEET SOLUTIONS

This is the solution sheet for the collection of extra practice differentiation problems. The solutions show steps, although the amount of detail varies.

1. Find $\frac{dy}{dx}$ for $y = 2c(x^4 + 7)(3x^6 - 2)$, where c is a constant.

Solution: We use the product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

as well as the rule for scalar multiples. So

$$\frac{dy}{dx} = 2c(4x^3(3x^6 - 2) + (x^4 + 7)(3 \cdot 6x^5)) = 8cx^3(3x^6 - 2) + 36cx^5(x^4 + 7).$$

2. Find $f'(x)$, where $f(x) = \frac{x}{x^2 + p^2}$, and where p is a constant.

Solution: Use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

This gives

$$f'(x) = \frac{1 \cdot (x^2 + p^2) - x \cdot (2x)}{(x^2 + p^2)^2} = \frac{x^2 + p^2 - 2x^2}{(x^2 + p^2)^2} = \frac{p^2 - x^2}{(x^2 + p^2)^2}.$$

(The simplification is required.)

3. Differentiate $f(x) = 3 \arcsin(cx^{1/3}) + \frac{\ln(x)}{2} + \sqrt{\pi}$, where c is a constant.

Solution: Use the chain rule on the first term and the rule for constant multiples on the second term:

$$f'(x) = 3 \frac{1}{\sqrt{1 - (cx^{1/3})^2}} \cdot c \cdot \frac{1}{3} x^{-2/3} + \frac{1}{2} \cdot \frac{1}{x} = \frac{cx^{-2/3}}{\sqrt{1 - c^2 x^{2/3}}} + \frac{1}{2x}.$$

Note that $\sqrt{\pi}$ is a constant, so that its derivative is zero. Also, don't waste time using the quotient rule to differentiate $\frac{\ln(x)}{2}$. Rewrite it as $\frac{1}{2} \ln(x)$ and use the rule for constant multiples instead—it is much faster.

4. Differentiate $h(x) = 2 \tan(cx + d) + \frac{x}{\sin(x)} + \frac{2}{11}$, where c and d are constants.

Solution: Use the chain rule on the first term and the quotient rule on the second term:

$$\begin{aligned} f'(x) &= 2 \sec^2(cx + d) \cdot c + \frac{\sin(x) - x \cos(x)}{\sin^2(x)} \\ &= 2c \sec^2(cx + d) + \frac{\sin(x) - x \cos(x)}{\sin^2(x)}. \end{aligned}$$

Note that $\frac{2}{11}$ is a constant, so that its derivative is zero. Don't waste time using the quotient rule to differentiate it!

You can also differentiate the second term using the product and chain rules, by rewriting it as $x[\sin(x)]^{-1}$.

5. Differentiate $g(a) = c \arctan(2a^{7/3})$, where c is a constant.

Solution: Use the chain rule:

$$g'(a) = c \cdot \frac{1}{1 + (2a^{7/3})^2} \cdot 2 \cdot \frac{7}{3} a^{4/3} = \frac{14ca^{4/3}}{3 + 12a^{14/3}}.$$

6. Differentiate $f(t) = \sqrt{a + \arcsin(t)} + \pi^3$, where a is a constant.

Solution: Rewrite

$$\sqrt{a + \arcsin(t)} = (a + \arcsin(t))^{1/2}$$

before differentiating. Then use the chain rule. Note that the derivative of π^3 is zero because π^3 is a constant. This gives:

$$f'(t) = \frac{1}{2}(a + \arcsin(t))^{-1/2} \frac{1}{\sqrt{1-t^2}} = \frac{1}{2\sqrt{a + \arcsin(t)}\sqrt{1-t^2}}.$$

7. Differentiate $f(t) = \sqrt[3]{ct + \arctan(t)} + e^3$, where c is a constant.

Solution: Rewrite

$$\sqrt[3]{ct + \arctan(t)} = (ct + \arctan(t))^{1/2}$$

before differentiating. Then use the chain rule. Note that the derivative of e^3 is zero because e^3 is a constant. This gives:

$$f'(t) = \frac{1}{2}(ct + \arctan(t))^{-1/2} \left(c + \frac{1}{1+t^2} \right) = \frac{c}{2\sqrt{ct + \arctan(t)}} + \frac{1}{2(1+t^2)\sqrt{ct + \arctan(t)}}.$$

8. Differentiate $y = x \tan(2x) + 3\sqrt{x} - \frac{1}{2}$.

Solution:

$$y'(x) = \tan(2x) + x \sec^2(2x) \cdot 2 + 3 \cdot \frac{1}{2} x^{-1/2} = \tan(2x) + 2x \sec^2(2x) + \frac{3}{2} x^{-1/2}.$$

Don't forget to use the chain rule when differentiating $\tan(2x)$. To differentiate the second term, rewrite it as $3x^{1/2}$. The derivative of $-\frac{1}{2}$ is immediately seen to be zero because $-\frac{1}{2}$ is a constant. (Don't waste time differentiating $-\frac{1}{2}$ via the quotient rule!)

9. Differentiate the function $q(x) = \cos(cx e^x + 7\pi^2)$, where c is a constant.

Solution: Use the chain rule. The product rule is needed to differentiate $x e^x$.

$$q'(x) = -\sin(cx e^x + 7\pi^2) \cdot c(e^x + x e^x) = -c(1+x)e^x \sin(cx e^x + 7\pi^2).$$

(The rearrangement is not really necessary.) The derivative of $7\pi^2$ is zero because $7\pi^2$ is a constant.

10. Differentiate $y = \frac{x}{\arctan(kx)} + \frac{\sqrt{\pi}}{2}$, where k is a constant.

Solution: Use the quotient rule. For $\frac{d}{dx}(\arctan(kx))$, it is necessary to use the chain rule, noting that, since k is a constant, we have $\frac{d}{dx}(kx) = k$. Thus

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(x) \cdot \arctan(kx) - x \frac{d}{dx}(\arctan(kx))}{[\arctan(kx)]^2} \\ &= \frac{\arctan(kx) - x \cdot \frac{1}{1+(kx)^2} \frac{d}{dx}(kx)}{[\arctan(kx)]^2} \\ &= \frac{\arctan(kx) - \frac{kx}{1+(kx)^2}}{[\arctan(kx)]^2}.\end{aligned}$$

The derivative of $\frac{\sqrt{\pi}}{2}$ is zero because $\frac{\sqrt{\pi}}{2}$ is a constant. (Don't waste time using the quotient rule to differentiate $\frac{\sqrt{\pi}}{2}$!)

11. Differentiate $f(x) = \left(\left((x^2 + \pi^2)^7 + \frac{9}{7} \right)^3 - 1 \right)^{100}$

Solution: This one requires repeated use of the chain rule.

$$\begin{aligned}f'(x) &= 100 \left(\left((x^2 + \pi^2)^7 + \frac{9}{7} \right)^3 - 1 \right)^{99} \cdot 3 \left((x^2 + \pi^2)^7 + \frac{9}{7} \right)^2 \cdot 7(x^2 + \pi^2)^6 \cdot 2x \\ &= 4200x \left(\left((x^2 + \pi^2)^7 + \frac{9}{7} \right)^3 - 1 \right)^{99} \left((x^2 + \pi^2)^7 + \frac{9}{7} \right)^2 (x^2 + \pi^2)^6.\end{aligned}$$

The derivatives of π^2 and $\frac{9}{7}$ are zero because π^2 and $\frac{9}{7}$ are constants. (Don't waste time using the quotient rule to differentiate $\frac{9}{7}$!)

12. Differentiate $w(t) = \arctan(\sqrt{3t+1}) - \ln(2\pi)$.

Solution: Use the chain rule twice on the first term. The derivative of the second term is zero because $\ln(2\pi)$ is a constant. This gives the answer

$$w'(t) = \frac{1}{1 + (\sqrt{3t+1})^2} \cdot \frac{d}{dt}(\sqrt{3t+1}) = \frac{1}{1 + (3t+1)} \cdot \left(\frac{1}{2}(3t+1)^{-1/2} \cdot 3 \right) = \frac{3}{2} \cdot \frac{1}{3t+2} \cdot (3t+1)^{-1/2}.$$

13. Let f be a function such that $f'(x) = 6e^x - 7f(x)$. Find the derivative of the function $h(z) = \frac{f(z)}{3z}$. (Your answer might involve the function f .)

Solution: Use the quotient rule:

$$h'(y) = \frac{f(z)}{3z} = \frac{f'(z) \cdot 3z - f(z) \cdot 3}{(3z)^2} = \frac{[6e^z - 7f(z)] \cdot 3z - 3f(z)}{9z^2} = \frac{18ze^z - 21zf(z) - 3f(z)}{9z^2}.$$

14. Let g be a function such that $g'(x) = e^{x^2} - e^x$. Find the derivative of the function $h(t) = t + \pi^2 - g(t)$. (Your answer might involve the function g .)

Solution:

$$h'(t) = 1 - g'(t) = 1 - e^{t^2} + e^t.$$

(Note that π^2 is a *constant*, so its derivative is zero.)

15. Let f be a function such that $f'(x) = 6xf(x)$. Find the derivative of the function $w(x) = f(\sqrt{7})$. (Your answer might involve the function f .)

Solution: $w'(x) = 0$ because $f(\sqrt{7})$ is a constant.

16. Differentiate $f(x) = x^2g(x) + \sqrt{\pi}$, where g is a function such that $g'(x) = \sqrt[3]{3x^2 - 6} - 13$. (Your answer might involve the function g .)

Solution: Use the product rule:

$$f'(x) = 2xg(x) + x^2g'(x) = 2xg(x) + x^2(\sqrt[3]{3x^2 - 6} - 13) = 2xg(x) + x^2\sqrt[3]{3x^2 - 6} - 13x^2.$$

(Note that $\sqrt{\pi}$ is a *constant*, so its derivative is zero.)

17. Let h be a function such that $h'(x) = \frac{\sin(3x)}{x}$. Find the derivative $\frac{d}{dt} \left(\frac{\sin(3t)}{h(t)} \right)$. (Your answer might involve the function h .)

Solution: Use the quotient rule:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\sin(3t)}{h(t)} \right) &= \frac{3 \cos(3t)h(t) - \sin(3t)h'(t)}{[h(t)]^2} = \frac{3 \cos(3t)h(t) - \sin(3t)\frac{\sin(3t)}{t}}{[h(t)]^2} \\ &= \frac{3t \cos(3t)h(t) - \sin^2(3t)}{t[h(t)]^2}. \end{aligned}$$

18. Let g be a function such that $g'(z) = e^{2z} - g(z)$. Find the derivative of the function $h(t) = 6g(2t+1)$. (Your answer might involve the function g .)

Solution: Use the chain rule:

$$h'(t) = 6g'(2t+1) \cdot \frac{d}{dt}(2t+1) = 6 \left[e^{2(2t+1)} - g(2t+1) \right] \cdot 2 = 12 \left[e^{4t+2} - g(2t+1) \right].$$

19. Let f be a function such that $f'(x) = (2x^2 + \sqrt{3})^{117}$. Find the derivative of the function $g(t) = f(t + \cos(t))$. (Your answer might involve the function f .)

Solution: Use the chain rule:

$$g'(t) = f'(t + \cos(t)) \cdot \frac{d}{dt}(t + \cos(t)) = \left(2(t + \cos(t))^2 + \sqrt{3} \right)^{117} [1 - \sin(t)].$$

20. Let g be a function such that $g'(x) = 3x + \frac{1}{\cos(x) + 9} + \ln(2)$. Find the derivative of the function $h(x) = g(2-x)$. (Your answer might involve the function g .)

Solution: Use the chain rule:

$$h'(x) = g'(2-x) \cdot \frac{d}{dt}(2-t) = \left(3x + \frac{1}{\cos(x) + 9} + \ln(2) \right) (-1) = -3x - \frac{1}{\cos(x) + 9} - \ln(2).$$

21. Let q be a function such that $q'(x) = xq(x)$. Find the derivative of the function $f(x) = q(2x + \sin(3))$. (Your answer might involve the function q .)

Solution: Use the chain rule:

$$\begin{aligned} f'(x) &= q'(2x + \sin(3)) \cdot \frac{d}{dx}(2x + \sin(3)) = [2x + \sin(3)]q'(2x + \sin(3)) \cdot 2 \\ &= 2[2x + \sin(3)]q'(2x + \sin(3)). \end{aligned}$$

(Note that $\sin(3)$ is a *constant*, so its derivative is zero.)

22. Let f be a function such that $f'(x) = \sqrt[3]{x^2 + 9} - 11$. Find the derivative of the function $h(x) = f(1 - x) + \pi^3$. (Your answer might involve the function f .)

Solution: Use the chain rule:

$$h'(x) = f'(1 - x) \cdot \frac{d}{dx}(1 - x) = (\sqrt[3]{(1 - x)^2 + 9} - 11) \cdot (-1) = -\sqrt[3]{(1 - x)^2 + 9} + 11.$$

The last expression can be rewritten as

$$11 - \sqrt[3]{x^2 - 2x + 10},$$

but this is not necessary. (Note that π^3 is a *constant*, so its derivative is zero.)

23. Let f be a function such that $f'(x) = \sqrt{x^3 + 7} + 6$. Find the derivative of the function $g(t) = f(3t) + \sqrt{2}$. (Your answer might involve the function f .)

Solution: Use the chain rule:

$$g'(t) = f'(3t) \cdot \frac{d}{dt}(3t) = (\sqrt{(3t)^3 + 7} + 6) \cdot 3 = 3\sqrt{27t^3 + 7} + 18.$$

(Note that $\sqrt{2}$ is a *constant*, so its derivative is zero.)

24. Differentiate $q(x) = \frac{7}{11} - 3f(\sin(x))$, given that $f'(x) = \sqrt[3]{1 + x^2}$. (Your answer might involve the function f .)

Solution: The derivative of $\frac{7}{11}$ is zero because $\frac{7}{11}$ is a constant. (Don't waste time using the quotient rule to differentiate it!) For the other term, we use the chain rule, and then substitute the formula for f' :

$$q'(x) = -3f'(\sin(x)) \cdot \cos(x) = -3\sqrt[3]{1 + \sin^2(x)} \cdot \cos(x).$$

(This looks nicer if rearranged as $-3\cos(x)\sqrt[3]{1 + \sin^2(x)}$, but that is not necessary.)

25. Let h be a function such that $h'(s) = 2s + \frac{1}{\sin(s) + 7} + e^4$. Find the derivative of the function $g(t) = \ln(h(t))$. (Your answer might involve the function h .)

Solution: Use the chain rule:

$$g'(t) = \frac{1}{h(t)} \cdot h'(t) = \frac{1}{h(t)} \left(2t + \frac{1}{\sin(t) + 7} + e^4 \right).$$

26. Let f be a function such that $f'(x) = f(x) - \sin(2x)$. Find the derivative of the function $h(y) = [f(y)]^2$. (Your answer might involve the function f .)

Solution: Use the chain rule:

$$h'(y) = 2f(y)f'(y) = 2f(y)[f(y) - \sin(2y)].$$

27. Let f be a function such that $f'(x) = e^{-x^2}$. Find the derivative of the function $g(t) = tf(2t)$. (Your answer might involve the function f .)

Solution: Use the product and chain rules:

$$g'(t) = f(2t) + t \frac{d}{dt} (f(2t)) = f(2t) + t f'(2t) \cdot 2 = f(2t) + 2te^{-(2t)^2} = f(2t) + 2te^{-4t^2}.$$

28. Let q be a function such that $q'(x) = (2x^2 + \sqrt{11})^{98}$. Find the derivative of the function $g(t) = q(\cos(2t))$. (Your answer might involve the function q .)

Solution: Use the chain rule twice:

$$\begin{aligned} g'(t) &= q'(\cos(2t)) \cdot \frac{d}{dt}(\cos(2t)) = (2[\cos(2t)]^2 + \sqrt{11})^{98} (-2 \sin(2t)) \\ &= -2 \sin(2t) \left(2 \cos^2(2t) + \sqrt{11} \right)^{98}. \end{aligned}$$

29. Let h be a function such that $h'(t) = -11h(t)$. Find the derivative of the function $f(t) = h(\sin^2(t))$. (Your answer might involve the function h .)

Solution: Use the chain rule (twice):

$$\begin{aligned} f'(t) &= h'(\sin^2(t)) \cdot \frac{d}{dt}(\sin^2(t)) = -11h(\sin^2(t)) \cdot 2 \sin(t) \cos(t) \\ &= -22 \sin(t) \cos(t) h(\sin^2(t)). \end{aligned}$$

30. Differentiate $y = \cos(2g(x)) - \frac{2}{\pi}$, where $g'(t) = \sqrt[3]{t^2 + 1}$. (Your answer might involve the function g .)

Solution: Use the chain rule, noting that the second term is a constant so that its derivative is zero:

$$\frac{dy}{dx} = -\sin(2g(x)) \cdot 2g'(x) = -2 \sin(2g(x)) \sqrt[3]{x^2 + 1}.$$

31. Differentiate Let g be a function such that $g'(t) = \sqrt[3]{t^2 + 1}$. Find the derivative of the function $f(x) = e^{xg(x)+1}$. (Your answer might involve the function g .)

Solution: Use the product and chain rules:

$$f'(x) = e^{xg(x)+1} \frac{d}{dx} (xg(x) + 1) = e^{xg(x)+1} (g(x) + xg'(x)) = e^{xg(x)+1} \left(g(x) + x \sqrt[3]{x^2 + 1} \right).$$

32. Let h be a function such that $h'(x) = \frac{\sin(2x)}{x}$. Find the derivative of the function $f(x) = 3x^2 h(x + \ln(7))$. (Your answer might involve the function h .)

Solution: Use the product and chain rules, noting that $\ln(7)$ is a constant so that its derivative is zero:

$$\begin{aligned} f'(x) &= 6xh(x + \ln(7)) + 3x^2 h'(x + \ln(7)) \frac{d}{dx} (x + \ln(7)) \\ &= 6xh(x + \ln(7)) + 3x^2 \frac{\sin(2(x + \ln(7)))}{x + \ln(7)} \cdot 1 \\ &= 6xh(x + \ln(7)) + \frac{3x^2 \sin(2(x + \ln(7)))}{x + \ln(7)}. \end{aligned}$$