Lecture 3: Crossed Products by Finite Groups; the Rokhlin Property

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- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

Recall: Group actions on C*-algebras

Definition

Let G be a group and let A be a C*-algebra. An *action of G on A* is a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g \colon A \to A$, and $\alpha_1 = \mathrm{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

When G is a topological group, we require that the action be continuous: $(g, a) \mapsto \alpha_g(a)$ is jointly continuous from $G \times A$ to A.

Recall: The action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus

Recall: Every action of a group G on a compact space X gives an action of G on $C(X)$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$.
$\mathrm{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Restriction gives actions of these on $S^1 \times S^1$.

Reminder: The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra A_θ, the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta} uv$. Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then A_θ is simple. In particular, *any* two unitaries u and v in any C*-algebra satisfying $vu = e^{2\pi i\theta} uv$ generate a copy of A_θ.
- If $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then A_θ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber M_n.
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).

The action of $\mathrm{SL}_2(\mathbb{Z})$ on the rotation algebra

Recall: A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta} uv$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

Exercise: Check that α_n is an automorphism, and that $n \mapsto \alpha_n$ is a group homomorphism.

This action is the analog of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. It reduces to that action when $\theta = 0$.

The action of $\mathrm{SL}_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta} uv$.

Recall that $\mathrm{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of A_θ, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*,$$

and the action of $\mathbb{Z}_4$ is generated by

$$u \mapsto v \quad \text{and} \quad v \mapsto u^*.$$

Exercise: Find the analogous formulas for $\mathbb{Z}_3$ and $\mathbb{Z}_6$, and check that they give actions of these groups.

Another example: The tensor flip

Assume (for convenience) that A is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the "tensor flip" $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group S_n acts on $A^{\otimes n}$.

The tensor flip on the 2^∞ UHF algebra $A = \bigotimes_{n=1}^\infty M_2$ turns out to be essentially the product type action generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad \bigotimes_{n=1}^{\infty} M_4.$$

Exercise: Prove this. (Hint: Look at the tensor flip on $M_2 \otimes M_2$.)

Another interesting example is gotten by taking A to be the Jiang-Su algebra Z. It is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $Z \otimes Z \cong Z$.

The free flip

Let A be a C*-algebra, and let $A \star A$ be the free product of two copies of A. (Use $A \star_{\mathbb{C}} A$ to get a unital C*-algebra.) Then there is an automorphism $\alpha \in \mathrm{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of a in the first free factor to the copy of the same element in the second free factor, and similarly the copy of a in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the "free flip". It generates a actions of $\mathbb{Z}_2$ on $A \star A$ and $A \star_{\mathbb{C}} A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of A), or the reduced free product $A \star_{\mathrm{r}} A$ (using the same state on both copies of A). There is a permutation action of S_n on the free product of n copies of A. And one can make any combination of these generalizations.

See the appendix for some actions on Cuntz algebras, along with a reminder of the definition of Cuntz algebras.

Recall: Construction of the crossed product by a finite group

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in G with coefficients in A:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g \colon c_g \in A \text{ for } g \in G \right\}.$$

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_g^{-1}]) \cdot u_g u_h = (a \alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. We saw that there is a unique norm which makes this a C*-algebra.

Examples of crossed products by finite groups

Let G be a finite group, and let $\iota \colon G \to \mathrm{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of G. (So far, G could be any locally compact group.)

Since we are assuming that G is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \mathrm{card}(G)$. If G is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\mathrm{card}(G)}$. If G is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand M_k for each unitary equivalence class of k-dimensional irreducible representations of G.

Now let A be any C*-algebra, and let $\iota_A \colon G \to \mathrm{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of A "factor out", since $A[G]$ is just the ordinary group ring:

$$(a \cdot u_g)(b \cdot u_h) = (a \iota_g(b)) \cdot u_{gh} = (ab) \cdot u_{gh}.$$

Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)

Examples of crossed products (continued)

Let G be a finite group, acting on $C(G)$ via the translation action on G. That is, the action $\alpha\colon G \to \mathrm{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$.

Set $n = \mathrm{card}(G)$. We describe how to prove that $C^*(G, \, C(G)) \cong M_n$. This calculation plays a key role later.

Recall multiplication in the crossed product: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$.

For $g \in G$, we let u_g be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi_{\{g\}}$. Thus $\sum_{g \in G} \delta_g = 1$ in $C(G)$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.) For $g, h \in G$, set

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, \, C(G), \, \alpha).$$

These elements form a system of matrix units. We calculate:

$$\begin{aligned} v_{g_1,h_1} v_{g_2,h_2} &= \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} \\ &= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}} u_{g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}. \end{aligned}$$

Examples of crossed products (continued)

G is a finite group, $n = \mathrm{card}(G)$, and $\alpha\colon G \to \mathrm{Aut}(C(G))$ is $\alpha_g(a)(h) = a(g^{-1}h)$ for $g, h \in G$ and $a \in C(G)$. We want to get $C^*(G, \, C(G)) \cong M_n$.

We defined

$$\delta_g = \chi_{\{g\}} \in C(G) \qquad \text{and} \qquad v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G, \, C(G), \, \alpha),$$

and we got

$$v_{g_1,h_1} v_{g_2,h_2} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}.$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. This is what matrix units are supposed to do. Similarly (do it as an exercise), $v_{g,h}^* = v_{h,g}$.

Since the elements δ_g span $C(G)$, the elements $v_{g,h}$ span $C^*(G, \, C(G), \, \alpha)$. So $C^*(G, \, C(G), \, \alpha) \cong M_n$ with $n = \mathrm{card}(G)$. Exercise: Write out a complete proof.

Examples of crossed products (continued)

Let G be a finite group, acting on $C(G)$ via the translation action on G. (That is, $\alpha_g(f)(h) = f(g^{-1}h)$.) Set $n = \mathrm{card}(G)$. Then $C^*(G, \, C(G)) \cong M_n$.

Now consider G acting on $G \times X$, by translation on G and trivially on X. Exercise: Use the same method to prove that $C^*(G, \, C_0(G \times X)) \cong C_0(X, M_n)$ (which is $M_n \otimes C_0(X)$).

A harder exercise: Prove that for any action of G on X, and using the diagonal action on $G \times X$, we still have $C^*(G, \, C_0(G \times X)) \cong C_0(X, M_n)$. Hint: A trick reduces this to the previous exercise.

This result generalizes greatly: for any locally compact group G, one gets $C^*(G, \, C_0(G)) \cong K(L^2(G))$, etc.

Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$, we say that a homomorphism $\varphi\colon A \to B$ is *equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. That is, for all $g \in G$, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \alpha_g}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_g} \\ A & \xrightarrow{\varphi} & B \end{array}$$

An equivariant homomorphism $\varphi\colon A \to B$ induces a homomorphism

$$\overline{\varphi}\colon C^*(G, A, \alpha) \to C^*(G, B, \beta),$$

just by applying φ to the algebra elements.

Equivariant homomorphisms (continued)

$\varphi\colon A \to B$ is *equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$. We get

$$\overline{\varphi}\colon C^*(G, A, \alpha) \to C^*(G, B, \beta)$$

by applying φ to the algebra elements.

Thus, if G is discrete, the standard unitaries in $C^*(G, A, \alpha)$ are called u_g, and the standard unitaries in $C^*(G, B, \beta)$ are called v_g, then

$$\overline{\varphi}\left(\sum_{g \in G} c_g u_g\right) = \sum_{g \in G} \varphi(c_g) v_g.$$

Exercises: Assume that G is finite. Prove that $\overline{\varphi}$ is a *-homomorphism, that if φ is injective then so is $\overline{\varphi}$, and that if φ is surjective then so is $\overline{\varphi}$. (Warning: the surjectivity result is true for general G, but the injectivity result can fail if G is not amenable.)

Digression: Conjugacy

For $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$, we say that a homomorphism $\varphi\colon A \to B$ is *equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If φ is an isomorphism, we say it is a *conjugacy*. If there is such a map, the C* dynamical systems (G, A, α) and (G, B, β) are *conjugate*. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if G is locally compact and φ is a conjugacy, then φ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^{\infty} M_2$ generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are "essentially the same". The correct statement is that these actions are conjugate. Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that $w \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) w^* = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and take $\varphi = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w)$.

Equivariant exact sequences

The homomorphism φ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

Theorem

Let G be a locally compact group. Let $0 \to J \to A \to B \to 0$ be an exact sequence of C*-algebras with actions γ of G on J, α of G on A, and β of G on B, and equivariant maps. Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

When G is finite, the proof is easy: consider

$$0 \longrightarrow J[G] \longrightarrow A[G] \longrightarrow B[G] \longrightarrow 0$$

Exercise: Do it. (You already proved exactness at $J[G]$ and $B[G]$ in a previous exercise.)

Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle S^1 by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x\colon k = 0, 1, \ldots, n-1\} \qquad \text{and} \qquad U_x = S^1 \setminus L_x.$$

Then L_x is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation, and U_x is equivariantly homeomorphic to

$$\mathbb{Z}_n \times \left\{e^{2\pi i t/n} x\colon 0 < t < 1\right\} \cong \mathbb{Z}_n \times (0, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U_x) \longrightarrow C(S^1) \longrightarrow C(L_x) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((0,1),\, M_n) \longrightarrow C^*(\mathbb{Z}_n,\, C(S^1)) \longrightarrow M_n \longrightarrow 0.$$

With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_n,\, C(S^1)) \cong C(S^1, M_n)$. The copy of S^1 on the right arises as the orbit space $S^1/\mathbb{Z}_n$.

Examples of crossed products (continued)

We use the standard abbreviation $C^*(G,X) = C^*(G,\, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle S^1 by rotation, we got

$$C^*(\mathbb{Z}_n, S^1) \cong C(S^1/\mathbb{Z}_n,\, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle S^1 by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2,\, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber M_2, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in X gives a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. We illustrate this when G is finite (so that all orbits are closed) and H is either $\{1\}$ (above) or G ($C^*(G)$, and see the next slide).

Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.

Set

$$L = \{-1, 1\} \subset S^1 \qquad \text{and} \qquad U = S^1 \setminus L.$$

Then the action on L is trivial, and U is equivariantly homeomorphic to

$$\mathbb{Z}_2 \times \{x \in U \colon \mathrm{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((-1,1),\, M_2) \longrightarrow C^*(\mathbb{Z}_2,\, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,$$

in which $C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4$. With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_2,\, C(S^1))$ is isomorphic to

$$\big\{ f \in C([-1,\,1],\, M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices} \big\}.$$

Crossed products by inner actions

Recall the inner action $\alpha_g = \mathrm{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from G to the unitary group of a C*-algebra A. The crossed product is the same as for the trivial action, in a canonical way.

Assume G is finite. Let $\iota \colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism φ sends $a \cdot u_g$ to $a z_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that φ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = b v_g$). So we need $\varphi(u_g)\varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$$

and, using $z_g b = \alpha_g(b) z_g$,

$$\varphi(u_g)\varphi(b) = z_g v_g b = z_g b v_g = \alpha_g(b) z_g v_g.$$

Exercise: When G is finite, give a detailed proof that φ is an isomorphism. (This is written out in my crossed product notes.)

Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Reminder: $\mathrm{Ad}(v)(a) = v a v^*$. Set

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad z_n = v^{\otimes n} \in (M_2)^{\otimes n} \cong M_{2^n}.$$

Then write this action as $\alpha = \varinjlim_n \mathrm{Ad}(z_n)$ on $A = \varinjlim_n M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Do it for finite G.) Since $\mathrm{Ad}(z_n)$ is inner, we get

$$C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \cong C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong M_{2^n} \oplus M_{2^n}.$$

Now we have to use the explicit form of these isomorphisms to compute the maps in the direct system of crossed products, and then find the direct limit.

Crossed products by product type actions (continued)

The action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra is generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

We wrote this action as $\alpha = \varinjlim_n \mathrm{Ad}(z_n)$ on $A = \varinjlim_n M_{2^n}$. Then

$$\begin{aligned} C^*(\mathbb{Z}_2, A, \alpha) &\cong \varinjlim_n C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \\ &\cong \varinjlim_n C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \varinjlim_n (M_{2^n} \oplus M_{2^n}). \end{aligned}$$

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right),$$

and a computation with Bratteli diagrams shows that the direct limit is again the 2^∞ UHF algebra. (For general product type actions, the direct limit will be more complicated, and usually not a UHF algebra.)

Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra generated by $\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \varinjlim_n \mathrm{Ad}(z_n)$ on $A = \varinjlim_n M_{2^n}$, with maps $\varphi_n \colon M_{2^n} \to M_{2^{n+1}}$.

Exercise: Find isomorphisms $\sigma_n \colon C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$ and homomorphisms $\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ such that, with $\overline{\varphi}_n$ being the map induced by φ_n on the crossed products, the following diagram commutes for all n:

$$\begin{tikzcd} C^*(\mathbb{Z}_2,\, M_{2^n},\, \mathrm{Ad}(z_n)) \arrow[r, "\sigma_n"] \arrow[d, "\overline{\varphi}_n"'] & M_{2^n} \oplus M_{2^n} \arrow[d, "\psi_n"] \\ C^*(\mathbb{Z}_2,\, M_{2^{n+1}},\, \mathrm{Ad}(z_{n+1})) \arrow[r, "\sigma_{n+1}"] & M_{2^{n+1}} \oplus M_{2^{n+1}}. \end{tikzcd}$$

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \to \mathbb{C} \oplus \mathbb{C}$.) Then prove that, using the maps ψ_n, one gets $\varinjlim_n (M_{2^n} \oplus M_{2^n}) \cong A$. (This part doesn't have anything to do with crossed products.) Conclude that $C^*(\mathbb{Z}_2, A, \alpha) \cong A$.

Motivation for the Rokhlin property

Recall that an action $(g, x) \mapsto gx$ of a group G on a set X is *free* if every $g \in G \setminus \{1\}$ acts on X with no fixed points. Equivalently, whenever $g \in G$ and $x \in X$ satisfy $gx = x$, then $g = 1$. Equivalently, every orbit is isomorphic to G acting on G by translation. (Examples: G acting on G by translation, $\mathbb{Z}_n$ acting on S^1 by rotation by $e^{2\pi i/n}$, and $\mathbb{Z}$ acting on S^1 by an irrational rotation.)

Let X be the Cantor set, let G be a finite group, and let G act freely on X. Fix $x_0 \in X$. Then the points gx_0, for $g \in G$, are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set $K \subset X$ such that $x_0 \in K$ and the sets gK, for $g \in G$, are all disjoint.

By repeating this process, one can find a compact open set $L \subset X$ such that the sets $L_g = gL$, for $g \in G$, are all disjoint, and such that their union is X.

Exercise: Carry out the details. (It isn't quite trivial.)

The Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.

The Rokhlin property (continued)

The conditions in the definition of the Rokhlin prperty:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g\in G} e_g = 1$.

The projections e_g are the analogs of the characteristic functions of the compact open sets gL from the Cantor set example.

Condition (1) is an approximate version of $gL_h = L_{gh}$. (Recall that $L_g = gL$.)

Condition (3) is the requirement that X be the disjoint union of the L_g.

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3). Without (2) the inner action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(M_2)$ generated by $\mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ would have the Rokhlin property. (Exercise: Prove this statement.) We don't want this. For example, M_2 is simple but $C^*(\mathbb{Z}_2, M_2, \alpha)$ isn't. (There is more on outerness in Lecture 5.)

Examples

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g\in G} e_g = 1$.

Exercise: Let G be finite. Let act G on G by translation. Prove that the action of G on $C(G)$ (namely $\alpha_g(f)(h) = f(g^{-1}h)$) has the Rokhlin property.

Exercise: Let G be finite. Let A be any unital C*-algebra. Prove that the action of G on $\bigoplus_{g\in G} A$ by translation of the summands has the Rokhlin property.

Exercise: Let G be finite, and let G act freely on the Cantor set X. Prove that the corresponding action of G on $C(X)$ has the Rokhlin property. (Use the earlier exercise on free actions on the Cantor set.)

In the exercises above, condition (2) is trivial. Can it be satisfied in a nontrivial way? In particular, are there any actions on simple C*-algebras with the Rokhlin property?

An example using a simple C*-algebra (more in the next lecture)

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g\in G} e_g = 1$.

We want an example in which A is simple. Thus, we won't be able to satisfy condition (2) by choosing e_g to be in the center of A.

From Lecture 1, recall the product type action of $\mathbb{Z}_2$ generated by

$$\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

We will show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in place of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Reasons for using $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ will appear in Lecture 4.

An example (continued)

We had

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The action α of $\mathbb{Z}_2$ is generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$w p_0 w^* = p_1, \quad w p_1 w^* = p_0, \qquad \text{and} \qquad p_0 + p_1 = 1.$$

The action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $w p_0 w^* = p_1$, $w p_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in A, we can assume $F \subset A_n$ for some n. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that e_0 and e_1 actually commute with everything in F. (Proofs: See the next slide.) This proves the Rokhlin property.

An example (continued)

The projections e_0 and e_1 actually commute with everything in F, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of F have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \mathrm{Ad}\big(w^{\otimes n} \otimes w\big)$, so

$$\beta(e_0) = \big(w^{\otimes n} \otimes w\big)(1 \otimes p_0)\big(w^{\otimes n} \otimes w\big)^* = 1 \otimes w p_0 w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same.

Appendix: Cuntz algebras and some actions on them

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.
Let $d \in \{2, 3, \ldots\}$. Recall that the *Cuntz algebra* $\mathcal{O}_d$ is the universal C*-algebra on generators $s_1, s_2, \ldots, s_d$ satisfying the relations

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \qquad \text{and} \qquad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

Thus, $s_1, s_2, \ldots, s_d$ are isometries with orthogonal ranges which add up to 1. The Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges. Thus, $s_1^* s_1 = s_2^* s_2 = \cdots = 1$ and $s_j^* s_k = 0$ for $j \neq k$.

These algebras are purely infinite, simple, and nuclear. Details and other properties are on the next slide.

Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

  $$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \qquad \text{and} \qquad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,$$

  then they generate a copy of $\mathcal{O}_d$.
- $\mathcal{O}_d$ is purely infinite and nuclear.
- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, for $d \in \{2, 3, \ldots\}$.
- If A is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.
- If A is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$.

The last two facts are Kirchberg's absorption theorems. They are much harder.

Actions on Cuntz algebras

For d finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho \colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\beta^\rho_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.) For $d = \infty$, a similar formula works for any unitary representation of G on $l^2(\mathbb{N})$.

Actions on Cuntz algebras (continued)

The Cuntz relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$. (For $d = \infty$, $s_1, s_2, \ldots$ are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For $G = \mathbb{Z}_n$, choose n-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply s_j by ζ_j.
- Let G be a finite group. Take $d = \mathrm{card}(G)$, and label the generators s_g for $g \in G$. Then define $\beta^G \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ by $\beta^G_g(s_h) = s_{gh}$ for $g, h \in G$. (This is the quasifree action coming from regular representation of G.)
- Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{N}$. Define $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{N}$. (This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation of G.)

Actions on Cuntz algebras: The tensor flips on $\mathcal{O}_2$ and $\mathcal{O}_\infty$

There are tensor flip actions of $\mathbb{Z}_2$ on $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$. Since

$$\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2 \qquad \text{and} \qquad \mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty,$$

one gets actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ and $\mathcal{O}_\infty$.

More generally, any subgroup of S_n acts on the n-fold tensor products $(\mathcal{O}_2)^{\otimes n}$ and $(\mathcal{O}_\infty)^{\otimes n}$. This gives actions of these groups on $\mathcal{O}_2$ and $\mathcal{O}_\infty$.