Lecture 3: Crossed Products by Finite Groups; the Rokhlin Property

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The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
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A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
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Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is just the algebraic tensor product.)

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Let G be a finite group, and let $\iota\colon G \to \mathrm{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of G. (So far, G could be any locally compact group.)

Since we are assuming that G is finite, $C^*(G)$ is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \mathrm{card}(G)$. If G is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\mathrm{card}(G)}$. If G is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand M_k for each unitary equivalence class of k-dimensional irreducible representations of G.

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$$\begin{aligned} C^*(\mathbb{Z}_2, A, \alpha) &\cong \varinjlim_n C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \\ &\cong \varinjlim_n C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong \varinjlim_n (M_{2^n} \oplus M_{2^n}). \end{aligned}$$

The maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right),$$

and a computation with Bratteli diagrams shows that the direct limit is again the 2^∞ UHF algebra. (For general product type actions, the direct limit will be more complicated, and usually not a UHF algebra.)

Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the 2^{∞} UHF algebra generated by $\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ on $A = \bigotimes_{n=1}^{\infty} M_2$.

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Let A be a unital C*-algebra, and let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.

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Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\| \beta_g(e_h) - e_{gh} \| < \varepsilon$ for all $g, h \in G$.
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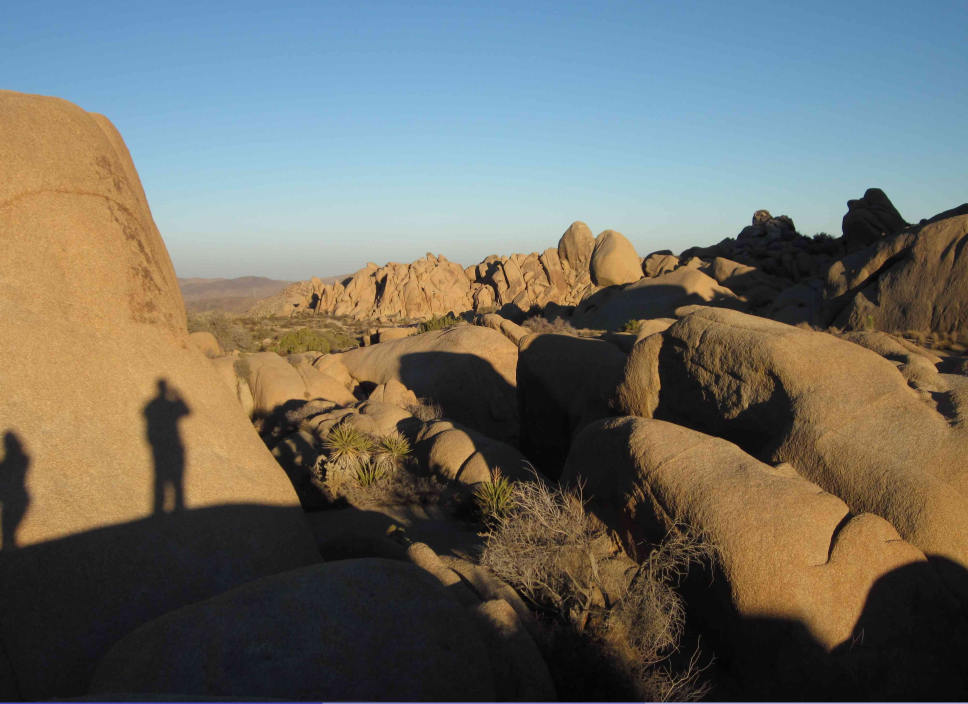

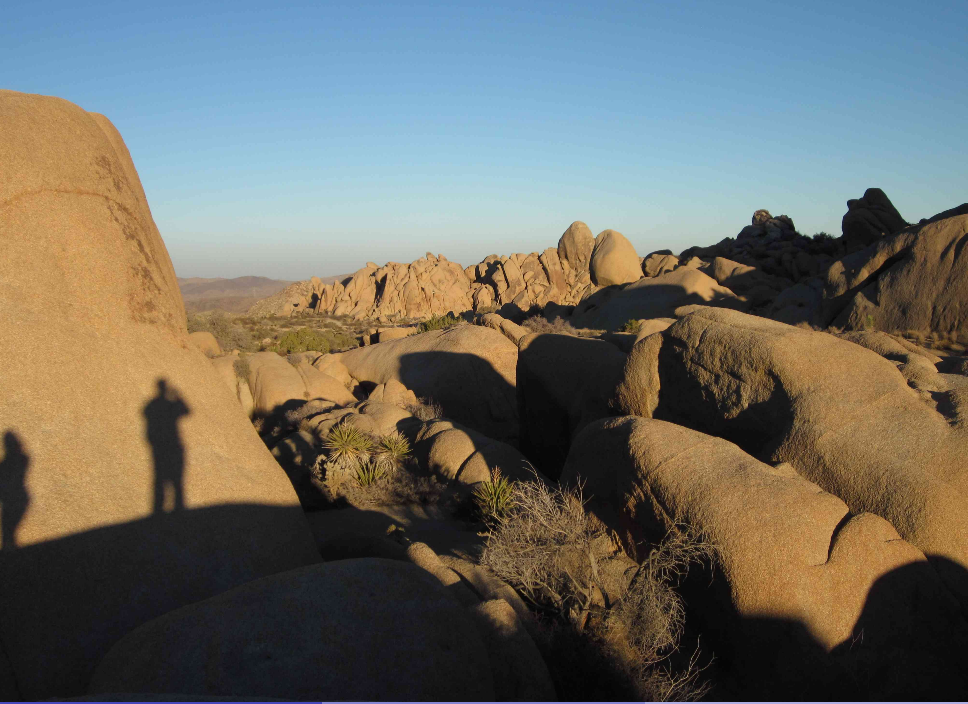

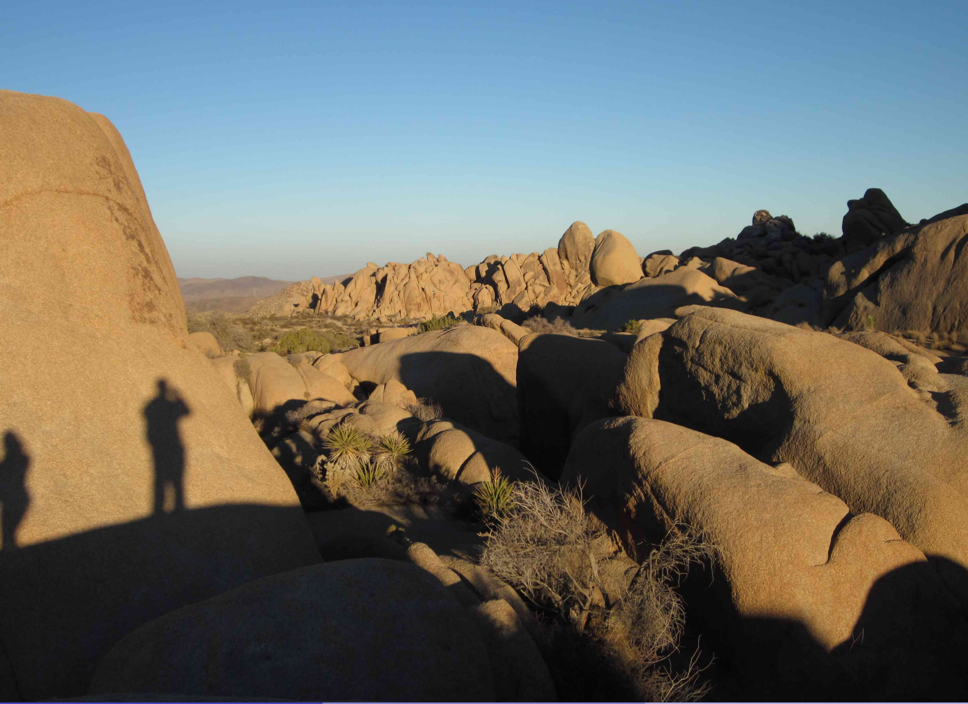

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We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

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There are tensor flip actions of $\mathbb{Z}_2$ on $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$.

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