Lecture 1: Group C*-algebras and Actions of Finite Groups on C*-Algebras

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- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

General motivation

The material to be described is part of the structure and classification theory for simple nuclear C*-algebras (the Elliott program). More specifically, it is about proving that C*-algebras which appear in other parts of the theory (in these lectures, certain kinds of crossed product C*-algebras) satisfy the hypotheses of known classification theorems.

To keep things from being too complicated, we will consider crossed products by actions of finite groups. Nevertheless, even in this case, one can see some of the techniques which are important in more general cases.

Crossed product C*-algebras have long been important in operator algebras, for reasons having nothing to do with the Elliott program. It has generally been difficult to prove that crossed products are classifiable, and there are really only three cases in which there is a somewhat satisfactory theory: actions of finite groups on simple C*-algebras, free minimal actions of groups which are not too complicated (mostly, not too far from $\mathbb{Z}^d$) on compact metric spaces, and "strongly outer" actions of such groups on simple C*-algebras.

Background

These lectures assume some familiarity with the basic theory of C*-algebras, as found, for example, in Murphy's book. K-theory will be occasionally used, but not in an essential way. A few other concepts will be important, such as tracial rank zero. They will be defined as needed, and some basic properties mentioned, usually without proof. Various side comments will assume more background, but these can be skipped. (Many side comments which should be made will be omitted entirely, for lack of time.)

Do the exercises!

There will be many exercises given, with varying levels of difficulty. To really get to understand this material, **please do them!**

I am happy to talk to people about the exercises.

The beginning: The group ring

Let G be a finite group. Its *group ring* $\mathbb{C}[G]$ (a standard construction in algebra) is, as a vector space, the set of formal linear combinations

$$\sum_{g \in G} a_g \cdot g \qquad (1)$$

of group elements with coefficients $a_g \in \mathbb{C}$. (Formally: the free $\mathbb{C}$-module on G.) The multiplication is $(a \cdot g)(b \cdot h) = (ab) \cdot (gh)$ for $g, h \in G$ and $a, b \in \mathbb{C}$, extended linearly. That is, the product comes from the group.

G need not be finite (but must be discrete), provided that in (1) one uses only finite sums ($a_g = 0$ for all but finitely many $g \in G$).

Exercise

Prove that $\mathbb{C}[G]$ is an associative unital algebra over $\mathbb{C}$.

One can use any field (even ring) in place of $\mathbb{C}$. The algebraists actually do this.

Motivation: Representation theory (brief comments below).

The beginning: The group ring (continued)

G is a discrete group, and

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g \cdot g \colon a_g \in \mathbb{C}, \ a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Multiplication: $(a \cdot g)(b \cdot h) = (ab) \cdot (gh)$, extended linearly.

Recall the usual polynomial ring $\mathbb{C}[x]$. Let $\langle S \rangle$ denote the ideal generated by a set S. Also, abbreviate $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}_n$. (We won't use p-adic integers.)

Example

Take $G = \mathbb{Z}_2$. Then $\mathbb{C}[G] \cong \mathbb{C}[x]/\langle x^2 - 1 \rangle$. The identity of the group is 1 and the nontrivial element is x.

Exercise (easy)

Check the statements made in the previous example.

Exercise (easy)

Generalize the previous example to $\mathbb{Z}_n$ for $n \in \mathbb{N}$.

The beginning: The group ring (continued)

G is a discrete group, and

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g \cdot g \colon a_g \in \mathbb{C},\ a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Multiplication: $(a \cdot g)(b \cdot h) = (ab) \cdot (gh)$, extended linearly.

Recall: $\mathbb{C}[\mathbb{Z}_2] \cong \mathbb{C}[x]/\langle x^2 - 1 \rangle$, with the nontrivial group element being x.

Example

Take $G = \mathbb{Z}$. Then $\mathbb{C}[G] \cong \mathbb{C}[x, x^{-1}]$, the ring of Laurent polynomials in one variable. The group element $k \in \mathbb{Z}$ is x^k.

Exercise (easy)

Check the statements made in the previous example.

The beginning: The C*-algebra of a group

G is a discrete group, and

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g \cdot g \colon a_g \in \mathbb{C},\ a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Multiplication: $(a \cdot g)(b \cdot h) = (ab) \cdot (gh)$, extended linearly.

Make $\mathbb{C}[G]$ a *-algebra by making the group elements unitary: $g^* = g^{-1}$. C*-algebraists thus usually write u_g for the element $g \in \mathbb{C}[G]$. So

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g \cdot u_g \colon a_g \in \mathbb{C},\ a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

The multiplication is $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$ for $g, h \in G$ and $a, b \in \mathbb{C}$, extended linearly, and the adjoint is $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$.

Exercise

Show that this adjoint in $\mathbb{C}[G]$ is well defined, conjugate linear, reverses multiplication: $(xy)^* = y^* x^*$, and satisfies $(x^*)^* = x$.

The beginning: The C*-algebra of a group (continued)

G is a discrete group, and

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g \cdot u_g \colon a_g \in \mathbb{C},\ a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Multiplication: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$. Adjoint: $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$.

We still need a norm. We will assume G is finite; things are otherwise more complicated. First, a bit of representation theory.

Let G be a discrete group, and let H be a Hilbert space. Let w be a unitary representation of G on H: a group homomorphism $g \mapsto w_g$ from G to the group $U(H)$ of unitary operators on H. Then we define a linear map $\pi_w \colon \mathbb{C}[G] \to L(H)$ by

$$\pi_w\left(\sum_{g \in G} a_g \cdot u_g \right) = \sum_{g \in G} a_g \cdot w_g.$$

Exercise

Prove that π_w is a unital *-homomorphism.

The beginning: The C*-algebra of a group (continued)

G is a discrete group. Multiplication in $\mathbb{C}[G]$: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$. Adjoint: $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$.

If w is a unitary representation of G on a Hilbert space H (a group homomorphism $g \mapsto w_g$ from G to the group $U(H)$ of unitary operators on H), then the unital *-homomorphism $\pi_w \colon \mathbb{C}[G] \to L(H)$ is

$$\pi_w\left(\sum_{g \in G} a_g \cdot u_g \right) = \sum_{g \in G} a_g \cdot w_g.$$

Theorem

The assignment $w \mapsto \pi_w$ is a bijection from unitary representations of G on H to unital *-homomorphisms $\mathbb{C}[G] \to L(H)$.

Exercise

Prove this theorem. (To recover w from π_w, look at $\pi_w(u_g)$.)

The left regular representation

G is a discrete group. Multiplication in $\mathbb{C}[G]$: $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$. Adjoint: $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$.

If w is a unitary representation of G on H, then $\pi_w \colon \mathbb{C}[G] \to L(H)$ is the unital *-homomorphism $\pi_w\left(\sum_{g \in G} a_g \cdot u_g\right) = \sum_{g \in G} a_g \cdot w_g$.

Take $H = l^2(G)$, and let z be the *left regular representation*: $(z_g \xi)(h) = \xi(g^{-1}h)$. (On standard basis vectors, it is $z_g \delta_h = \delta_{gh}$.)

Exercise

Prove that z is a unitary representation, and that the claimed formula for $z_g \delta_h$ is correct. (What goes wrong if you use $\xi(gh)$ in place of $\xi(g^{-1}h)$? Which of the formulas $\xi(hg)$ and $\xi(hg^{-1})$ also gives a representation?)

Proposition

If z is the left regular representation, then $\pi_z \colon \mathbb{C}[G] \to L(l^2(G))$ is injective.

The norm on $C^*(G)$

G is a discrete group. If w is a unitary representation of G on H, then $\pi_w \colon \mathbb{C}[G] \to L(H)$ is $\pi_w\left(\sum_{g \in G} a_g \cdot u_g\right) = \sum_{g \in G} a_g \cdot w_g$. Take $H = l^2(G)$, and let z be the left regular representation: $(z_g \xi)(h) = \xi(g^{-1}h)$.

Proposition

If z is the left regular representation, then π_z is injective.

Exercise

Prove this proposition. (You need to show that the elements $\pi_z(u_g)$ are linearly independent; this ultimately reduces to linear independence of the standard basis vectors $\delta_h \in l^2(G)$.)

Definition

If G is finite, $C^*(G)$ is $\mathbb{C}[G]$ equipped with the norm $\|x\| = \|\pi_z(x)\|$.

$C^*(G)$ is complete since $\mathbb{C}[G]$ is finite dimensional. $\pi_z(\mathbb{C}[G]) \subset L(l^2(G))$, so $C^*(G)$ is a C*-algebra, and this is the unique C* norm.

Group C*-algebras and representation theory

Recall:

Theorem

If G is discrete, the assignment $w \mapsto \pi_w$ is a bijection from unitary representations of G to unital *-homomorphisms $\mathbb{C}[G] \to L(H)$.

When G is finite, get the unique C* norm on $\mathbb{C}[G]$ by choosing w so that π_w is injective. (For example, take w to be the left regular representation.)

Information about w is often much easier to see from π_w. A very primitive example: one can have $\mathrm{Ker}(v) = \mathrm{Ker}(w)$ but $\mathrm{Ker}(\pi_v) \neq \mathrm{Ker}(\pi_w)$. So $\mathbb{C}[G]$ is very important in the algebraic study of representations.

If G is not finite, one must choose a C* norm on $\mathbb{C}[G]$ and complete. This makes things harder. If G is locally compact, then one uses the convolution algebra $L^1(G)$ in place of $\mathbb{C}[G]$. (For discrete G, one can use $l^1(G)$.) Now the group elements aren't even in the algebra. The resulting C*-algebras are very important in representation theory, and this is an important direction in C*-algebras, but I will say no more here.

Examples of C*-algebras of finite groups: $\mathbb{Z}_2$

Recall: $C^*(G)$ is $\mathbb{C}[G]$ with a (unique) C* norm.

Example

$G = \mathbb{Z}_2$. Then $C^*(\mathbb{Z}_2)$ is commutative because G is, and $\dim(C^*(\mathbb{Z}_2)) = \mathrm{card}(\mathbb{Z}_2) = 2$. There is only one commutative C*-algebra A with $\dim(A) = 2$, namely $\mathbb{C} \oplus \mathbb{C}$, so $C^*(\mathbb{Z}_2) \cong \mathbb{C} \oplus \mathbb{C}$.

To see this more directly, let h be the nontrivial group element. Then $u_h^2 = 1$, so one can calculate: $p_0 = \frac{1}{2}(1 + u_h)$ and $p_1 = \frac{1}{2}(1 - u_h)$ are orthogonal projections with $p_0 + p_1 = 1$, and which span $C^*(\mathbb{Z}_2)$. Since $\dim(C^*(\mathbb{Z}_2)) = 2$, we get an isomorphism $\mathbb{C} \oplus \mathbb{C} \to C^*(\mathbb{Z}_2)$ by $(1, 0) \mapsto p_0$ and $(0, 1) \mapsto p_1$.

p_j is functional calculus of u_h: $p_j = f_j(u_h)$ with

$$f_0(1) = 1, \qquad f_0(-1) = 0, \qquad \text{and} \qquad f_1(1) = 0, \qquad f_1(-1) = 1.$$

Exercise

Find an explicit isomorphism $C^*(\mathbb{Z}_n) \to \mathbb{C}^n$. (Take h to be a generator of the group.)

Recall: For G finite, $C^*(G)$ is $\mathbb{C}[G]$ with a (unique) C* norm.

We found an explicit isomorphism $\mathbb{C} \oplus \mathbb{C} \to C^*(\mathbb{Z}_2)$. Exercise from the last slide: find an explicit isomorphism $\mathbb{C}^n \to C^*(\mathbb{Z}_n)$.

Exercise (easy)

Let G be finite. Prove that $C^*(G)$ is commutative if and only if G is commutative.

This is true in general. (You have seen enough to do this for discrete G, even without knowing the definition of the norm.)

Example

$G = S_3$, the permutation group on 3 symbols. Then $C^*(S_3)$ is noncommutative because G is, and $\dim(C^*(S_3)) = \mathrm{card}(S_3) = 6$. There is only one noncommutative C*-algebra A with $\dim(A) = 6$, namely $\mathbb{C} \oplus \mathbb{C} \oplus M_2$, so $C^*(S_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2$.

Exercise (messy from scratch; I have not done it)

Find an explicit isomorphism $\mathbb{C} \oplus \mathbb{C} \oplus M_2 \to C^*(S_3)$.

Examples of C*-algebras of finite groups

Recall: For G finite, $C^*(G)$ is $\mathbb{C}[G]$ with a (unique) C* norm.

$C^*(\mathbb{Z}_n) \cong \mathbb{C}^n$.

$C^*(S_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2$.

Fact: For general finite G, $C^*(G)$ is given by the representation theory: there is one summand M_d for every unitary equivalence class of d-dimensional irreducible unitary representations of G.

This fact generalizes to compact groups.

Examples without justification: $C^*(\mathbb{Z}) \cong C(S^1)$; $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$; if G is commutative, then $C^*(G) \cong C_0\big(\widehat{G}\big)$.

Computing $C^*(G)$ is very hard in general. See Higson's later lectures for how to get some information (usually K-theory).

Towards crossed products

We will look at crossed products of actions of (finite) groups on C*-algebras. These are a generalization of C*-algebras of groups: $C^*(G)$ is gotten by letting G act trivially on $\mathbb{C}$.

Possibly the original motivation: if G is a semidirect product $G = N \rtimes H$, then H acts on $C^*(N)$ and the crossed product is $C^*(G)$.

Before defining crossed products (see Lecture 2), we give the definition of an action and some examples.

Group actions on C*-algebras

Definition

Let G be a group and let A be a C*-algebra. An *action of G on A* is a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g \colon A \to A$, and $\alpha_1 = \mathrm{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

In these lectures, almost all groups will be discrete (usually finite). If the group has a topology, one requires that the function $g \mapsto \alpha_g(a)$, from G to A, be continuous for all $a \in A$.

We give examples of actions of groups (mainly finite groups), considering first actions on commutative C*-algebras. These come from actions on locally compact spaces, as described next.

Group actions on spaces

Definition

Let G be a group and let X be a set. Then an *action of G on X* is a map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

If G and X have topologies, then $(g, x) \mapsto gx$ is required to be (jointly) continuous.

Without topologies, an action of G on X is just a homomorphism from G to the automorphism group of X. Since X has no structure, its automorphism group consists of all permutations of X.

When G is discrete, continuity means that $x \mapsto gx$ is continuous for all $g \in G$. Since the action of g^{-1} is also continuous, this map is in fact a homeomorphism. Thus, an action is a homomorphism from G to the automorphism group of X. Here, the automorphisms of X are the homeomorphisms.

Group actions on commutative C*-algebras

An action of G on X is a continuous map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Lemma

Let G be a topological group and let X be a locally compact Hausdorff space. Suppose G acts continuously on X. Then there is an action $\alpha\colon G \to \mathrm{Aut}(C_0(X))$ such that $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C_0(X)$, and $x \in X$.

Every action of G on $C_0(X)$ comes this way from an action of G on X.

Compare the formula for $\alpha_g(f)$ with that for the left regular representation.

If G is discrete, the lemma is obvious from the correspondence between maps of locally compact spaces and homomorphisms of commutative C*-algebras. (In the general case, one needs to check that the two continuity conditions correspond properly.)

Examples of group actions on spaces

An action of G on X is a continuous map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Every action on this list of a group G on a compact space X gives an action of G on $C(X)$.

- Any group G has a trivial action on any space X, given by $gx = x$ for all $g \in G$ and $x \in X$.
- Any group G acts on itself by (left) translation: gh is the usual product of g and h.
- The finite cyclic group $\mathbb{Z}_n$ acts on the unit circle $S^1 \subset \mathbb{C}$ by rotation: the standard generator acts as multiplication by $e^{2\pi i/n}$.
- $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.
- $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.

More examples of group actions on spaces

- Fix $\theta \in \mathbb{R}$. Then there is an action of $\mathbb{Z}$ on S^1, given by $n \cdot \zeta = e^{2\pi i n\theta}\zeta$ for $n \in \mathbb{Z}$ and $\zeta \in S^1$. (This action is generated by the rotation homeomorphism $\zeta \mapsto e^{2\pi i\theta}\zeta$.)
- If G is a group and H is a (closed) subgroup (not necessarily normal), then G has a translation action on $X = G/H$, given by $g \cdot (kH) = (gk)H$ for $g, k \in G$.
- If G is a group and $\sigma\colon G \to H$ is a continuous homomorphism to another group H, then there is an action of G on $X = H$ given by $g \cdot h = \sigma(g)h$ for $g \in G$ and $h \in H$. For example, G might be a closed subgroup of H. (The action on the previous slide of $\mathbb{Z}_n$ on S^1 by rotation comes this way.) The previous example comes from the homomorphism $\mathbb{Z} \to S^1$ given by $n \mapsto e^{2\pi i\theta}$. If $\theta \notin \mathbb{Q}$, this homomorphism has dense range.
- Let Y be a compact space, and set $X = Y \times Y$. Then $G = \mathbb{Z}_2$ acts on X via the order two homeomorphism $(y_1, y_2) \mapsto (y_2, y_1)$. Similarly, the symmetric group S_n acts on Y^n.

Group actions on noncommutative C*-algebras

Some elementary examples:

- For every group G and every C*-algebra A, there is a trivial action $\iota\colon G \to \mathrm{Aut}(A)$, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in A$.
- Suppose $g \mapsto z_g$ is a (continuous) homomorphism from G to the unitary group $U(A)$ of a unital C*-algebra A. Then $\alpha_g(a) = z_g a z_g^*$ defines an action of G on A. (We write $\alpha_g = \mathrm{Ad}(z_g)$.) This is an *inner* action. (If A is not unital, use the multiplier algebra $M(A)$, and the strict topology on its unitary group.)
- As a special case, let G be a finite group, and let $g \mapsto z_g$ be a unitary representation of G on $\mathbb{C}^n$. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on M_n.

Exercise

Prove that, in the second example above, $g \mapsto \mathrm{Ad}(z_g)$ really is a continuous action of G on A.

Pointwise inner does not imply inner

Let A be a unital C*-algebra. An automorphism $\varphi \in \mathrm{Aut}(A)$ is *inner* if there is a unitary $z \in A$ such that $\varphi = \mathrm{Ad}(z)$. Recall also that $\alpha\colon G \to \mathrm{Aut}(A)$ is *inner* if there is a homomorphism $g \mapsto z_g$ from G to $U(A)$ such that $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$.

Let $A = M_2$, let $G = (\mathbb{Z}_2)^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \qquad \alpha_{g_1} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\alpha_{g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \alpha_{g_1 g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These define an action $\alpha\colon G \to \mathrm{Aut}(A)$ such that α_g is inner for all $g \in G$, but for which there is no homomorphism $g \mapsto z_g \in U(A)$ for which $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$. The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly. Exercise: Prove this.

Infinite tensor product actions

We describe a particular infinite tensor product action. (It is an example of what is called a "product type action" in the literature.). Let $A_n = (M_2)^{\otimes n}$, the tensor product of n copies of the algebra M_2 of 2×2 matrices. Thus $A_n \cong M_{2^n}$. Define

$$\varphi_n\colon A_n \to A_{n+1} = A_n \otimes M_2$$

by $\varphi_n(a) = a \otimes 1$. Let A be the (completed) direct limit $\varinjlim_n A_n$. (This is just the 2^∞ UHF algebra.) Define a unitary $v \in M_2$ by

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $z_n \in A_n$ by $z_n = v^{\otimes n}$. Define $\alpha_n \in \mathrm{Aut}(A_n)$ by $\alpha_n = \mathrm{Ad}(z_n)$. Then α_n is an inner automorphism of order 2. Using $z_{n+1} = z_n \otimes v$, one can easily check that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$ for all n (diagram on next slide; exercise: prove this), and it follows that the α_n determine an order 2 automorphism α of A. Thus, we have an action of $\mathbb{Z}_2$ on A.

Infinite tensor product action example (continued)

Recall: $A_n = (M_2)^{\otimes n} \cong M_{2^n}$.

$\varphi_n\colon A_n \to A_{n+1} = A_n \otimes M_2$ is $\varphi_n(a) = a \otimes 1$, and $A = \varinjlim_n A_n$.

$v = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in U(M_2)$, and $z_n = v^{\otimes n} \in U(A_n)$.

$\alpha_n \in \mathrm{Aut}(A_n)$ is $\alpha_n = \mathrm{Ad}(z_n)$.

Commutative diagram to define the order 2 automorphism $\alpha \in \mathrm{Aut}(A)$:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\varphi_0"] \arrow[d, "\alpha_0"] & M_2 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"] & M_4 \arrow[r, "\varphi_2"] \arrow[d, "\alpha_2"] & M_8 \arrow[r, "\varphi_3"] \arrow[d, "\alpha_3"] & \cdots \arrow[r] & A \arrow[d, "\alpha"] \\
\mathbb{C} \arrow[r, "\varphi_0"] & M_2 \arrow[r, "\varphi_1"] & M_4 \arrow[r, "\varphi_2"] & M_8 \arrow[r, "\varphi_3"] & \cdots \arrow[r] & A
\end{tikzcd}$$

The action of $\mathbb{Z}_2$ is not inner (see later), although it is "approximately inner" (that is, a pointwise limit of inner actions).