Large Subalgebras and the Structure of Crossed Products, Lecture 3: Large Subalgebras and the Radius of Comparison

N. Christopher Phillips

University of Oregon

4 June 2015

Rocky Mountain Mathematics Consortium Summer School

University of Wyoming, Laramie

1–5 June 2015

- Lecture 1 (1 June 2015): Introduction, Motivation, and the Cuntz Semigroup.
- Lecture 2 (2 June 2015): Large Subalgebras and their Basic Properties.
- Lecture 3 (4 June 2015): Large Subalgebras and the Radius of Comparison.
- Lecture 4 (5 June 2015 [morning]): Large Subalgebras in Crossed Products by $\mathbb{Z}$.
- Lecture 5 (5 June 2015 [afternoon]): Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms.

A rough outline of all five lectures

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if B is a large subalgebra of A, then A and B have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.

Definition

Let A be a C*-algebra, and let $a, b \in (K \otimes A)_+$. We say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Reminder: The Cuntz semigroup

Definition

Let A be a C*-algebra, and let $a, b \in (K \otimes A)_+$.

1. We say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.
2. We define $a \sim_A b$ if $a \precsim_A b$ and $b \precsim_A a$.

Definition

Let A be a C*-algebra.

1. The *Cuntz semigroup* of A is $\mathrm{Cu}(A) = (K \otimes A)_+ / \sim_A$, together with the commutative semigroup operation $\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A$ (using an isomorphism $M_2(K) \to K$; the result does not depend on which one) and the partial order $\langle a \rangle_A \leq \langle b \rangle_A$ if and only if $a \precsim_A b$.
2. We also define the subsemigroup $W(A) = M_\infty(A)_+ / \sim_A$, with the same operations and order.

Comparison

Let A be a stably finite simple unital C*-algebra. Recall that $\mathrm{T}(A)$ is the set of tracial states on A and that $\mathrm{QT}(A)$ is the set of normalized 2-quasitraces on A.

We say that *the order on projections over A is determined by traces* if, as happens for type II_1 factors, whenever $p, q \in M_\infty(A)$ are projections such that for all $\tau \in \mathrm{T}(A)$ we have $\tau(p) < \tau(q)$, then p is Murray-von Neumann equivalent to a subprojection of q.

Simple C*-algebras need not have very many projections, so a more definitive version of this condition is to ask for *strict comparison of positive elements*, that is, whenever $a, b \in M_\infty(A)$ (or $K \otimes A$) are positive elements such that for all $\tau \in \mathrm{QT}(A)$ we have $d_\tau(a) < d_\tau(b)$ (recall $d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$), then $a \precsim_A b$. (It turns out that it does not matter whether one uses $M_\infty(A)$ or $K \otimes A$, but this is not as easy to see as with projections.)

(Note: We have also switched from traces to quasitraces. For exact C*-algebras, this makes no difference.)

Comparison (continued)

From the previous slide: A has strict comparison of positive elements if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.

Simple AH algebras with slow dimension growth have strict comparison, but other simple AH algebras need not. Strict comparison is necessary for any reasonable hope of classification in the sense of the Elliott program. According to the Toms-Winter Conjecture, when A is simple, separable, nuclear, unital, and stably finite, strict comparison should imply Z-stability, and this is known to hold in a number of cases.

The radius of comparison $\mathrm{rc}(A)$ of A measures the failure of strict comparison. For context, we point out that $\mathrm{rc}(C(X))$ is roughly $\frac{1}{2}\dim(X)$ (at least for reasonable spaces X, such as finite complexes).

Radius of comparison

Definition

Let A be a stably finite unital C*-algebra.

1. Let $r \in [0, \infty)$. We say that A has *r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.
2. The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\left(\left\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\right\}\right).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

(It is equivalent to use $K \otimes A$ in place of $M_\infty(A)$.)

The following is a special case of a result stated in the first lecture.

Theorem

Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

Special case of a theorem from the first lecture (on previous slide):

Theorem

Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

The extra assumption is that A is exact, so that every quasitrace is a trace.

We describe a proof directly from the definition of a large subalgebra. We give a heuristic argument first, using the following simplifications:

1. The algebra A, and therefore also B, has a unique tracial state τ.
2. We consider elements of A_+ and B_+ instead of elements of $M_\infty(A)_+$ and $M_\infty(B)_+$.
3. For $a \in A_+$, when needed, instead of getting $(1-g)c(1-g) \in B$ for some $c \in A_+$ which is close to a, we can actually get $(1-g)a(1-g) \in B$. Similarly, for $a \in A$ we can get $(1-g)a \in B$.
4. For $a, b \in A_+$ with $a \precsim_A b$, we can find $v \in A$ such that $v^* b v = a$ (not just such that $\|v^* b v - a\|$ is small).
5. None of the elements we encounter are Cuntz equivalent to projections, that is, we never encounter anything for which 0 is an isolated point of, or not in, the spectrum.

The heuristic argument

Simplifications for the heuristic argument (on previous slide):

1. The algebra A, and therefore also B, has a unique tracial state τ.
2. We consider only elements of A_+ and B_+.
3. For $a \in A_+$, we can actually get $(1-g)a(1-g) \in B$, and for $a \in A$ we can get $(1-g)a \in B$.
4. For $a, b \in A_+$ with $a \precsim_A b$, we can find $v \in A$ such that $v^* b v = a$.
5. We never encounter anything for which 0 is isolated in the spectrum.

The most drastic simplification is (3). In the actual proof, since we only get approximation, we will need to make systematic use of elements $(a - \varepsilon)_+$ for carefully chosen, and varying, values of $\varepsilon > 0$.

We first consider the inequality $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. So let $a, b \in A_+$ satisfy $d_\tau(a) + \mathrm{rc}(B) < d_\tau(b)$. The essential idea is to replace b by something slightly smaller which is in B_+, say y, and replace a by something slightly larger which is in B_+, say x, in such a way that we still have $d_\tau(x) + \mathrm{rc}(B) < d_\tau(y)$. Then use the definition of $\mathrm{rc}(B)$.

The heuristic argument (continued)

We want to show $\mathrm{rc}(A) \leq \mathrm{rc}(B)$.

So let $a, b \in A_+$ satisfy $d_\tau(a) + \mathrm{rc}(B) < d_\tau(b)$. Choose $\delta > 0$ such that

$$d_\tau(a) + \mathrm{rc}(B) + \delta \leq d_\tau(b).$$

Applying (3) of our simplification, find $g \in B$ with $0 \leq g \leq 1$, such that

$$(1-g)a(1-g) \in B \quad \text{and} \quad (1-g)b(1-g) \in B,$$

and so small in $W(A)$ that $d_\tau(g) < \frac{\delta}{3}$. Using basic result (4) on Cuntz comparison, we get

$$(1-g)b(1-g) \sim_A b^{1/2}(1-g)^2 b^{1/2} \leq b.$$

Similarly, $(1-g)a(1-g) \precsim_A a$, and this relation implies

$$d_\tau\big((1-g)a(1-g)\big) \leq d_\tau(a).$$

Also, $b \precsim_A (1-g)b(1-g) \oplus g$ by the second lemma on the list of basic results on Cuntz equivalence, so

$$d_\tau\big((1-g)b(1-g)\big) + d_\tau(g) \geq d_\tau(b).$$

The heuristic argument (continued)

We have $a, b \in A_+$, we want to show $a \precsim_A b$, and we got:

$$d_\tau(a) + \mathrm{rc}(B) + \delta \leq d_\tau(b). \tag{1}$$

$$d_\tau(g) < \tfrac{\delta}{3}. \tag{2}$$

$$(1-g)b(1-g) \precsim_A b. \tag{3}$$

$$d_\tau\big((1-g)a(1-g)\big) \leq d_\tau(a). \tag{4}$$

$$d_\tau\big((1-g)b(1-g)\big) + d_\tau(g) \geq d_\tau(b). \tag{5}$$

Using, in order, (4), (1), (5), (2), we get

$$\begin{aligned} d_\tau\big((1-g)a(1-g) \oplus g\big) + \mathrm{rc}(B) + \tfrac{\delta}{3} &\leq d_\tau(a) + d_\tau(g) + \mathrm{rc}(B) + \tfrac{\delta}{3} \\ &\leq d_\tau(b) + d_\tau(g) - \tfrac{2\delta}{3} \leq d_\tau\big((1-g)b(1-g)\big) + 2d_\tau(g) - \tfrac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big). \end{aligned}$$

Use the definition of $\mathrm{rc}(B)$ in the middle, the second lemma on the list of basic results on Cuntz equivalence at the first step, and (3) at the end:

$$a \precsim_A (1-g)a(1-g) \oplus g \precsim_B (1-g)b(1-g) \precsim_A b,$$

that is, $a \precsim_A b$, as desired.

Heuristic for $\mathrm{rc}(B) \leq \mathrm{rc}(A)$

Let $a, b \in B_+$ satisfy $d_\tau(a) + \mathrm{rc}(A) < d_\tau(b)$. Choose $\delta > 0$ such that $d_\tau(a) + \mathrm{rc}(B) + \delta \leq d_\tau(b)$. By lower semicontinuity of d_τ, we always have $d_\tau(b) = \sup_{\varepsilon>0} d_\tau((b-\varepsilon)_+)$. So there is $\varepsilon > 0$ such that

$$d_\tau((b-\varepsilon)_+) > d_\tau(a) + \mathrm{rc}(A). \tag{6}$$

Define $f\colon [0,\infty) \to [0,\infty)$ by $f(\lambda) = \max(0, \varepsilon^{-1}\lambda(\varepsilon - \lambda))$ for $\lambda \in [0,\infty)$. Then $f(b)$ and $(b-\varepsilon)_+$ are orthogonal positive elements such that $f(b) + (b-\varepsilon)_+ \leq b$, and $f(b) \neq 0$ (since we assume $0 \in \mathrm{sp}(b)$ is not isolated). We have $a \precsim_A (b-\varepsilon)_+$ by (6) and the definition of $\mathrm{rc}(A)$. We are assuming for simplification that we can find $v \in A$ such that $v^*(b-\varepsilon)_+ v = a$. Similarly, we are assuming we can find $g \in B$ with $0 \leq g \leq 1$ such that $(1-g)v^* \in B$ and $g \precsim_B f(b)$. Since

$$v(1-g) \in B \quad \text{and} \quad [v(1-g)]^*(b-\varepsilon)_+[v(1-g)] = (1-g)a(1-g),$$

we get $(1-g)a(1-g) \precsim_B (b-\varepsilon)_+$. Therefore, using the second lemma on the list of basic results on Cuntz equivalence at the first step,

$$a \precsim_B (1-g)a(1-g) \oplus g \precsim_B (b-\varepsilon)_+ \oplus g \precsim_B (b-\varepsilon)_+ \oplus f(b) \precsim_B b.$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$: Preliminary results

Theorem (For Cu(A): Blackadar-Robert-Tikuisis-Toms-Winter)

Let A be a stably finite simple unital C*-algebra. Then:

1. $\mathrm{rc}(A)$ is the least $s \in [0,\infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > s$, and $a, b \in M_\infty(A)_+$ satisfy $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ in $W(A)$, then $a \precsim_A b$.
2. $\mathrm{rc}(A)$ is the least $s \in [0,\infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > t$, and $a, b \in M_\infty(A)_+$ satisfy $(n+1)\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ in $W(A)$, then $a \precsim_A b$.

The second part has $n+1$ in one of the places the first part has n.

Lemma (Lemma on functional calculus)

Let $M \in (0,\infty)$, let $f\colon [0,\infty) \to \mathbb{C}$ be a continuous with $f(0) = 0$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever A is a C*-algebra and $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$, then $\|f(a) - f(b)\| < \varepsilon$.

Standing assumptions

Throughout, A is an infinite dimensional stably finite simple separable unital exact C*-algebra and $B \subset A$ is a large subalgebra. We want to prove that $\mathrm{rc}(A) \leq \mathrm{rc}(B)$.

Since A is stably finite, $M_n(B)$ is large in $M_n(A)$ for all n.

Since A is exact, $\mathrm{QT}(A) = T(A)$ (all quasitraces are traces). Being a subalgebra of A, the algebra B is also exact, so $\mathrm{QT}(B) = T(B)$. We know from the previous lecture that (abusing notation) $T(B) = T(A)$.

Also, recall from the first lecture (the second lemma on the list of basic results on Cuntz equivalence):

Lemma (Cutdown comparison)

Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then

$$(a-\varepsilon)_+ \precsim_A \big[(1-g)a(1-g) - \varepsilon\big]_+ \oplus g.$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$

We use the first criterion in the theorem above. Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(B)$, and let $a, b \in M_\infty(A)_+$ satisfy $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ in $W(A)$. We want to prove that $a \precsim_A b$. Without loss of generality $\|a\|, \|b\| \leq 1$. It suffices to prove that $(a-\varepsilon)_+ \precsim_A b$ for every $\varepsilon > 0$.

So let $\varepsilon > 0$. We may assume $\varepsilon < 1$. Let $x \in M_\infty(A)_+$ be the direct sum of n copies of a, let $y \in M_\infty(A)_+$ be the direct sum of n copies of b, and let $q \in M_\infty(A)_+$ be the direct sum of m copies of the identity of A. The relation $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ means that $x \oplus q \precsim_A y$. By (11) on the Cuntz semigroup handout, there exists $\delta > 0$ such that

$$\left((x \oplus q) - \tfrac{1}{3}\varepsilon\right)_+ \precsim_A (y-\delta)_+.$$

Since $\varepsilon < 3$, this is equivalent to

$$\left(x - \tfrac{1}{3}\varepsilon\right)_+ \oplus q \precsim_A (y-\delta)_+.$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

Choose $l \in \mathbb{Z}_{>0}$ so large that $a, b \in M_l \otimes A$. Since $m/n > \mathrm{rc}(B)$, there is $k \in \mathbb{Z}_{>0}$ such that

$$\mathrm{rc}(B) < \frac{m}{n} - \frac{2}{k}.$$

Set

$$\varepsilon_0 = \min\left(\tfrac{1}{3}\varepsilon, \tfrac{1}{2}\delta\right).$$

Using the functional calculus lemma above, choose $\varepsilon_1 > 0$ with $\varepsilon_1 \leq \varepsilon_0$ and so small whenever D is a C*-algebra and $z \in D_+$ satsifies $\|z\| \leq 1$, then $\|z_0 - z\| < \varepsilon_1$ implies

$$\|(z_0 - \varepsilon_0)_+ - (z - \varepsilon_0)_+\| < \varepsilon_0, \quad \left\|\left(z_0 - \tfrac{1}{3}\varepsilon\right)_+ - \left(z - \tfrac{1}{3}\varepsilon\right)_+\right\| < \varepsilon_0,$$

and

$$\left\|\left(z_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right)_+ - \left(z - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right)_+\right\| < \varepsilon_0.$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

We have from above

$$a, b \in M_l \otimes A, \quad n\langle a\rangle_A = \langle x\rangle_A, \quad n\langle b\rangle_A = \langle y\rangle_A, \quad m\langle 1\rangle_A = \langle q\rangle_A,$$

$$\left(x - \tfrac{1}{3}\varepsilon\right)_+ \oplus q \precsim_A (y - \delta)_+, \quad \text{and} \quad \mathrm{rc}(B) < \frac{m}{n} - \frac{2}{k},$$

and we want to prove that $a \precsim_A b$.

Since A is infinite dimensional and simple, the third lemma on the Cuntz semigroup handout provides $z \in A_+ \setminus \{0\}$ such that $(k+1)\langle z\rangle_A \leq \langle 1\rangle_A$. Since $M_l(B)$ is large in $M_l(A)$, there are $g \in M_l(B)_+$ and $a_0, b_0 \in M_l(A)_+$ satisfying

$$0 \leq g, a_0, b_0 \leq 1, \quad \|a_0 - a\| < \varepsilon_1, \quad \|b_0 - b\| < \varepsilon_1, \quad g \precsim_A z,$$

and such that

$$(1-g)a_0(1-g), \ (1-g)b_0(1-g) \in M_l \otimes B.$$

From $g \precsim_A z$ and $(k+1)\langle x\rangle_A \leq \langle 1\rangle_A$ we get $\sup_{\tau \in \mathrm{T}(A)} d_\tau(g) < \frac{1}{k}$.

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

We have from above

$$0 \leq g, a_0, b_0 \leq 1, \quad \|a_0 - a\| < \varepsilon_1, \quad \|b_0 - b\| < \varepsilon_1, \quad g \precsim_A x,$$

and

$$(1-g)a_0(1-g), \ (1-g)b_0(1-g) \in M_l \otimes B.$$

Set

$$a_1 = \left[(1-g)a_0(1-g) - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \quad \text{and} \quad b_1 = \left[(1-g)b_0(1-g) - \varepsilon_0\right]_+,$$

which are in $M_l \otimes B$. We claim that a_0, a_1, b_0, and b_1 satisfy:

1. $(a - \varepsilon)_+ \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+$.
2. $\left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \precsim_B a_1 \oplus g$.
3. $a_1 \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+$.
4. $(b - \delta)_+ \precsim_A (b_0 - \varepsilon_0)_+$.
5. $(b_0 - \varepsilon_0)_+ \precsim_B b_1 \oplus g$.
6. $b_1 \precsim_A b$.

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

We will prove the first three claims (involving a, a_0, and a_1); the last three (involving b, b_0, and b_1) are similar but easier.

Recall:

$$\|a_0 - a\| < \varepsilon_1 \quad \text{and} \quad a_1 = \left[(1-g)a_0(1-g) - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+.$$

We prove claim 1: $(a - \varepsilon)_+ \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+$. The choice of ε_1 implies

$$\left\|\left[a_0 - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+ - \left[a - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+\right\| < \varepsilon_0 \leq \tfrac{1}{3}\varepsilon.$$

At the last step in the following computation use this and (10) on the Cuntz semigroup handout, at the first step use $\varepsilon_0 \leq \frac{1}{3}\varepsilon$, and at the second step use (8) on the Cuntz semigroup handout:

$$\begin{aligned}(a - \varepsilon)_+ &\leq \left[a - \left(\tfrac{2}{3}\varepsilon + \varepsilon_0\right)\right]_+ \\ &= \left[\left(a - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right)_+ - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left[a_0 - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+.\end{aligned}$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

Recall:

$$a_1 = \left[(1-g)a_0(1-g) - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+.$$

Claim 2 ($\left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \precsim_B a_1 \oplus g$) is an instance of the second lemma on the list of basic results on Cuntz equivalence:

Lemma (Cutdown comparison)

Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then

$$(a-\varepsilon)_+ \precsim_A \left[(1-g)a(1-g) - \varepsilon\right]_+ \oplus g.$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

Recall:

$$\|a_0 - a\| < \varepsilon_1 \quad \text{and} \quad a_1 = \left[(1-g)a_0(1-g) - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+.$$

For claim 3 ($a_1 \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+$), note $\|a_0 - a\| < \varepsilon_1$ implies

$$\|(1-g)a_0(1-g) - (1-g)a(1-g)\| < \varepsilon_1.$$

Therefore

$$\left\| \left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \right\| < \varepsilon_0.$$

Using (8) on the Cuntz semigroup handout at the first step, this fact and (10) on the Cuntz semigroup handout at the second step, (6) on the Cuntz semigroup handout at the third step, and (17) on the Cuntz semigroup handout and $a^{1/2}(1-g)^2a^{1/2} \leq a$ at the last step, we get

$$a_1 = \left[\left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \varepsilon_0\right]_+$$
$$\precsim_A \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \sim_A \left[a^{1/2}(1-g)^2a^{1/2} - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+,$$

as desired.

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$: What we have so far

We have

$$a, b \in M_l \otimes A, \quad n\langle a\rangle_A = \langle x\rangle_A, \quad n\langle b\rangle_A = \langle y\rangle_A, \quad m\langle 1\rangle_A = \langle q\rangle_A,$$
$$\left(x - \tfrac{1}{3}\varepsilon\right)_+ \oplus q \precsim_A (y-\delta)_+, \quad \text{and} \quad \mathrm{rc}(B) < \tfrac{m}{n} - \tfrac{2}{k},$$

and we want to prove that $a \precsim_A b$.
We got g, a_0, a_1, b_0, b_1 such that

$$(1-g)a_0(1-g),\ (1-g)b_0(1-g) \in M_l \otimes B \quad \text{and} \quad \sup_{\tau \in \mathrm{T}(A)} d_\tau(g) < \tfrac{1}{k}.$$

and the following hold:

1. $(a-\varepsilon)_+ \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+$.
2. $\left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \precsim_B a_1 \oplus g$.
3. $a_1 \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+$.
4. $(b-\delta)_+ \precsim_A (b_0 - \varepsilon_0)_+$.
5. $(b_0 - \varepsilon_0)_+ \precsim_B b_1 \oplus g$.
6. $b_1 \precsim_A b$.

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

Now let $\tau \in \mathrm{T}(A)$. Since x and y are the direct sums of n copies of a and b, it follows that $\left(x - \tfrac{1}{3}\varepsilon\right)_+$ is the direct sum of n copies of $\left(a - \tfrac{1}{3}\varepsilon\right)_+$ and $(y-\delta)_+$ is the direct sum of n copies of $(b-\delta)_+$. So the relation

$$\left(x - \tfrac{1}{3}\varepsilon\right)_+ \oplus q \precsim_A (y-\delta)_+$$

implies

$$n \cdot d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right) + m \leq n \cdot d_\tau\left((b-\delta)_+\right). \tag{7}$$

Using claim 4 and claim 5 at the first step and $\sup_{\tau \in \mathrm{T}(A)} d_\tau(g) < \frac{1}{k}$ at the third step, we get the estimate

$$d_\tau\left((b-\delta)_+\right) \leq d_\tau(b_1) + d_\tau(g) < d_\tau(b_1) + k^{-1}. \tag{8}$$

Claim 3 implies

$$d_\tau(a_1) \leq d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right). \tag{9}$$

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

Using $\sup_{\tau \in \mathrm{T}(A)} d_\tau(g) < \frac{1}{k}$ at the second step, (9) at the third step, (7) at the fourth step, and (8) at the fifth step, we get

$$\begin{aligned} n \cdot d_\tau(a_1 \oplus g) + m &= n \cdot d_\tau(a_1) + m + n \cdot d_\tau(g) \\ &\leq n \cdot d_\tau(a_1) + m + nk^{-1} \\ &\leq n \cdot d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right) + m + nk^{-1} \\ &\leq n \cdot d_\tau\left((b - \delta)_+\right) + nk^{-1} \\ &\leq n \cdot d_\tau(b_1) + 2nk^{-1}. \end{aligned}$$

It follows that

$$d_\tau(a_1 \oplus g) + \frac{m}{n} - \frac{2}{k} \leq d_\tau(b_1).$$

This holds for all $\tau \in \mathrm{T}(A)$, and therefore, since A and B have the same traces, for all $\tau \in \mathrm{T}(B)$.

Proving $\mathrm{rc}(A) \leq \mathrm{rc}(B)$ (continued)

Since $\mathrm{QT}(A) = \mathrm{T}(A)$, since

$$\frac{m}{n} - \frac{2}{k} > \mathrm{rc}(B),$$

and since $a_1, b_1, g \in M_l \otimes B$, it follows that $a_1 \oplus g \precsim_B b_1$. Using this relation at the third step, claim 1 at the first step, claim 2 at the second step, and claim 6 at the last step, we then get

$$(a - \varepsilon)_+ \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_+ \precsim_A a_1 \oplus g \precsim_B b_1 \precsim_A b.$$

This completes the proof that $\mathrm{rc}(A) \leq \mathrm{rc}(B)$.