A rough outline of all five lectures

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if $B$ is a large subalgebra of $A$, then $A$ and $B$ have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.
Reminder: The Cuntz semigroup

**Definition**

Let $A$ be a C*-algebra, and let $a, b \in (K \otimes A)_{+}$.

1. We say that $a$ is *Cuntz subequivalent* to $b$ over $A$, written $a \sim_{A} b$, if there is a sequence $(v_{n})_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_{n}b v_{n}^{*} = a$.

2. We define $a \sim_{A} b$ if $a \preceq_{A} b$ and $b \preceq_{A} a$.

**Comparison (continued)**

From the previous slide: $A$ has strict comparison of positive elements if whenever $a, b \in M_{\infty}(A)_{+}$ satisfy $d_{\tau}(a) < d_{\tau}(b)$ for all $\tau \in QT(A)$, then $a \preceq_{A} b$.

Simple AH algebras with slow dimension growth have strict comparison, but other simple AH algebras need not. Strict comparison is necessary for any reasonable hope of classification in the sense of the Elliott program. According to the Toms-Winter Conjecture, when $A$ is simple, separable, nuclear, unital, and stably finite, strict comparison should imply Z-stability, and this is known to hold in a number of cases.

The radius of comparison $rc(A)$ of $A$ measures the failure of strict comparison. For context, we point out that $rc(C(X))$ is roughly $\frac{1}{2} \dim(X)$ (at least for reasonable spaces $X$, such as finite complexes).

**Comparison**

Let $A$ be a stably finite simple unital C*-algebra. Recall that $T(A)$ is the set of tracial states on $A$ and that $QT(A)$ is the set of normalized 2-quasitraces on $A$.

We say that the *order on projections over $A$ is determined by traces if, as happens for type II$_1$ factors, whenever $p, q \in M_{\infty}(A)$ are projections such that for all $\tau \in T(A)$ we have $\tau(p) < \tau(q)$, then $p$ is Murray-von Neumann equivalent to a subprojection of $q$.

Simple C*-algebras need not have very many projections, so a more definitive version of this condition is to ask for strict comparison of positive elements, that is, whenever $a, b \in M_{\infty}(A)$ (or $K \otimes A$) are positive elements such that for all $\tau \in QT(A)$ we have $d_{\tau}(a) < d_{\tau}(b)$ (recall $d_{\tau}(a) = \lim_{n \to \infty} \tau(a^{1/n})$), then $a \preceq_{A} b$. (It turns out that it does not matter whether one uses $M_{\infty}(A)$ or $K \otimes A$, but this is not as easy to see as with projections.)

(For exact C*-algebras, this makes no difference.)

**Radius of comparison**

**Definition**

Let $A$ be a stably finite unital C*-algebra.

1. Let $r \in [0, \infty)$. We say that $A$ has $r$-comparison if whenever $a, b \in M_{\infty}(A)_{+}$ satisfy $d_{\tau}(a) < d_{\tau}(b)$ for all $\tau \in QT(A)$, then $a \preceq_{A} b$.

2. The *radius of comparison* of $A$, denoted $rc(A)$, is

$$rc(A) = \inf \{ r \in [0, \infty) : A \text{ has } r\text{-comparison} \}.$$ 

(We take $rc(A) = \infty$ if there is no $r$ such that $A$ has $r$-comparison.)

(It is equivalent to use $K \otimes A$ in place of $M_{\infty}(A)$.)

The following is a special case of a result stated in the first lecture.

**Theorem**

Let $A$ be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $rc(A) = rc(B)$. 
The heuristic argument (continued)

We want to show \( \text{rc}(A) \leq \text{rc}(B) \).

So let \( a, b \in A_+ \) satisfy \( d_r(a) + \text{rc}(B) < d_r(b) \). Choose \( \delta > 0 \) such that

\[
d_r(a) + \text{rc}(B) + \delta \leq d_r(b).
\]

Applying (3) of our simplification, find \( g \in B \) with \( 0 \leq g \leq 1 \), such that

\[
(1 - g)a(1 - g) \in B \quad \text{and} \quad (1 - g)b(1 - g) \in B,
\]

and so small in \( W(A) \) that \( d_r(g) < \frac{\delta}{3} \). Using basic result (4) on Cuntz comparison, we get

\[
(1 - g)b(1 - g) \sim_A b^{1/2}(1 - g)^2b^{1/2} \leq b.
\]

Similarly, \( (1 - g)a(1 - g) \precsim_A a \), and this relation implies

\[
d_r((1 - g)a(1 - g)) \leq d_r(a).
\]

Also, \( b \precsim_A (1 - g)b(1 - g) \oplus g \) by the second lemma on the list of basic results on Cuntz equivalence, so

\[
d_r((1 - g)b(1 - g)) + d_r(g) \geq d_r(b).
\]

The heuristic argument

Simplifications for the heuristic argument (on previous slide):

\begin{itemize}
  \item The algebra \( A \), and therefore also \( B \), has a unique tracial state \( \tau \).
  \item We consider only elements of \( A_+ \) and \( B_+ \).
  \item For \( a \in A_+ \), we can actually get \( (1 - g)a(1 - g) \in B \), and for \( a \in A \) we can get \( (1 - g)a \in B \).
  \item For \( a, b \in A_+ \) with \( a \precsim_A b \), we can find \( v \in A \) such that \( v^*bv = a \).
  \item We never encounter anything for which \( 0 \) is isolated in the spectrum.
\end{itemize}

The most drastic simplification is (3). In the actual proof, since we only get approximation, we will need to make systematic use of elements \( (a - \varepsilon)_+ \) for carefully chosen, and varying, values of \( \varepsilon > 0 \).

We first consider the inequality \( \text{rc}(A) \leq \text{rc}(B) \). So let \( a, b \in A_+ \) satisfy \( d_r(a) + \text{rc}(B) < d_r(b) \). The essential idea is to replace \( b \) by something slightly smaller which is in \( B_+ \), say \( y \), and replace \( a \) by something slightly larger which is in \( B_+ \), say \( x \), in such a way that we still have \( d_r(x) + \text{rc}(B) < d_r(y) \). Then use the definition of \( \text{rc}(B) \).
Heuristic for $rc(B) \leq rc(A)$
Let $a, b \in B_+$. satisfy $d_\tau(a) + rc(A) < d_\tau(b)$. Choose $\delta > 0$ such that $d_\tau(a) + rc(B) + \delta \leq d_\tau(b)$. By lower semicontinuity of $d_\tau$, we always have $d_\tau(b) = \sup_{\varepsilon > 0} d_\tau((b - \varepsilon)_+)$. So there is $\varepsilon > 0$ such that

$$d_\tau((b - \varepsilon)_+) > d_\tau(a) + rc(A).$$

(6)

Define $f : [0, \infty) \to [0, \infty]$ by $f(\lambda) = \max(0, \varepsilon^{-1}(\varepsilon - \lambda))$ for $\lambda \in [0, \infty)$. Then $f(b)$ and $(b - \varepsilon)_+$ are orthogonal positive elements such that $f(b) + (b - \varepsilon)_+ \leq b$, and $f(b) \neq 0$ (since we assume $0 \in \sp(b)$ is not isolated). We have $a \preceq_A (b - \varepsilon)_+$ by (6) and the definition of $rc(A)$. We are assuming for simplification that we can find $v \in A$ such that $v^*(b - \varepsilon)_+ v = a$. Similarly, we are assuming we can find $g \in B$ with $0 \leq g \leq 1$ such that $(1 - g)v^* \in B$ and $g \preceq_B f(b)$. Since $v(1 - g) \in B$ and $[v(1 - g)]^*(b - \varepsilon)_+[v(1 - g)] = (1 - g)a(1 - g)$, we get $(1 - g)a(1 - g) \preceq_B (b - \varepsilon)_+$. Therefore, using the second lemma on the list of basic results on Cuntz equivalence at the first step,

$$a \preceq_B (1 - g)a(1 - g) \oplus g \preceq_B (b - \varepsilon)_+ \oplus f(b) \preceq_B b.$$

Proving $rc(A) \leq rc(B)$: Preliminary results
Let $A$ be a stably finite simple unital C*-algebra. Then:

- $rc(A)$ is the least $s \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > s$, and $a, b \in M_\infty(A)_+$ satisfy $n(a)_A + m(1)_A \leq n(b)_A$ in $W(A)$, then $a \preceq_A b$.

- $rc(A)$ is the least $s \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > s$, and $a, b \in M_\infty(A)_+$ satisfy $(n + 1)(a)_A + m(1)_A \leq n(b)_A$ in $W(A)$, then $a \preceq_A b$.

The second part has $n + 1$ in one of the places the first part has $n$.

Lemma (Lemma on functional calculus)
Let $M \in (0, \infty)$, let $f : [0, \infty) \to \mathbb{C}$ be a continuous with $f(0) = 0$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever $A$ is a C*-algebra and $a, b \in A_{sa}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$, then $\|f(a) - f(b)\| < \varepsilon$.

Standing assumptions
Throughout, $A$ is an infinite dimensional stably finite simple separable unital exact C*-algebra and $B \subset A$ is a large subalgebra. We want to prove that $rc(A) \leq rc(B)$.

Since $A$ is stably finite, $M_n(B)$ is large in $M_n(A)$ for all $n$.

Since $A$ is exact, $QT(A) = T(A)$ (all quasitraces are traces). Being a subalgebra of $A$, the algebra $B$ is also exact, so $QT(B) = T(B)$. We know from the previous lecture that (abusing notation) $T(B) = T(A)$.

Also, recall from the first lecture (the second lemma on the list of basic results on Cuntz equivalence):

Lemma (Cutdown comparison)
Let $A$ be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then

$$(a - \varepsilon)_+ \preceq_A [(1 - g)a(1 - g) - \varepsilon]_+ \oplus g.$$
Proving $\text{rc}(A) \leq \text{rc}(B)$ (continued)

We have from above

$$0 \leq g, a_0, b_0 \leq 1, \quad \|a_0 - a\| < \varepsilon_1, \quad \|b_0 - b\| < \varepsilon_1, \quad g \precsim_A x,$$

and

$$(1-g)a_0(1-g), (1-g)b_0(1-g) \in M_l \otimes B.$$

Set

$$a_1 = [(1-g)a_0(1-g) - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+ \quad \text{and} \quad b_1 = [(1-g)b_0(1-g) - \varepsilon_0]_+,$$

which are in $M_l \otimes B$. We claim that $a_0, a_1, b_0,$ and $b_1$ satisfy:

- $(a - \varepsilon)_+ \precsim_A [a_0 - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+$.
- $[a_0 - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+ \precsim_B a_1 \oplus g$.
- $a_1 \precsim_A (a - \frac{1}{3}\varepsilon)_+$.
- $(b - \delta)_+ \precsim_A (b_0 - \varepsilon_0)_+$.
- $(b_0 - \varepsilon_0)_+ \precsim_B b_1 \oplus g$.
- $b_1 \precsim_A b$.

Proving $\text{rc}(A) \leq \text{rc}(B)$ (continued)

We will prove the first three claims (involving $a, a_0,$ and $a_1$); the last three (involving $b, b_0,$ and $b_1$) are similar but easier.

Recall:

$$\|a_0 - a\| < \varepsilon_1 \quad \text{and} \quad a_1 = [(1-g)a_0(1-g) - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+.$$

We prove claim 1: $(a - \varepsilon)_+ \precsim_A [a_0 - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+$. The choice of $\varepsilon_1$ implies

$$\| [a_0 - (\frac{1}{3}\varepsilon + \varepsilon_0)]_+ - [a - (\frac{1}{3}\varepsilon + \varepsilon_0)]_+ \| < \varepsilon_0 \leq \frac{1}{3}\varepsilon.$$

At the last step in the following computation use this and (10) on the Cuntz semigroup handout, at the first step use $\varepsilon_0 \leq \frac{1}{3}\varepsilon$, and at the second step use (8) on the Cuntz semigroup handout:

$$(a - \varepsilon)_+ \leq [(a - (\frac{1}{3}\varepsilon + \varepsilon_0))_+ - \frac{1}{3}\varepsilon]_+ \precsim_A [a_0 - (\frac{1}{3}\varepsilon + \varepsilon_0)]_+.$$
Proving \( rc(A) \leq rc(B) \) (continued)

Recall:

\[
a_1 = [(1 - g)a_0(1 - g) - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+.
\]

Claim 2 \( ([a_0 - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+ \preceq_B a_1 \oplus g) \) is an instance of the second lemma on the list of basic results on Cuntz equivalence:

**Lemma (Cutdown comparison)**

Let \( A \) be a C*-algebra, let \( a \in A_+ \), let \( g \in A_+ \) satisfy \( 0 \leq g \leq 1 \), and let \( \varepsilon \geq 0 \). Then

\[
(a - \varepsilon)_+ \preceq_A [(1 - g)a(1 - g) - \varepsilon]_+ \oplus g.
\]

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Proving \( rc(A) \leq rc(B) \): What we have so far

We have

\[
a, b \in M_l \otimes A, \quad n(a)_A = \langle x \rangle_A, \quad n(b)_A = \langle y \rangle_A, \quad m(1)_A = \langle q \rangle_A,
\]

\[
(x - \frac{1}{3}\varepsilon)_+ \oplus q \preceq_A (y - \delta)_+, \quad \text{and } \quad \text{rc}(B) \leq \frac{m}{n} - \frac{2}{k},
\]

and we want to prove that \( a \preceq_A b \).

We got \( g, a_0, a_1, b_0, b_1 \) such that

\[
(1 - g)a_0(1 - g), (1 - g)b_0(1 - g) \in M_l \otimes B \quad \text{and} \quad \sup_{\tau \in T(A)} d_\tau(g) \leq \frac{1}{k}.
\]

and the following hold:

1. \( (a - \varepsilon)_+ \preceq_A [a_0 - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+ \).
2. \( [a_0 - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+ \preceq_B a_1 \oplus g \).
3. \( a_1 \preceq_A (a - \frac{1}{3}\varepsilon)_+ \).
4. \( (b - \delta)_+ \preceq_A (b_0 - \varepsilon_0)_+ \).
5. \( (b_0 - \varepsilon_0)_+ \preceq_B b_1 \oplus g \).
6. \( b_1 \preceq_A b \).

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Proving \( rc(A) \leq rc(B) \) (continued)

Recall:

\[
\|a_0 - a\| < \varepsilon_1 \quad \text{and} \quad a_1 = [(1 - g)a_0(1 - g) - (\varepsilon_0 + \frac{1}{3}\varepsilon)]_+.
\]

For claim 3 \( (a_1 \preceq_A (a - \frac{1}{3}\varepsilon)_+ ) \), note \( \|a_0 - a\| < \varepsilon_1 \) implies

\[
\|(1 - g)a_0(1 - g) - (1 - g)a(1 - g)\| < \varepsilon_1.
\]

Therefore

\[
\|[(1 - g)a_0(1 - g) - \frac{1}{3}\varepsilon]_+ - [(1 - g)a(1 - g) - \frac{1}{3}\varepsilon]_+\| < \varepsilon_0.
\]

Using (8) on the Cuntz semigroup handout at the first step, this fact and (10) on the Cuntz semigroup handout at the second step, \( (6) \) on the Cuntz semigroup handout at the third step, and (17) on the Cuntz semigroup handout and \( a^{1/2}(1 - g)^2a^{1/2} \leq a \) at the last step, we get

\[
a_1 = [(1 - g)a_0(1 - g) - \frac{1}{3}\varepsilon]_+ - \varepsilon_0_+ \preceq_A [a^{1/2}(1 - g)^2a^{1/2} - \frac{1}{3}\varepsilon]_+ \preceq_A (a - \frac{1}{3}\varepsilon)_+,
\]

as desired.

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Proving \( rc(A) \leq rc(B) \) (continued)

Now let \( \tau \in T(A) \). Since \( x \) and \( y \) are the direct sums of \( n \) copies of \( a \) and \( b \), it follows that \( (x - \frac{1}{3}\varepsilon)_+ \) is the direct sum of \( n \) copies of \( (a - \frac{1}{3}\varepsilon)_+ \) and \( (y - \delta)_+ \) is the direct sum of \( n \) copies of \( (b - \delta)_+ \). So the relation

\[
(x - \frac{1}{3}\varepsilon)_+ \oplus q \preceq_A (y - \delta)_+
\]

implies

\[
n \cdot d_\tau((a - \frac{1}{3}\varepsilon)_+) + m \leq n \cdot d_\tau((b - \delta)_+).
\]

Using claim 4 and claim 5 at the first step and \( \sup_{\tau \in T(A)} d_\tau(g) < \frac{1}{k} \) at the third step, we get the estimate

\[
d_\tau((b - \delta)_+) \leq d_\tau(b_1) + d_\tau(g) < d_\tau(b_1) + k^{-1}.
\]

Claim 3 implies

\[
d_\tau(a_1) \leq d_\tau((a - \frac{1}{3}\varepsilon)_+).
\]
Proving $\text{rc}(A) \leq \text{rc}(B)$ (continued)

Using $\sup_{\tau \in T(A)} d_\tau(g) < \frac{1}{k}$ at the second step, (9) at the third step, (7) at the fourth step, and (8) at the fifth step, we get

$$n \cdot d_\tau(a_1 \oplus g) + m = n \cdot d_\tau(a_1) + m + n \cdot d_\tau(g)$$

$$\leq n \cdot d_\tau(a_1) + m + nk^{-1}$$

$$\leq n \cdot d_\tau((a - \frac{1}{3} \varepsilon)_{+}) + m + nk^{-1}$$

$$\leq n \cdot d_\tau((b - \delta)_{+}) + nk^{-1}$$

$$\leq n \cdot d_\tau(b_1) + 2nk^{-1}.$$

It follows that

$$d_\tau(a_1 \oplus g) + \frac{m}{n} - \frac{2}{k} \leq d_\tau(b_1).$$

This holds for all $\tau \in T(A)$, and therefore, since $A$ and $B$ have the same traces, for all $\tau \in T(B)$.

Proving $\text{rc}(A) \leq \text{rc}(B)$ (continued)

Since $\text{QT}(A) = T(A)$, since

$$\frac{m}{n} - \frac{2}{k} > \text{rc}(B),$$

and since $a_1, b_1, g \in M_1 \otimes B$, it follows that $a_1 \oplus g \preceq_B b_1$. Using this relation at the third step, claim 1 at the first step, claim 2 at the second step, and claim 6 at the last step, we then get

$$(a - \varepsilon)_{+} \preceq_A [a_0 - (\varepsilon_0 + \frac{1}{3} \varepsilon)]_{+} \preceq_A a_1 \oplus g \preceq_B b_1 \preceq_A b.$$

This completes the proof that $\text{rc}(A) \leq \text{rc}(B)$.