**Large Subalgebras and the Structure of Crossed Products, Lecture 2: Large Subalgebras and their Basic Properties**

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**A rough outline of all five lectures**

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if $B$ is a large subalgebra of $A$, then $A$ and $B$ have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.

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**Definition**

Let $A$ be a C*-algebra, and let $a, b \in (K \otimes A)_+$. We say that $a$ is **Cuntz subequivalent to** $b$ over $A$, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

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**Definition**

Let $A$ be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be **large** in $A$ if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$. 

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**Lecture 1 (1 June 2015): Introduction, Motivation, and the Cuntz Semigroup.**

**Lecture 2 (2 June 2015): Large Subalgebras and their Basic Properties.**

**Lecture 3 (4 June 2015): Large Subalgebras and the Radius of Comparison.**

**Lecture 4 (5 June 2015 [morning]): Large Subalgebras in Crossed Products by $\mathbb{Z}$.**

**Lecture 5 (5 June 2015 [afternoon]): Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms.**
Dense subsets

$B \subset A$ is large in $A$ if for $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \preceq_B y$ and $g \preceq_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Lemma

In the definition, it suffices to let $S \subset A$ be a subset whose linear span is dense in $A$, and verify the hypotheses only when $a_1, a_2, \ldots, a_m \in S$.

Unlike other approximation properties (such as tracial rank), it seems not to be possible to take $S$ to be a generating subset, or even a selfadjoint generating subset. (We can do this for the definition of a centrally large subalgebra.)

When $A$ is finite

$B \subset A$ is large in $A$ if for $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \preceq_B y$ and $g \preceq_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Proposition

Let $A$ be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra satisfying the definition of a large subalgebra except for condition (5). Then $B$ is large in $A$.

When $A$ is finite (continued)

From the previous slide:

Proposition

Let $A$ be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra satisfying the definition of a large subalgebra except for the condition $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$. Then $B$ is large in $A$.

It suffices to prove:

Lemma

Let $A$ be a finite simple infinite dimensional unital C*-algebra. Let $x \in A_+$ satisfy $\|x\| = 1$. Then for every $\varepsilon > 0$ there is $x_0 \in (xA) \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \preceq_A x_0$, then $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

If we also require $x_0 \preceq_A x$, then we can use $x_0$ in place of $x$ in the definition.

When $A$ is finite (continued)

To show: $x \in A_+$ with $\|x\| = 1$, $\varepsilon > 0$. Then there is $y \in (\overline{xA})_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \preceq_A y$, then $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Choose a sufficiently small number $\varepsilon_0 > 0$. Choose $f : [0, 1] \to [0, 1]$ such that $f = 0$ on $[0, 1 - \varepsilon_0]$ and $f(1) = 1$. Construct $a, b_j, c_j, d_j \in \overline{f(x)Af(x)}$ for $j = 1, 2$ such that

$0 \leq d_j \leq c_j \leq b_j \leq a_j$, \hspace{1em} $ab_j = b_j$, \hspace{1em} $b_jc_j = c_j$, \hspace{1em} $cjd_j = d_j$, \hspace{1em}$d_j \neq 0$, \hspace{1em}$b_1b_2 = 0$. Take $x_0 = d_1$.

If $\varepsilon_0$ is small enough, $g \preceq_A d_1$, and $\|(1 - g)x(1 - g)\| \leq 1 - \varepsilon$, use

$\|(1 - g)(b_1 + b_2)(1 - g)\| = \|(b_1 + b_2)^{1/2}(1 - g)^2(b_1 + b_2)^{1/2}\|$, \hspace{1em} $\|(1 - g)x(1 - g)\| = \|x^{1/2}(1 - g)^2x^{1/2}\|$, \hspace{1em} $(b_1 + b_2)^{1/2}x^{1/2} \approx x^{1/2}$ to get (details omitted)

$\|(1 - g)(b_1 + b_2)(1 - g)\| > 1 - \frac{\varepsilon}{3}$. 
When $A$ is finite (continued)

We assumed $g \preceq_A d_1$ and $\|(1 - g)x(1 - g)\| \leq 1 - \varepsilon$, and we want a contradiction. We have

$$0 \leq d_j \leq c_j \leq b_j \leq a \leq 1,$$

for $j = 1, 2$, and $b_1 b_2 = 0$. We also have

$$\|(1 - g)(b_1 + b_2)(1 - g)\| > 1 - \frac{\varepsilon}{3}. \quad (1)$$

From $(b_1 + b_2)(c_1 + c_2) = c_1 + c_2$ one gets, for any $\beta \in [0, 1],

$$c_1 + c_2 \preceq_A [(b_1 + b_2) - \beta]_+.$$

(2)

(If we are in $C(X)$, whenever $(c_1 + c_2)(x) \neq 0$, we have $(b_1 + b_2)(x) = 1 \geq \beta$.) Take $\beta = 1 - \frac{\varepsilon}{3}$. Combine (2) with the second lemma on the list of basic results on Cuntz equivalence at the first step, at the second step, and $g \preceq_A d_1$ at the last step, to get

$$c_1 + c_2 \preceq_A [(1 - g)(b_1 + b_2)(1 - g) - \beta]_+ \oplus g = 0 \oplus g \preceq_A d_1.$$

When $A$ is finite (continued)

In search of a contradiction, we have gotten

$$c_1 + c_2 \preceq_A d_1$$

with

$$c_1 d_1 = d_1, \quad c_1 c_2 = 0, \quad \text{and} \quad c_2 \neq 0.$$

This looks rather suspicious.

Set $r = (1 - c_1 - c_2) + d_1$. Use basic result (12) at the first step, $c_1 + c_2 \preceq_A d_1$ at the second step, and basic result (13) and $d_1(1 - c_1 - c_2) = 0$ at the third step, to get

$$1 \preceq_A (1 - c_1 - c_2) \oplus (c_1 + c_2) \preceq_A (1 - c_1 - c_2) \oplus d_1 \sim_A (1 - c_1 - c_2) + d_1 = r.$$

Thus, there is $v \in A$ such that $\|vrv^* - 1\| < \frac{1}{2}$. It follows that $vrv^{1/2}$ has a right inverse. Recall that $c_2 d_2 = d_2$ and $d_2 \neq 0$. So $rd_2 = 0$, whence $vrv^{1/2} d_2 = 0$. Thus $vrv^{1/2}$ is not invertible. We have contradicted finiteness of $A$, and thus proved the lemma.

### Lemma

Let $A$ be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $m, n \in \mathbb{Z}_{\geq 0}$, let $a_1, a_2, \ldots, a_m \in A$, let $b_1, b_2, \ldots, b_n \in A_+$, let $\varepsilon > 0$, let $z \in A_+$ satisfy $\|z\| = 1$, and let $y \in B_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$, $d_1, d_2, \ldots, d_n \in A_+$, and $g \in B$ such that:

- $0 \leq g \leq 1$.
- $\|c_j - a_j\| < \varepsilon$ and $\|d_j - b_j\| < \varepsilon$.
- $\|c_j\| \leq \|a_j\|$ and $\|d_j\| \leq \|b_j\|$.
- $(1 - g)c_j \in B$ and $(1 - g)d_j(1 - g) \in B$.
- $g \preceq_B y$ and $g \preceq_A x$.

### Sketch of proof.

To get $\|c_j\| \leq \|a_j\|$ one takes $\varepsilon > 0$ to be a bit smaller, and scales down $c_j$ for any $j$ for which $\|c_j\|$ is too big. To get $d_j$, approximate $b_j^{1/2}$ sufficiently well by $r_j$ (without increasing the norm), and take $d_j = r_j r_j^*$. \[\square\]

### Simplicity of a large subalgebra

Recall from Lecture 1:

**Proposition**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then $B$ is simple.

(The result stated in Lecture 1 also included infinite dimensionality. Once one has simplicity, infinite dimensionality is easy to prove, and we omit it.) The proof of this proposition uses two preliminary lemmas.

**Lemma**

Let $A$ be a C*-algebra, let $n \in \mathbb{Z}_{\geq 0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^n a_k$ and $x = \sum_{k=1}^n a_k^* a_k$. Then $a^* a \in \mathbb{R}^A$.

**Lemma**

Let $A$ be a unital C*-algebra and let $a \in A_+$. Suppose $AaA = A$. Then there exist $n \in \mathbb{Z}_{\geq 0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\sum_{k=1}^n x_k^* a x_k = 1$. 

[Note: The content continues with more propositions and lemmas related to C*-algebras and their properties, but the focus here is on the structural aspects of the document.]
The first lemma

From the previous slide:

**Lemma**

Let $A$ be a C*-algebra, let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^n a_k$ and $x = \sum_{k=1}^n a_k^* a_k$. Then $a^* a \in xA \bar{x}$.

**Sketch of proof.**

Assume $\|a_k\| \leq 1$ for $k = 1, 2, \ldots, n$. Choose $c \in xA \bar{x}$ such that $\|c\| \leq 1$ and $\|ca_k^* a_k - a_k^* a_k\|$ is small for $k = 1, 2, \ldots, n$. Check that $\|ca_k^* a_k - a_k^* a_k\|^2 \leq 2\|ca_k^* a_k - a_k^* a_k\|$, so $\|ca_k^* a_k - a_k^* a_k\|$ is small. Then $\|ca^* - a^*\|$ is small, so that $\|ca^* a - a^* a\|$ is small. Therefore $a^* a$ is arbitrarily close to $xA \bar{x}$.

The second lemma

From the slide before the previous slide:

**Lemma**

Let $A$ be a unital C*-algebra and let $a \in A_+$. Suppose $\bar{A}aA = A$. Then there exist $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\sum_{k=1}^n x_k^* a x_k = 1$.

**Proof.**

Choose $n \in \mathbb{Z}_{>0}$ and $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in A$ such that the element $c = \sum_{k=1}^n y_k a z_k$ satisfies $\|c - I\| < 1$. Set $r = \sum_{k=1}^n z_k^* a y_k^* a z_k$, $M = \max_k \|y_k\|$, and $s = M^2 \sum_{k=1}^n z_k^* a^2 z_k$.

The previous lemma implies that $c^* c$ is in the hereditary subalgebra generated by $r$. The relation $\|c - I\| < 1$ implies that $c$ is invertible, so $r$ is invertible. Since $r \leq s$, it follows that $s$ is invertible. Set $x_k = Ma_{1/2} s^{-1/2}$. Then check that $\sum_{k=1}^n x_k^* a x_k = s^{-1/2} ss^{-1/2} = 1$.

Proof of simplicity of $B$

Recall that we want to prove:

**Proposition**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then $B$ is simple.

Let $b \in B_+ \setminus \{0\}$. We show that there are $n \in \mathbb{Z}_{>0}$ and $r_1, r_2, \ldots, r_n \in B$ such that $\sum_{k=1}^n r_k b r_k^*$ is invertible.

Use the previous lemma to find $x_1, x_2, \ldots, x_m \in A$ such that $\sum_{k=1}^m x_k b x_k^* = 1$. Set $M = \max(1, \|x_1\|, \ldots, \|x_m\|, \|b\|)$ and $\delta = \min\left(1, \frac{1}{3MM(2M+1)}\right)$.

By definition, there are $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, $\|y_j - x_j\| < \delta$, $(1-g)y_j \in B$, and $g \lesssim_B b$. Set $z = \sum_{j=1}^m y_j b y_j^*$. The number $\delta$ has been chosen to ensure that $\|z - 1\| < \frac{1}{3}$; we omit details. Then $\|(1-g)z(1-g) - (1-g)^2\| < \frac{1}{3}$.

We took $b \in B_+ \setminus \{0\}$. We got $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, $\|y_j - x_j\| < \delta$, $(1-g)y_j \in B$, and $g \lesssim_B b$. We defined $z = \sum_{j=1}^m y_j b y_j^*$, and got $\|(1-g)z(1-g) - (1-g)^2\| < \frac{1}{3}$.

Set $h = 2g - g^2$. Use basic result (3) on Cuntz comparison on the map $\lambda \mapsto 2\lambda - \lambda^2$ on $[0, 1]$, to get $h \sim_B g$. So $h \lesssim_B b$. Choose $v \in B$ such that $\|v b v^* - h\| < \frac{1}{3}$.

Take $n = m + 1$, take $r_j = (1-g) y_j$ for $j = 1, 2, \ldots, m$, and take $r_{m+1} = v$. Then $r_1, r_2, \ldots, r_n \in B$. One can now check, using $(1-g)^2 + h = 1$, that $\|1 - \sum_{k=1}^n r_k b r_k^*\| < \frac{2}{3}$. Therefore $\sum_{k=1}^n r_k b r_k^*$ is invertible. This proves simplicity of $B$.

Proof of simplicity of $B$ (continued)

We are proving:

**Proposition**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then $B$ is simple.

We took $b \in B_+ \setminus \{0\}$. We got $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, $\|y_j - x_j\| < \delta$, $(1-g)y_j \in B$, and $g \lesssim_B b$. We defined $z = \sum_{j=1}^m y_j b y_j^*$, and got $\|(1-g)z(1-g) - (1-g)^2\| < \frac{1}{3}$.

Set $h = 2g - g^2$. Use basic result (3) on Cuntz comparison on the map $\lambda \mapsto 2\lambda - \lambda^2$ on $[0, 1]$, to get $h \sim_B g$. So $h \lesssim_B b$. Choose $v \in B$ such that $\|v b v^* - h\| < \frac{1}{3}$.

Take $n = m + 1$, take $r_j = (1-g) y_j$ for $j = 1, 2, \ldots, m$, and take $r_{m+1} = v$. Then $r_1, r_2, \ldots, r_n \in B$. One can now check, using $(1-g)^2 + h = 1$, that $\|1 - \sum_{k=1}^n r_k b r_k^*\| < \frac{2}{3}$. Therefore $\sum_{k=1}^n r_k b r_k^*$ is invertible. This proves simplicity of $B$. 
Traces

For a unital C*-algebra $A$, we denote by $T(A)$ the set of tracial states on $A$. We denote by $QT(A)$ the set of normalized 2-quasitraces on $A$.

If you haven’t heard of quasitraces, just pretend they are all tracial states. This is true on exact C*-algebras (in particular, on nuclear ones), and it is possible that it is always true.

Let $A$ be a stably finite unital C*-algebra, and let $\tau \in QT(A)$. Define $d_\tau : M_\infty(A) \to [0, \infty)$ by $d_\tau(a) = \lim_n \tau(a^{1/n})$.

To understand this, take $A = C(X)$ and $g \in C(X)$ with $0 \leq g \leq 1$, and take $\tau$ to be given by a probability measure $\mu$ on $X$. $(\tau(f) = \int_X f \, d\mu)$

Set $U = \{x \in X : g(x) \neq 0\}$. Then $g^{1/n} \to \chi_U$ and $d_\tau(g) = \mu(U)$.

Some facts: $d_\tau$ gives a well defined functional $d_\tau : W(A) \to [0, \infty)$ (and also $d_\tau : Cu(A) \to [0, \infty]$) such that $d_\tau((\langle a \rangle_A)$ is “the trace of the open support of $a$”. It preserves order and addition, and commutes with countable increasing suprema when they exist. In particular, $d_\tau(a) = \sup_{\varepsilon > 0} d_\tau((a - \varepsilon)_+)$. Also, $0 \leq a \leq 1$ implies $\tau(a) \leq d_\tau(a)$.

From the previous slide:

**Lemma**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in T(B)$. Then there exists a unique state $\omega$ on $A$ such that $\omega|_B = \tau$.

Existence of $\omega$ follows from the Hahn-Banach Theorem.

For uniqueness, let $\omega_1$ and $\omega_2$ be states with $\omega_1|_B = \omega_2|_B = \tau$, let $a \in A_+$, and let $\varepsilon > 0$. We show $|\omega_1(a) - \omega_2(a)| < \varepsilon$. We can assume $\|a\| \leq 1$.

We saw above that $B$ is simple and infinite dimensional. The third lemma on the list of basic results on Cuntz equivalence can be used to find $y \in B_+ \setminus \{0\}$ such that $\sup_{\varepsilon \in QT(B)} d_\tau(y)$ is as small as we want. (For orthogonal elements with $b_1 \sim_B b_2 \sim_B \cdots \sim_B b_n$, we must have $d_\tau(b_1) = d_\tau(b_2) = \cdots = d_\tau(b_n)$, so $nd_\tau(b_1) \leq 1$.) Choose $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \varepsilon^2/64$. Since $B$ is large, there are $c \in A_+$ and $g \in B_+$ such that $\|c\| \leq 1$, $\|g\| \leq 1$, $\|c - a\| < \varepsilon/2$, $(1 - g)c(1 - g) \in B$, and $g \lesssim_B y$.

So $\omega_j(g^2) = \tau(g^2) \leq d_\tau(g^2) < \varepsilon^2/64$.

**Bijection on traces**

Recall from Lecture 1:

**Theorem**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $T(A) \to T(B)$ is bijective.

(The result stated in Lecture 1 also included the same thing for quasitraces. That result requires much more work, since it depends on the fact that the inclusion of $A$ in $B$ induces an isomorphism on the subsemigroups of purely positive elements.)

The proof of this proposition uses a preliminary lemma.

**Lemma**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in T(B)$. Then there exists a unique state $\omega$ on $A$ such that $\omega|_B = \tau$.

We have $a \in A$ and we want to prove that $|\omega_1(a) - \omega_2(a)| < \varepsilon$. We have $\|c\| \leq 1$, $\|g\| \leq 1$, $\|c - a\| < \varepsilon$, $(1 - g)c(1 - g) \in B$, and $\omega_j(g^2) < \varepsilon^2/64$.

The Cauchy-Schwarz inequality gives

$$\omega_j((c - a)g^2) \leq \omega_j((c - a)g^2) \leq \omega_j(g^2) < \varepsilon^2/64.$$
Bijection on traces
Recall that we want to prove:

**Theorem**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $T(A) \to T(B)$ is bijective.

Let $\tau \in T(B)$. We show that there is a unique $\omega \in T(A)$ such that $\omega|_B = \tau$. We know that there is a unique state $\omega$ on $A$ such that $\omega|_B = \tau$, and it suffices to show that $\omega$ is a trace. Thus let $a_1, a_2 \in A$ satisfy $\|a_1\| \leq 1$ and $\|a_2\| \leq 1$, and let $\varepsilon > 0$. We show that $\|\omega(a_1 a_2) - \omega(a_2 a_1)\| < \varepsilon$.

As in the proof of the lemma, find $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \varepsilon^2/54$. Since $B$ is large, there are $c_1, c_2 \in A$ and $g \in B_+$ such that

\[
\|c_j\| \leq 1, \quad \|c_j - a_j\| < \frac{\varepsilon}{8}, \quad \text{and} \quad (1 - g)c_j \in B
\]

for $j = 1, 2$, and such that $\|g\| \leq 1$ and $g \not\succ B$ $y$. As before, $\omega(g^2) \leq d_\tau(y) < \varepsilon^2/54$.

Bijection on traces (continued)
We got $\|a_j\| \leq 1$, $\|c_j\| \leq 1$, $\|c_j - a_j\| < \frac{\varepsilon}{8}$, $(1 - g)c_j \in B$, and (at the bottom of the previous slide)

\[
\|((1 - g)c_1(1 - g)c_2) - \omega(c_1 c_2)\| < \frac{\varepsilon}{4}.
\]

A similar argument gives

\[
\|((1 - g)c_2(1 - g)c_1) - \omega(c_2 c_1)\| < \frac{\varepsilon}{4}.
\]

Since $(1 - g)c_1, (1 - g)c_2 \in B$ and $\omega|_B$ is a tracial state, we get

\[
\omega((1 - g)c_1(1 - g)c_2) = \omega((1 - g)c_2(1 - g)c_1).
\]

Therefore $\|\omega(c_1 c_2) - \omega(c_2 c_1)\| < \frac{\varepsilon}{2}$.

One checks that $\|c_1 c_2 - a_1 a_2\| < \frac{\varepsilon}{4}$ and $\|c_2 c_1 - a_2 a_1\| < \frac{\varepsilon}{4}$. It now follows that $\|\omega(a_1 a_2) - \omega(a_2 a_1)\| < \varepsilon$.

We have $\|\omega(a_1 a_2) - \omega(a_2 a_1)\| < \varepsilon$ for all $\varepsilon > 0$, so $\omega(a_1 a_2) = \omega(a_2 a_1)$.

Bijection on traces

We have thus proved:

**Theorem**

Let $A$ be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $T(A) \to T(B)$ is bijective.