Large Subalgebras and the Structure of Crossed Products, Lecture 1: Introduction, Motivation, and the Cuntz Semigroup

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- Lecture 1 (1 June 2015): Introduction, Motivation, and the Cuntz Semigroup.
- Lecture 2 (2 June 2015): Large Subalgebras and their Basic Properties.
- Lecture 3 (4 June 2015): Large Subalgebras and the Radius of Comparison.
- Lecture 4 (5 June 2015 [morning]): Large Subalgebras in Crossed Products by $\mathbb{Z}$.
- Lecture 5 (5 June 2015 [afternoon]): Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms.

A rough outline of all five lectures

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if B is a large subalgebra of A, then A and B have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.

Introduction

Main references:

- N. C. Phillips, *Large subalgebras*, preprint (arXiv: 1408.5546v1 [math.OA]).
- D. Archey and N. C. Phillips, *Permanence of stable rank one for centrally large subalgebras and crossed products by minimal homeomorphisms*, preprint (arXiv: 1505.00725v1 [math.OA]).
- T. Hines, N. C. Phillips, and A. S. Toms, *Mean dimension and radius of comparison for minimal homeomorphisms with Cantor factors*, in preparation.
- N. C. Phillips, *Large subalgebras and applications*, lecture notes.

The first four lectures are mostly from the first paper, with a small amount of material from the second paper. The last lecture is from the third paper. The proof of the result in the third lecture is quite different from that in the first paper. The lecture notes contain a substantial amount of material not in the actual lectures, but condensed considerably from the first paper.

Applications

The first large subalgebra was used by Putnam in 1989 (not by name) to study the order on $K_0(C^*(\mathbb{Z}, X, h))$ when h is a minimal homeomorphism of the Cantor set. They have been used in a number of places (still without the name) to study the structure of crossed products by minimal homeomorphisms. (Some references are in the notes.) The main recent uses are as follows:

1. The "extended" irrational rotation algebras, obtained by "cutting" each of the standard unitary generators at one or more points in its spectrum, are AF (Elliott-Niu).
2. If $h \colon X \to X$ is a minimal homeomorphism of an infinite compact metric space with mean dimension zero, then $C^*(\mathbb{Z}, X, h)$ is Z-stable (Elliott-Niu).
3. If $h \colon X \to X$ is a minimal homeomorphism and X has a surjective map to the Cantor set K, then $C^*(\mathbb{Z}, X, h)$ has stable rank one, regardless of the mean dimension of h (joint with Archey).
4. If $h \colon X \to X$ is a minimal homeomorphism and X has a surjective map to K, then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$ (with Hines and Toms).

Applications (continued)

From the previous slide: Large subalgebras are used to prove that if $h \colon X \to X$ is a minimal homeomorphism and X has a surjective map to the Cantor set, then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

We also show that for minimal homeomorphisms of the type considered by Giol and Kerr, we actually have $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$.

The applications to $C^*(\mathbb{Z}, X, h)$ use the "orbit breaking subalgebra" $C^*(\mathbb{Z}, X, h)_Y$ (defined below). Other applications (such as the first proof that if $\mathbb{Z}^d$ acts freely and minimally on a finite dimensional compact metric space, then $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements) require large subalgebras for which we don't have a formula, only an existence proof. (We won't get to such examples in this course.)

The result on $C^*(\mathbb{Z}^d, X)$ has been superseded by Rokhlin dimension methods. There unfortunately is no time in this course to say anything about Rokhlin dimension, but in many problems one should consider both Rokhlin dimension and large subalgebras as possible methods.

Definition

Let A be a C*-algebra, and let $a, b \in (K \otimes A)_+$. We say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

About the definitions

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $v_n b v_n^* \to a$.

More about Cuntz comparison later.

From the previous slide: A unital subalgebra $B \subset A$ is *large* in A if for $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

B being unital means $1_A \in B$.

The Cuntz subequivalence involving y in (4) is relative to B, not A.

About the definitions (continued)

From the previous slide: A unital subalgebra $B \subset A$ is *large* in A if for $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Condition (5) is needed to avoid triviality when A is purely infinite and simple. In the stably finite case, we will see that it is automatic.

Even in the stably finite case, we need both $g \precsim_B y$ and $g \precsim_A x$ in (4).

One can (with some functional calculus) replace (2) and (3) by $\operatorname{dist}((1 - g)a_j, B) < \varepsilon$. (The value of ε is different.)

Centrally large subalgebras

The difference in the definitions is approximate commutation (6).

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *centrally large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.
6. For $j = 1, 2, \ldots, m$ we have $\|g a_j - a_j g\| < \varepsilon$.

A big difference between (central) largeness and other related conditions is that g is not required to be a projection.

Stably large subalgebras

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *stably large* in A if $M_n(B)$ is large in $M_n(A)$ for all $n \in \mathbb{Z}_{>0}$.

Proposition

Let A_1 and A_2 be infinite dimensional simple unital C*-algebras, and let $B_1 \subset A_1$ and $B_2 \subset A_2$ be large subalgebras. Assume that $A_1 \otimes_{\min} A_2$ is finite. Then $B_1 \otimes_{\min} B_2$ is a large subalgebra of $A_1 \otimes_{\min} A_2$.

In particular, if A is stably finite and $B \subset A$ is large, then B is stably large. This is easy to prove directly. (The condition $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$ causes problems, but it is not needed here.) We don't know whether stable finiteness of A is needed.

Orbit breaking subalgebras

We will say more about crossed products by $\mathbb{Z}$ in Lecture 3. Here, if A is unital and $\alpha \in \operatorname{Aut}(A)$, we let $u \in A[\mathbb{Z}] \subset C^*(\mathbb{Z}, A, \alpha)$ be the standard unitary generator, corresponding to $1 \in \mathbb{Z}$.

For a compact Hausdorff space X and a closed subset $Y \subset X$, identify

$$C_0(X \setminus Y) = \big\{ f \in C_0(X) \colon f(x) = 0 \text{ for all } x \in Y \big\} \subset C_0(X).$$

Definition

Let X be a locally compact Hausdorff space and let $h \colon X \to X$ be a homeomorphism. Let $Y \subset X$ be a nonempty closed subset, and define

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C_0(X), \, C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

We call it the *Y-orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

In the past, one usually took $C^*(\mathbb{Z}, X, h)_Y = C^*\big(C_0(X), \, uC_0(X \setminus Y)\big)$. Our choice has the advantage that, when used in connection with Rokhlin towers, the bases of the towers are subsets of Y rather than of $h(Y)$.

Orbit breaking subalgebras are large

Recall: $C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\ C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h)$.

Theorem

Let X be a compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$.

The key fact about $C^*(\mathbb{Z}, X, h)_Y$ which makes this theorem useful is that it is a direct limit of recursive subhomogeneous C*-algebras whose base spaces are closed subsets of X. The structure of $C^*(\mathbb{Z}, X, h)_Y$ is therefore much more accessible than the structure of crossed products.

Large subalgebras, simplicity, traces, and finiteness

Proposition

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple and infinite dimensional.

Theorem

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$, on traces and quasitraces, are bijective.

The proofs of the parts are quite different. We prove the first part later.

Proposition

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then:

1. A is finite if and only if B is.
2. If B is stably large in A, then A is stably finite if and only if B is.
3. A is purely infinite if and only if B is.

Cuntz semigroup and radius of comparison

Let A be a C*-algebra. The Cuntz semigroup $\mathrm{Cu}(A)$ is the semigroup of Cuntz equivalence classes of positive elements in A (defined below). Let $\mathrm{Cu}_+(A)$ denote the set of elements $\eta \in \mathrm{Cu}(A)$ which are not the classes of projections. (Its elements are sometimes called *purely positive*.)

Theorem

Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\iota\colon B \to A$ be the inclusion map. Then $\mathrm{Cu}_+(B) \cup \{0\} \to \mathrm{Cu}_+(A) \cup \{0\}$ is an order and semigroup isomorphism.

Known examples show that $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ need not be injective, and probably it need not be surjective either.

Theorem

Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be a large subalgebra. Let $\mathrm{rc}(-)$ be the radius of comparison (defined in Lecture 3). Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

Stable rank

Theorem (Joint with Archey)

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Then:

1. If $\mathrm{tsr}(B) = 1$ then $\mathrm{tsr}(A) = 1$.
2. If $\mathrm{tsr}(B) = 1$ and $\mathrm{RR}(B) = 0$ then $\mathrm{RR}(A) = 0$.

In progress with Archey and Buck:

Let A be an infinite dimensional simple nuclear unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Let Z be the Jiang-Su algebra. If B is Z-stable ($Z \otimes B \cong B$), then so is A.

Nuclearity is needed because what we actually get is "tracial Z-stability", and other machinery (Hirshberg-Orovitz, via Sato etc.) is needed to get Z-stability.

Technical lemmas

Here are the key technical results behind many of the results (in particular, behind $\mathrm{Cu}_+(B) \cup \{0\} \cong \mathrm{Cu}_+(A) \cup \{0\}$, used to prove many of the others):

Lemma

Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a stably large subalgebra.

1. Let $a, b, x \in (K \otimes A)_+$ satisfy $x \neq 0$ and $a \oplus x \precsim_A b$, and let $\varepsilon > 0$. Then there are $n \in \mathbb{Z}_{>0}$, $c \in (M_n \otimes B)_+$, and $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A c \precsim_A (b - \delta)_+$.
2. Let $a, b \in (K \otimes B)_+$ and $c, x \in (K \otimes A)_+$ satisfy $x \neq 0$, $a \precsim_A c$, and $c \oplus x \precsim_A b$. Then $a \precsim_B b$.

We won't prove or use them in these lectures. Instead, we give a more direct proof that a large subalgebra has the same radius of comparison.

Application: Radius of comparison of crossed products

Theorem (Joint work with Hines and Toms)

Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

The number $\mathrm{rc}(A)$ is the radius of comparison of A, discussed in Lecture 3. The number $\mathrm{mdim}(h)$ is the mean dimension of h, discussed in Lecture 5. It is conjectured that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for all minimal homeomorphisms. We also prove that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \geq \frac{1}{2}\mathrm{mdim}(h)$ for a generalization of Giol and Kerr's examples. For such minimal homeomorphisms, there is a continuous surjective map to the Cantor set.

The proof uses a suitable orbit breaking subalgebra, the fact that the radius of comparison of large subalgebra is the same as for the containing algebra, the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ is the direct limit of an AH system with diagonal maps, and methods of Niu to estimate radius of comparison of simple direct limits of AH systems with diagonal maps.

Application: Stable rank of crossed products

Theorem (Joint work with Dawn Archey)

Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one.

There is no finite dimensionality assumption on X. We don't even assume that h has mean dimension zero. In particular, this theorem holds for the examples of Giol and Kerr, for which the crossed products are known not to be Z-stable and not to have strict comparison of positive elements. (In fact, by work with Hines and Toms [previous slide], $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for such systems, and Giol and Kerr show that $\mathrm{mdim}(h) \neq 0$.)

The proof uses a suitable orbit breaking subalgebra, the fact that stable rank one passes up from a centrally large subalgebra, the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ is the direct limit of an AH system with diagonal maps, and a result of Elliott, Ho, and Toms to show that simple AH algebras with diagonal maps always have stable rank one.

Two further applications

Theorem (Elliott and Niu)

The "extended" irrational rotation algebras, obtained by "cutting" the standard unitary generators at one or more points the spectrum, are AF.

We omit the precise descriptions of these algebras.

Cutting one unitary gives a crossed product by a minimal homeomorphism of the Cantor set, with the other unitary being the generator of the group. If both are cut, the algebra is no longer an obvious crossed product.

Theorem (Elliott and Niu)

Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. If $\mathrm{mdim}(h) = 0$, then $C^*(\mathbb{Z}, X, h)$ is Z-stable.

Z is the Jiang-Su algebra. Z-stability is one of the conditions in the Toms-Winter conjecture, and for simple separable nuclear C*-algebras it is hoped, and known in many cases, that Z-stability implies classifiability in the sense of the Elliott program.

The Cuntz semigroup

Let $M_\infty(A)$ denote the algebraic direct limit of the system $(M_n(A))_{n=1}^\infty$ using the usual embeddings $M_n(A) \to M_{n+1}(A)$.

Recall that if $a, b \in (K \otimes A)_+$, then $a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^\infty$ in $K \otimes A$ such that $v_n b v_n^* \to a$. (It is not hard to see that if a and b are in any of A, $M_n(A)$, or $M_\infty(A)$, we can take $(v_n)_{n=1}^\infty$ in the same algebra.)

We define $a \sim_A b$ if $a \precsim_A b$ and $b \precsim_A a$. This relation is an equivalence relation, and we write $\langle a \rangle_A$ for the equivalence class of a.

The *Cuntz semigroup* of A is $\mathrm{Cu}(A) = (K \otimes A)_+ / \sim_A$, together with the commutative semigroup operation

$$\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A = \langle \mathrm{diag}(a, b) \rangle_A$$

(using an isomorphism $M_2(K) \to K$; the result does not depend on which one) and the partial order $\langle a \rangle_A \leq \langle b \rangle_A$ if and only if $a \precsim_A b$.

We also define the subsemigroup $W(A) = M_\infty(A)_+ / \sim_A$, with the same operations and order. We write 0 for $\langle 0 \rangle_A$.

$\varphi \colon A \to B$ gives $\mathrm{Cu}(\varphi) \colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ and $W(\varphi) \colon W(A) \to W(B)$.

What is the Cuntz semigroup?

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^\infty$ in $K \otimes A$ such that $v_n b v_n^* \to a$.

For separable A, the Cuntz semigroup can be very roughly thought of as K-theory using open projections in matrices over A'', that is, open supports of positive elements in matrices over A, instead of projections in matrices over A. For example, if X is a compact Hausdorff space and $f, g \in C(X)_+$, then $f \precsim_{C(X)} g$ if and only if

$$\{x \in X \colon f(x) > 0\} \subset \{x \in X \colon g(x) > 0\}.$$

K-theory is "discrete": if $p, q \in A$ are projections such that $\|p - q\| < 1$, then p and q are Murray-von Neumann equivalent. The best we can do with Cuntz comparison is suggested by the the fact that $\|f - g\| < \varepsilon$ implies

$$\{x \in X \colon f(x) > \varepsilon\} \subset \{x \in X \colon g(x) > 0\},$$

so that the function $\max(f - \varepsilon, 0)$ satisfies $\max(f - \varepsilon, 0) \precsim_{C(X)} g$. On the other hand, taking $f = g + \frac{\varepsilon}{2}$ gives $\|f - g\| < \varepsilon$ and $\langle f \rangle_{C(X)} = \langle 1 \rangle_{C(X)}$, however small $\langle g \rangle_{C(X)}$ is.

The cutdown $(a - \varepsilon)_+$

Definition

Let A be a C*-algebra, let $a \in A_+$, and let $\varepsilon > 0$. Let $f \colon [0, \infty) \to [0, \infty)$ be the function

$$f(\lambda) = (\lambda - \varepsilon)_+ = \begin{cases} 0 & 0 \leq \lambda \leq \varepsilon \\ \lambda - \varepsilon & \varepsilon < \lambda. \end{cases}$$

Then define $(a - \varepsilon)_+ = f(a)$ (using continuous functional calculus).

The positive result from the previous slide then becomes: if $a, b \in C(X)_+$ and $\|a - b\| < \varepsilon$, then $(a - \varepsilon)_+ \precsim_{C(X)} b$. This is in fact true in a general C*-algebra, not just in $C(X)$.

Warnings:

- $a \leq b$ does not imply $(a - \varepsilon)_+ \leq (b - \varepsilon)_+$ (but does imply $(a - \varepsilon)_+ \precsim_A (b - \varepsilon)_+$).
- $a \precsim_A b$ does not imply any relation between $(a - \varepsilon)_+$ and $(b - \varepsilon)_+$. (Take $b = \delta a$ with $\delta > 0$ small).

Basic lemmas on Cuntz comparison

$a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^\infty$ in $K \otimes A$ such that $v_n b v_n^* \to a$.

$(a - \varepsilon)_+ = f(a)$ for

$$f(\lambda) = (\lambda - \varepsilon)_+ = \begin{cases} 0 & 0 \leq \lambda \leq \varepsilon \\ \lambda - \varepsilon & \varepsilon < \lambda. \end{cases}$$

There are many more "basic lemmas" for Cuntz comparison than for K-theory. We give a list. Not all will be needed, but most will be, and we include the others to give a fuller picture. Most are old, but a few are new in the paper *Large subalgebras*.

I recommend keeping a printed copy of the list to refer to during the remaining lectures. The numbered items on the list have the same numbers as in the parts of the corresponding lemma in the notes, and will be referred to on the slides.

We denote by A^+ the unitization of a C*-algebra A. (We add a new unit even if A is already unital.)

Statements of basic lemmas on Cuntz comparison

Let A be a C*-algebra.

(1) Let $a, b \in A_+$. Suppose $a \in \overline{bAb}$. Then $a \precsim_A b$.

(2) Let $a \in A_+$ and let $f \colon [0, \|a\|] \to [0, \infty)$ be a continuous function such that $f(0) = 0$. Then $f(a) \precsim_A a$.

(3) Let $a \in A_+$ and let $f \colon [0, \|a\|] \to [0, \infty)$ be a continuous function such that $f(0) = 0$ and $f(\lambda) > 0$ for $\lambda > 0$. Then $f(a) \sim_A a$.

(4) Let $c \in A$. Then $c^* c \sim_A c c^*$.

(5) Let $a \in A_+$, and let $u \in A^+$ be unitary. Then $u a u^* \sim_A a$.

(6) Let $c \in A$ and let $\alpha > 0$. Then $(c^* c - \alpha)_+ \sim_A (c c^* - \alpha)_+$.

(7) Let $v \in A$. Then there is an isomorphism $\varphi \colon \overline{v^* v A v^* v} \to \overline{v v^* A v v^*}$ such that, for every positive element $z \in \overline{v^* v A v^* v}$, we have $z \sim_A \varphi(z)$.

(8) Let $a \in A_+$ and let $\varepsilon_1, \varepsilon_2 > 0$. Then

$$\big((a - \varepsilon_1)_+ - \varepsilon_2\big)_+ = \big(a - (\varepsilon_1 + \varepsilon_2)\big)_+.$$

Statements of basic lemmas on Cuntz comparison

(9) Let $a, b \in A_+$ satisfy $a \precsim_A b$ and let $\delta > 0$. Then there is $v \in A$ such that $v^* v = (a - \delta)_+$ and $v v^* \in \overline{bAb}$.

(10) Let $a, b \in A_+$. Then $\|a - b\| < \varepsilon$ implies $(a - \varepsilon)_+ \precsim_A b$.

(11) Let $a, b \in A_+$. Then the following are equivalent:

1. $a \precsim_A b$.
2. $(a - \varepsilon)_+ \precsim_A b$ for all $\varepsilon > 0$.
3. For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A (b - \delta)_+$.

(12) Let $a, b \in A_+$. Then $a + b \precsim_A a \oplus b$.

(13) Let $a, b \in A_+$ be orthogonal (that is, $ab = 0$). Then $a + b \sim_A a \oplus b$.

(14) Let $a_1, a_2, b_1, b_2 \in A_+$, and suppose that $a_1 \precsim_A a_2$ and $b_1 \precsim_A b_2$. Then $a_1 \oplus b_1 \precsim_A a_2 \oplus b_2$. (So addition in $\mathrm{Cu}(A)$ respects order.)

(15) Let $a, b \in A$ be positive, and let $\alpha, \beta \geq 0$. Then

$$\big(a + b - (\alpha + \beta)\big)_+ \precsim_A (a - \alpha)_+ + (b - \beta)_+ \precsim_A (a - \alpha)_+ \oplus (b - \beta)_+.$$

More basic lemmas on Cuntz comparison

(16) Let $\varepsilon > 0$ and $\lambda \geq 0$. Let $a, b \in A$ satisfy $\|a - b\| < \varepsilon$. Then $(a - \lambda - \varepsilon)_+ \precsim_A (b - \lambda)_+$.

(17) Let $a, b \in A$ satisfy $0 \leq a \leq b$. Let $\varepsilon > 0$. Then $(a - \varepsilon)_+ \precsim_A (b - \varepsilon)_+$.

(18) Let $a \in (K \otimes A)_+$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$ and $b \in (M_n \otimes A)_+$ such that $(a - \varepsilon)_+ \sim_A b$.

Lemma

Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then

$$(a - \varepsilon)_+ \precsim_A \big[(1 - g) a (1 - g) - \varepsilon\big]_+ \oplus g.$$

From the last slide:

Lemma

Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then $(a - \varepsilon)_+ \precsim_A \big[(1 - g) a (1 - g) - \varepsilon\big]_+ \oplus g$.

This lemma is new in the paper, and is crucial to the relation between Cuntz comparison and large subalgebras, so we give the proof.

Proof.

Set $h = 2g - g^2$, so that $(1 - g)^2 = 1 - h$. Then $h \sim_A g$ by basic result (3). Set $b = \big[(1 - g) a (1 - g) - \varepsilon\big]_+$. Use basic result (15) at the second step, (6) and (4) at the third step, and (14) at the last step:

$$\begin{aligned} (a - \varepsilon)_+ &= \big[a^{1/2}(1 - h)a^{1/2} + a^{1/2} h a^{1/2} - \varepsilon\big]_+ \\ &\precsim_A \big[a^{1/2}(1 - h)a^{1/2} - \varepsilon\big]_+ \oplus a^{1/2} h a^{1/2} \\ &\sim_A b \oplus h^{1/2} a h^{1/2} \leq b \oplus \|a\| h \precsim_A b \oplus g. \end{aligned}$$

This completes the proof. $\square$

Cuntz comparison and simple C*-algebras

Let A be a simple infinite dimensional C*-algebra not of type I. (Not all of this is needed in all these results.) The last lemma is technical.

Lemma

Let $a \in A_+ \setminus \{0\}$, and let $l \in \mathbb{Z}_{>0}$. Then there exist orthogonal elements $b_1, \ldots, b_l \in A_+ \setminus \{0\}$ such that $b_1 \sim_A \cdots \sim_A b_l$ and $\sum_{j=1}^{n} b_j \in \overline{aAa}$.

Lemma

Let $B \subset A$ be a nonzero hereditary subalgebra. Let $a_1, \ldots, a_n \in A_+ \setminus \{0\}$. Then there is $b \in B_+ \setminus \{0\}$ such that $b \precsim_A a_j$ for $j = 1, 2, \ldots, n$.

Lemma

Let $b \in A_+ \setminus \{0\}$, let $\varepsilon > 0$, and let $n \in \mathbb{Z}_{>0}$. Then there are $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that, in $W(A)$, we have $n\langle (b-\varepsilon)_+ \rangle_A \leq (n+1)\langle c \rangle_A$ and $\langle c \rangle_A + \langle y \rangle_A \leq \langle b \rangle_A$.

The hard case of the last lemma is if 0 is isolated in $\mathrm{sp}(b)$.

Some open problems

Question

Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra. If B has any of the following properties, does it follow that A has that property?

1. Tracial rank zero.
2. Qusidiagonality.
3. Finite decomposition rank.
4. Finite nuclear dimension.
5. Real rank zero.
6. Stable rank at most n.

We think we have Z-stability in the centrally large case. (See above.) Also in this case, real rank zero goes up to A in the presence of stable rank one, but it should go up to A in general. The others, except maybe stable rank at most n, seem doubtful without additional assumptions.

Open problem: The tracial Rokhlin property

Question

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon \mathbb{Z} \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Is there a useful large or centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$?

We want a centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$ which "locally looks like matrices over corners of A". In a paper with Osaka, we proved that, under the hypotheses of this question, if A has real rank zero, stable rank one, and order on projections determined by traces, then $C^*(\mathbb{Z}, A, \alpha)$ also has these properties. The method was inspired by the use of large subalgebras for crossed products of actions of $\mathbb{Z}^d$ on the Cantor set, but involved choosing a different subalgebra (analogous to $C^*(\mathbb{Z}, X, h)_Y$ for a small closed subset $Y \subset X$ with $\mathrm{int}(Y) \neq \varnothing$) for each finite set $F \subset C^*(\mathbb{Z}, A, \alpha)$ and $\varepsilon > 0$, without being able to arrange them in a direct system.

Open problem: More general groups

Problem

Let X be a compact metric space, and let G be a countable amenable group which acts minimally and essentially freely on X. Construct a (centrally) large subalgebra $B \subset C^*(G, X)$ which is a direct limit of recursive subhomogeneous C*-algebras whose base spaces are closed subsets of X, and which is the (reduced) C*-algebra of an open subgroupoid of the transformation group groupoid obtained from the action of G on X.

We know how to do this when $G = \mathbb{Z}^d$ and X has finite covering dimension. (In this situation, one can also use finite Rokhlin dimension methods.) It should be possible to do this much more generally.

Open problem: The nonsimple case

Problem

Develop the theory of large subalgebras of not necessarily simple C*-algebras.

If the definition as given is applied, one finds that if B is a nontrivial large subalgebra of A, then $B \oplus A$ won't be large in $A \oplus A$. (See the notes for further discussion.)

An initial goal (which should not require a full theory) is to relate mean dimension and radius of comparison when $h \colon X \to X$ has a factor system which is minimal but is not minimal itself.

Open problem: The nonunital case

Problem

Develop the theory of large subalgebras of simple but not necessarily unital C*-algebras.

This is aimed at two situations:

1. Minimal homeomorphisms of noncompact locally compact metric spaces.
2. Automorphisms of $C(X, D)$ which "lie over" a minimal homeomorphism of X when D is simple but not unital. (Julian Buck already has interesting results when D is simple and unital.)

See the notes for more discussion of all of these problems and further open problems.