Increasing the Math Maturity of Elementary School Students and Their Teachers

Dave Moursund

Teacher Education, College of Education

University of Oregon

Eugene, Oregon 97403

moursund@uoregon.edu

This document can be accessed at http://darkwing.uoregon.edu/~moursund/Math/.

Date of this version: 9/21/04

Abstract

This document was written specifically for use in a Math Methods course for preservice elementary school teachers. However, it is also designed for use by inservice elementary school teachers.

A person's math expertise consists of math content knowledge and skills, and math maturity. Math content to be taught and learned is specified in a wide variety of state and national standards, and it is strongly based on the standards developed by the National Council of Teachers of Mathematics (NCTM, n.d.).

Math maturity focuses on areas such as understanding, problem solving, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning (being able to use one's mathematical knowledge over a wide range of disciplines and in novel settings), and interest (including intrinsic motivation) in math.

The goal of this document is to improve your preparation to be a good math teacher—and thus, to improve the math education of your future students. The main focus is on helping you better understand some ideas about math maturity and ways to increase both your own and your future students' math maturity.

Increasing the Math Maturity of Elementary School Students and Their Teachers

Introduction

This document was written specifically for use in a Math Methods course for preservice elementary school teachers. Its goal is to improve your preparation to be a good math teacher—and thus, to improve the math education of your future students.

This document has a strong focus on math maturity, which is something quite different than math content knowledge and skills. Math maturity focuses on areas such as understanding, problem solving, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning (being able to use one's mathematical knowledge over a wide range of disciplines and in novel settings), and interest (including intrinsic motivation) in math.

A typical student taking a Math Methods course has had 11 or 12 years of precollege math and perhaps as much as a full year sequence in Math for Elementary Teachers. Somewhat surprisingly, considering this extensive math content preparation, many preservice elementary school teachers feel uncomfortable about their math knowledge and skills. The thesis of this document is that although these preservice teachers have taken a lot of math coursework and "covered" a lot of math content, they have a relatively low level of math maturity. This is because the way they have been taught and the way that they have learned focused mainly on math content and did little to increase their math maturity.

Elementary school teachers tend to teach math in the way that they were taught. Thus, the cycle repeats itself. Year after year students go through their math courses in a manner and curriculum described by Mike Battista (1999):

> For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them. Though the same topics are taught and retaught year after year, the students do not learn them. Numerous scientific studies have shown that traditional methods of teaching mathematics not only are ineffective but also seriously stunt the growth of students' mathematical reasoning and problem-solving skills. Traditional methods ignore recommendations by professional organizations in mathematics education, and they ignore modern scientific research on how children learn mathematics (Battista, 1999).

This cycle can and must be broken if the quality of math education that our students receive is to be significantly improved. You, personally, can make a significant difference for your students. The ideas presented in this document will help you.

This document focuses on six related topics that I feel are important as we work to improve our math education system:

1. **Problem Solving.** Problem solving lies at the heart of each discipline, including math. Math is an aid to representing and solving problems in many

disciplines. I believe our education system is not doing very well in helping students learn to make use of math in the various disciplines they study.

2. **Mathematical Expertise**. Expertise in math consists of a combination of math content knowledge and math maturity. This document has a strong emphasis on math maturity. I believe that our current math education system is doing a relatively poor job in helping students to increase their math maturity.

3. **Mind Science (Psychology of the Mind).** This document has a strong emphasis on the concept of fluid and crystallized intelligence, and education to increase math intelligence. I believe that an appropriately designed and implemented math education system will significantly increase both the math-oriented fluid intelligence and the math-oriented crystallized intelligence of students.

4. **Math Aspects of Piagetian Developmental Theory.** This document has a strong emphasis on the idea of a Piagetian-type math developmental scale and how this relates to math teaching and learning. I believe that there are significant components of our math curriculum that are being taught well "over the heads" of students, because the students are not at a developmental level appropriate to the content being taught. The leads to a "memorize without understanding, regurgitate to do the homework and pass the test, and forget" approach to math learning.

5. **Brain Science (Cognitive Neuroscience).** This document has a strong emphasis on brain development, our rapidly increasing understanding of brain plasticity, and how education changes the brain. I believe that cognitive neuroscience holds one of the keys to significant improvements in our math education system.

6. **Information and Communication Technology (ICT).** This document explores ways in which computers (including artificial intelligence) add a new dimension to math teaching, learning, and expertise. I believe that ICT holds one of the keys to significant improvements in our math education system.

Taken together, these six topics provide a framework for research, development, and implementation of a more effective math education system.

Introduction

This document summarizes some ideas from my current areas of study in math education. It is designed to help share these ideas with people attending the September 10-11, 2004 Teachers of Teachers of Mathematics (TOTOM) meeting in Ashland, Oregon. This is not intended to be a comprehensive, completed, and polished article. Rather, it is intended to contain a brief introduction to my current understanding about a number of related ideas that are important in math education. I welcome feedback and I like to share my ideas!

The basic problems being addressed in this document is that our math education system is not as good as we want it to be and is not as good as it could be. This document provides a framework consisting of a set of related areas for study, research, and implementation. I believe that appropriate understanding and use of these ideas will lead to a significant improvement in our math education system. These ideas can all be integrated into Math Methods and Math Content courses designed for preservice and inservice teachers.

A fundamental aspect of math education is "nature" versus "nurture." It is evident that people vary considerably in their genetic disposition that relates to learning math. Howard Gardner identified Logical/Mathematical as one of the "intelligences" that people have (Gardner, n.d.). Brain imaging has identified specific brain regions that are involved in doing exact and approximate computations, and a brain region associated with dyscalculia (Halber, 1999; Stanescu-Cosson, 2000).

However, we know that nurture is very important. For example, the lack of appropriate vitamins, minerals, and food before birth and during early childhood can lead to a decrease of perhaps 20 IQ points (Nutrition, n.d.). In math education, as well as in other cognitive disciplines, cognitively poor home environment and poor teaching contribute substantially to slow progress in learning math. Many students enter the first grade at least a year behind the average in their mathematical development, and this is mainly attributable to poor cognitive-oriented home environments. Poor teaching of math in many of our schools compounds this problem.

While this document references quite a bit of research, it is not a comprehensive survey or summary of the math education literature. There is a huge math education literature. Interestingly, research in math education has not been as productive as research on reading (Gersten, 2002).

Math is often considered to be a language, and math education includes speaking, listening, reading, and writing math. However, math is clearly a different type of language than English and other "natural" languages. Interestingly, researchers have found that there tends to be comorbidity in difficulties in learning reading and in learning math. For example, students who have trouble learning to read typically have trouble learning math. This issue becomes compounded when we try to have students learn to solve word problems.

We now have electronic calculators and computers of immense and steadily growing capability. While the National Council of Teachers Mathematics has long endorsed use of calculators and computers, our math education system has yet to adequately deal with these powerful aids to learning and "doing" math. Of course, some progress has occurred. However, the exponential growth in computer capability (doubling time of less than two years) seems to be overwhelming the coping abilities of our math education system.

Problem Solving

Some people think that problem solving is the sole provenance of mathematics. However, problem solving lies at the core of each discipline. Each discipline can be defined by its unique combination of:

- The types of problems, tasks, and activities it addresses.
- Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
- Its history, culture, language (including notation and special vocabulary), methods of teaching and learning, and methods of assessment.

- Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, etc.

Think about this definition in terms of the discipline of mathematics and helping students to learn math. Then think about it for some other discipline that you know well. You will likely see some similarities and some major differences between the two disciplines.

One answer to "Why do I need to learn the discipline xxx?" is that by doing so you will be able to solve a variety of problems that you cannot currently solve. You will learn about some of the important accomplishments within the discipline, some of its history, and some of its language. As you learn the language and notation, you will get better in making use of and building on the accumulated knowledge of the discipline. You will learn to precisely represent problems to be solved and tasks to be accomplished so that you can communicate your needs and interests to other people and to Information and Communications Technology (ICT) systems. ICT provides powerful information retrieval systems (an aid to building on the previous work of others) as well as tools that can solve or greatly aid in solving a wide range of problems.

It is evident that disciplines overlap each other and that math is an important aspect of many disciplines. For example, Figure 1 is a Venn diagram showing the disciplines of Mathematics and Education. These two disciplines overlap in a discipline we call Math Education. Among other things this diagram suggests that a person can know a great deal of mathematics and still not know much about education, or vice versa. For example, a mathematician may not know much about teaching reading in mathematics (reading in the content area) or math-oriented pedagogy (content pedagogy). A mathematician may not know much about various theories of transfer of learning, constructivism, or situated learning. But, of course, all of these topics are important in math education. Such observations help to explain the ongoing battle between mathematicians and educators about how to best prepare a person to teach math at the PK-12 levels. To learn more about various learning theories and math education, see Moursund (n.d.).

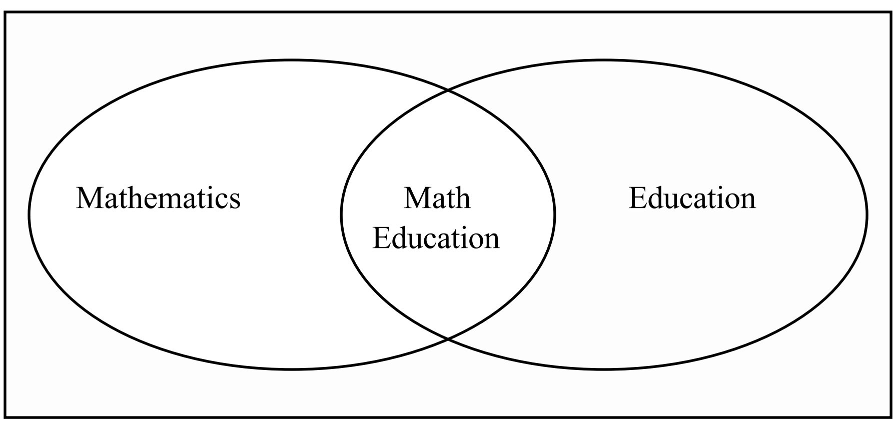

Figure 1: The discipline of Math Education.

If we assume a rather broad definition of the term *problem*, then problem solving lies at the heart of each discipline and is certainly a central focus in math. George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. *The Goals of Mathematical Education* (Polya, 1969) is a talk that he gave to a group of elementary school teachers.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher

aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.

…

However, we have a higher aim. We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures.

Polya's comments focus on problem solving and abstraction. Later in this document I will present some ideas from Piaget and others on developmental theory. Problem solving and abstraction lie at the Formal Operations end of the Piagetian scale for cognitive development. As we teach math, we are attempting to help students move up this cognitive development scale.

One of the most important ideas in problem solving is to build on the previous work of yourself and others. However, as math teachers, we often get confused between immediate (quick fix) goals of helping students score high on math problem solving tests, and longer term goals of helping students get better at more general math problem solving. In the short run, students will score higher on math problem solving tests if they memorize how to solve certain kinds of problems and to accomplish certain kinds of mathematical tasks. However, this memorize, regurgitate, and forget approach does little to develop higher-order mathematical cognitive and problem-solving skills.

Just for the fun of it, the next time you are teaching a class of students, ask them to define the term "problem." You may be disappointed about the nature of the answers your students provide. Since problem solving lies at the heart of every discipline, one might expect that students who are studying a particular discipline would learn a useful definition of what the term means in the discipline they are studying.

Here is a definition that I have found useful in my teaching of preservice and inservice teachers at all grade levels and in a variety of subject areas:

You (personally) have a problem if the following four conditions are satisfied:

1. You have a clearly defined given initial situation.
2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single goal problems.
3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. There may be specified limitations on resources, such as rules, regulations, and guidelines for what you are allowed to do in attempting to solve a particular problem.
4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, and energy, to achieve the desired final goal.

The fourth component of this definition is particularly important. Unless a student has ownership—an appropriate combination of intrinsic and extrinsic motivation—the student does not have a problem. Motivation, especially intrinsic motivation, is a huge topic in its own right, and I will not attempt to explore it in detail in this paper. Perhaps it suffices to say that many teachers are not very successful in helping their students to develop intrinsic motivation in their math studies. This is in spite of the fact that the human brain is naturally inquisitive and an intrinsic, natural learner

The research literature on problem solving is quite large, and math education includes a number of strategies for attaching math problems. This is a large topic and an important component of any math or math education course. While this topic is beyond the scope of this document, all readers should be interested in Polya's (1957) general strategy for attempting to solve any math problem. I have reworded his strategy so that it is applicable to a wide range of problems in a wide range of disciplines—not just in math. This six-step strategy can be called the Polya Strategy or the Six Step strategy. Note that there is no guarantee that use of the Six Step strategy will lead to success in solving a particular problem. You may lack the knowledge, skills, time, and other resources needed to solve a particular problem, or the problem might not be solvable.

1. Understand the problem. Among other things, this includes working toward having a well-defined (clearly defined) problem. You need an initial understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary. You need to make a personal commitment (Ownership) to solving the problem.

2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task?

3. Think carefully about possible consequences of carrying out your plan of action. Focus major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 as a consequence of this thinking.

4. Carry out your plan of action. Do so in a thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that this reflective thinking leads to increased expertise.

5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:

    A. If the problem has been solved, go to step 6.

    B. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.

    C. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.

6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think about how your increased knowledge and skills can be used in other problem-solving situations. (Work to increase your reflective intelligence!)

Many of the steps in this six-step strategy require careful thinking. However, there are a steadily growing number of situations in which much of the work of step 4 can be carried out by a computer. The person who is skilled at using a computer for this purpose may gain a significant advantage in problem solving, as compared to a person who lacks computer knowledge and skill.

I find the diagram given in Figure 2 to be particularly useful when I talk about computers and math problem solving at the precollege level. With some effort, this diagram can be modified to fit problem solving in other disciplines.

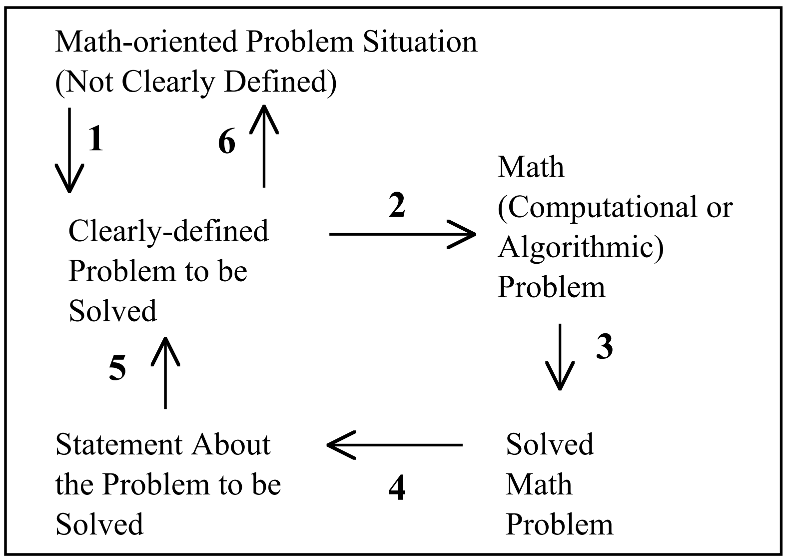

Figure 2. Math problem solving.

The six steps illustrated are 1) Problem posing and problem recognition; 2) Mathematical modeling; 3) Using a computational or algorithmic procedure to solve a computational or algorithmic math problem; 4) Mathematical "unmodeling"; 5) Thinking about the results to see if the Clearly-defined Problem has been solved; and 6) Thinking about whether the original Problem Situation has been resolved. Steps 5 and 6 also involve thinking about related problems and problem situations that one might want to address or that are created by the process or attempting to solve the original Clearly-defined Problem or resolve the original Problem Situation.

In some sense, all of the steps except (3) involve higher-order knowledge and skills. They require a significant level of math maturity. Step (3) lends itself to a rote memory approach. It is highly desirable that students develop speed and accuracy in certain types of mathematical operations. However, the human mind is not good at memorizing math procedures and then carrying them out rapidly and accurately with the assistance of pencil and paper. On the other hand, calculators and computers are really good at carrying out math procedures.

PK-12 teachers who teach math tend to estimate that about 75% of the math education curriculum focuses on (3). [Note: This is an estimate I have made based upon working with a very large number of teachers. I don't know of any published research that backs up my assertion.] This leaves about 25% of the learning time and effort focusing on the remaining five steps. Appropriate use of calculators and computers as tools, and Computer-Assisted Learning, could easily decrease the time spent on (3) to 50% or less of the total math education time. This would allow a doubling of the time devoted to instruction and practice on the higher-order knowledge and skill areas.

Math Expertise: Content and Maturity

An expertise scale like the one pictured in Figure 3 is applicable to each academic discipline. Roughly speaking, such a scale covers all the way from a beginner (novice) to a person who is considered to be a world-class expert. For most preservice and inservice teachers, their current level of expertise in any particular discipline they teach well above the novice level, and well below the world class level.

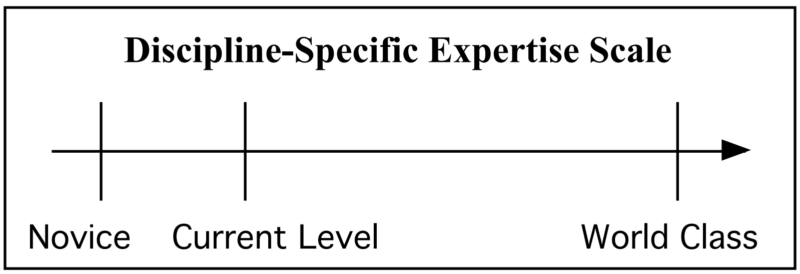

Figure 3: Expertise scale.

The terms fluency and proficiency are often used in talking about expertise. In the discipline of reading, fluency refers to speed, accuracy, and comprehension. Math education tends to use the term proficiency rather than fluency, placing heavy emphasis on understanding, problem solving, and proof. For example, quoting from an important Rand study:

> Developing proficiency in mathematics is important for all students. However, when considered in light of current standards, or compared with performance in other countries, evidence on student achievement in mathematics makes clear the need for substantial improvement. U.S. students do not, as a group, achieve high levels of mathematical proficiency. The nation must seek to narrow the achievement gaps between white students and students of color, between middle-class students and students living in poverty; gaps that have persisted over the past decade. To address these problems, the federal government and the nation's school systems have made and are continuing to make significant investments in the improvement of mathematics education. However, the knowledge base on which these efforts are founded has often been weak and speculative (Ball, 2002).

I find it interesting to think about a learner's mathematical needs versus the learner's current level of mathematical expertise. For example, when I was an undergraduate at the University of Oregon, I majored in math and took a lot of physics. Interestingly, the math demands in my upper division physics courses often seemed to exceed my current levels of math expertise. The opposite situation occurred for my precollege education and for all of the other coursework I took in college. Perhaps our current math education system is trying to push precollege students well above the math needs they are experiencing in their other courses and in their everyday lives?

To adequately discuss expertise within a discipline, one needs to have an understanding of the discipline and what separates it from other disciplines. For example:

> Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, 1992).

> Mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, logic, application, and principles. Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, mathematical education should be centered on encouraging students to think for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof—the demand for succinct argument that from a logical foundation for the veracity of a claim (Padraig & McLoughlin, 2002).

In Mathematics—as in each other discipline—expertise can be broken into a number of components. In this document, I break math expertise into Math Content Knowledge and Math Maturity. See Figure 4. A person may be at quite different points on the two scales.

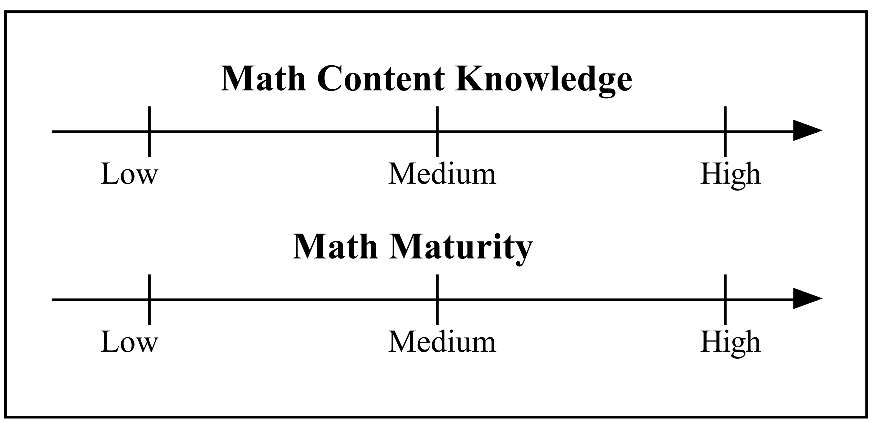

Figure 4. Scales for math content knowledge and math maturity.

The totality of Math Content Knowledge and huge and is continuing rapid growth. Our formal math education system, starting at Pre Kindergarten (PK) and extending through a doctorate, is relatively well defined. There is considerable agreement about the scope and sequence of PK-12 math education in the US, for example, and the undergraduate college math curriculum is relatively standardized throughout this country. Clearly, one measure of a person's progress toward increasing Math Expertise is the level of coursework that has been completed, the grades received in these courses, and the quality and rigor of the coursework. However, math can be learned through other ways than just taking courses. Moreover, there is a large amount of math that is not included in the commonly available coursework. And finally, many students forget most of the math they have "covered" in their math courses.

There is much less agreement on the meaning of Math Maturity, even though the term is widely used by mathematicians. The term is also often used in other disciplines when people are attempting to specify the math prerequisite for a course. (For example, the prerequisite for a particular freshman level computer science course might be stated as "Math Maturity.")

Math Maturity is not primarily knowledge of specific math content areas or skill in memorizing and accurately using arithmetic and other math procedures. Rather, the focus is on areas such as understanding, problem solving, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning (being able to use one's mathematical knowledge over a wide range of disciplines and in novel settings), and interest (including intrinsic motivation) in math.

Math Maturity is a scale, moving from very low to very high, and is only loosely coupled with Math Content Knowledge. Suppose that we had clear, commonly accepted measures in both Math Content Knowledge and in Math Maturity. We could then calculate a Math Content Quotient (divide Math Content Knowledge by Age) and a Math Maturity Quotient (divide Math Maturity by Age). Of course, it would prove useful to develop such measure to have a mean of 100 and a standard deviation of 15 or 16, as is done with widely used IQ measures. I am not aware of any such MCQ and MMQ instruments—perhaps these are relatively wide open areas for math education research.

Here is a list of some possible components of Math Maturity. Note that one can argue that each is "merely" a component of Math Content Knowledge. However, when people use the term Math Maturity they tend to be interested in those aspects of the topics listed below that are not dependent on specific Math Content Knowledge.

1. An understanding of the math that one has had adequate opportunities to "learn." A good way to think about this is in terms of lower-order versus higher-order knowledge and skills. Bloom's Taxonomy, developed about 50 years ago, is still a useful aid in understanding lower-order and higher-order (Bloom's Taxonomy, n.d.).

2. Considering mathematics as a language suggests three related components of Mathematical Maturity:

    A. Mathematical speaking and listening fluency.

    B. Mathematical reading and writing fluency.

    C. Thinking and reasoning in the language of mathematics. Gary Marcus (2004, p. 124) discusses roles of language and thought. Thought and language are loosely connected. Mathematicians and other people clearly develop and make use of mental representations that are not words. For example, Albert Einstein, when describing his discovery of special relativity said: "Words and sentences, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thoughts are certain signs or images, more or less clear, that I can reproduce and recombine at will." (Marcus, 2004, p219.)

3. Ability to pose and represent math problems, and to ask insightful mathematical questions. This includes the ability to recognize math aspects of a problem situation in a wide range of disciplines and represent them mathematically.
4. Ability to effectively use one's Math Content Knowledge to solve or help solve the types of math problems that arise in (3) above. Making connections within mathematics, and transfer of one's math learning to other disciplines.
5. Ability to learn mathematics, and to build upon one's current mathematical knowledge. In the field of reading, people talk about learning to read and then reading to learn. In our current education system approximately 70% of students learn to read well enough by the end of the third grade so that they can use their reading knowledge as a significant aid to learning in other disciplines. We can think about "learning to math and them mathing to learn."
6. Other factors affecting Math Maturity include attitude, interest, motivation, focused attention, perseverance, and acceptance of and fitting into the "culture" of mathematics.

When I am doing presentations to a general audience of teachers, I often ask for a show of hands on the question: Can you state and prove at least one theorem from high school geometry? Essentially every teacher has had such a course. Typically, the only people who answer "yes" are secondary school math teachers. One way to think about math maturity is that it's what is left after one forgets the details of the math content that one has studied. This is important to keep in mind as you teach math. What do you want your students to remember many years in the future? To what extent do you explicitly teach for and set the conditions for this type of long term retention? Note that this question is applicable to a first grade teacher, a secondary school math teacher, and a college math teacher. Discipline-specific variations of this question apply to teachers in other disciplines.

Mind Science; Intelligence

Historically, the study of the human brain (one of a person's organs) and the study of the human mind (think of the mind as a product of the brain) have been distinct disciplines. Computer-oriented people tend to think of the brain as hardware (wetware) and the mind as software. In this document, there is one section on the mind and a separate section on the brain. In essence, the study of the mind is part of the field of psychology, while the study of the brain is part of the discipline of neuroscience. In recent years, the mind and brain disciplines have begun to merge, so these two sections contain some overlap.

Intelligence is the ability to learn and to take actions that make use of one's learning. Clearly, intelligence is not limited just to humans. However, the ability to learn a natural language such as English demonstrates a very high level of intelligence on the intelligence scale of all life on earth.

For many years, psychologists studying the human brain/mind have tried to measure its capabilities. Quite a bit of this work has focused on defining “intelligence” and measuring a person’s intelligence.

> The concept that intelligence could be or should be tested began with a nineteenth-century British scientist, Sir Francis Galton. Galton was known as a dabbler in many different fields, including biology and early forms of psychology. After the shake-up from the 1859 publishing of Charles Darwin's "The Origin of Species,” Galton spent the majority of his time trying to discover the relationship between heredity and human ability (History of I.Q., n.d.).

Howard Gardner (1993), David Perkins (1995), and Robert Sternberg (1988) are researchers who have written widely sold books about intelligence. Of these three, Howard Gardner is probably the best known by PK-12 educators, because his theory of Multiple Intelligences has proven quite popular with such educators (Mckenzie). However, there are many researchers of have contributed to the extensive and continually growing collection of research papers on the intelligence (Yekovich 1994). The following definition of (human) intelligence is a composite from various authors, especially Gardner, Perkins, and Sternberg. Intelligence is a combination of the abilities to:

1. Learn. This includes all kinds of informal and formal learning via any combination of experience, education, and training.
2. Pose problems. This includes recognizing problem situations and transforming them into more clearly defined problems.
3. Solve problems. This includes solving problems, accomplishing tasks, and fashioning products.

Ways to measure intelligence were first developed more than 120 years ago, and this continues to be an active field of research and development. A very simplified summary of the current situation consists of:

1. There are a variety of IQ tests that produce one number or a small collection of numbers as measures of a person’s intelligence.
2. The “one number” approach (the general intelligence, or “g” factor) is usually attributed to Charles Spearman who proposed the idea in 1904, and it still has considerable prominence.
3. Expert estimates suggest that anywhere between 30 and 80 percent of the variation in IQ scores is determined by genetic factors, with 50 to 60 percent being the most commonly accepted range (Niabett, 1998). The 50 to 60 percent figure corresponds to a correlation of about .71 to .77.
4. There have been a number of studies of possible genetic differences that might affect IQ between “White” Americans and “African” Americans. In an analysis of this research literature, Niabett (1998) reports, “The studies most directly relevant to the question of whether the Black/White IQ gap is genetic in origin provide no evidence for a correlation between IQ and African (rather than European) ancestry.” That is, the differences are due to “nurture,” not genetics.
5. Many people have proposed and discussed the idea of multiple intelligences. In the past two decades, the work of Howard Gardner has helped to publicize

this idea. Logical/mathematical and spatial are two of the eight Multiple Intelligences identified by Gardner (n.d.), and that relate to learning and using mathematics. As noted earlier in this document, linguistic intelligence is also related to mathematics.

6. Intelligence comes from a combination of nature and nurture. Over the past few decades, IQ has been increasing at a significant pace (Sternberg, 1997). Reasons for this are not clear (Marcus, 2004, p176).

While Howard Gardner and Robert Sternberg have garnered a lot of publicity during the past couple of decades for their work on intelligence, many really important ideas have been developed by other people. One of these is the idea that "g" can be divided into two major components: fluid intelligence (gF) and crystallized intelligence (gC).

> Cognitive psychologists have re-framed the "fluid" and "crystallized" aspects of cognition into a model of a human cognitive system made-up of a long term memory which constitutes a knowledge base ("crystallized intelligence") for the person, a working memory which engages various processes ("fluid intelligence") that are going on at a given time using information picked-up from both the long term memory's knowledge base, and a sensory system that picks-up information from the external world that the person is in. Today, over thirty years of research has validated the usefulness of this simple three-part model for thinking about human cognition (Healy & McNamara, 1996).

In casual conversations about intelligence and IQ, people tend to forget about the meaning of the "Q" in IQ. The human brain grows considerably during a person's childhood, with full maturity being reached in the early 20s. Both gF and gC increase during this time. Recent research suggests that gF then begins a slow decline. However, with appropriate education and cognitive experiences, gC continues to grow well into a person's 60s (McArdle, et al. (2002). Quoting from the McArdle article:

> The theory of fluid and crystallized intelligence … proposes that primary abilities are structured into two principal dimensions, namely, fluid (***Gf***) and crystallized (***Gc***) intelligence. The first common factor, ***Gf***, represents a measurable outcome of the influence of biological factors on intellectual development (i.e., heredity, injury to the central nervous system), whereas the second common factor, ***Gc***, is considered the main manifestation of influence from education, experience, and acculturation. ***Gf-Gc*** theory disputes the notion of a unitary structure, or general intelligence, as well as, especially in the origins of the theory, the idea of a structure comprising many restricted, slightly different abilities

Robert Sternberg is well known for his triarchic model of intelligence. Very roughly speaking, he divides intelligence into the three parts: creativity, street smarts, and school smarts. Here is a somewhat different way of explaining his theory. Think of creativity as being gF, while street smarts and school smarts are two broad categories in which one develops gC. If a person is raised in a preliterate hunter-gather community living in a jungle, the person will develop a high level of "hunter-gather living in a jungle" street smarts. Since the person will not be exposed to reading, writing, and books, the person will not develop an appreciable level of school smarts.

Gene Maier (n.d.) was one of the founders of the Math Learning Center and served as its President for many years. One of his areas of interest is "folk math" versus school math. He notes that many people (including cabinet makers, carpenters, mill wrights, and people with little or no formal education) make routine use of math to help solve the types of problems they encounter on the job and in their day-to-day lives. By and large they make use of folk math (their math-oriented street smarts) rather than school math.

The street smarts versus school smarts analysis helps to explain why children raised in poverty (low socioeconomic environments) tend to be a year behind average in school smarts by the time they begin school. Their early childhood learning focuses on gaining street smarts knowledge and skills that help them survive and prosper in a poverty environment. Sanchez (2004) provides an interesting analysis of such data from the state of Colorado.

It is interesting to carry this line of thought a little further. Some children grow up in an environment that is school smarts mathematically "rich." I am an example of such a person, since both my father and mother were on the faculty in the Department of Mathematics at the University of Oregon. I grew up in a culture that placed high value on knowing and using math. This environment helped to "grow" my math oriented gF and gC.

My conclusion is that one of the reasons for the relatively poor success of our formal, school smarts math education system is that the math environment most of children grow up in before they start school and the math environment they encounter both at home and in school during the early years of their formal education is not particularly "rich" in its support of school mathematical development.

I also conclude that many people grow up rather weak in their folk math development, because they are not raised and taught in environments that are explicitly designed to foster cognitive growth of street smarts mathematics (folk math).

Developmental Theory

You are probably familiar with the four-stage Piagetian Developmental Scale shown in Figure 5 (Huitt and Hummel, 1998).

Approximate Age	Stage	Major Developments
Level 1. Birth to 2 years	Sensorimotor	Infants use sensory and motor capabilities to explore and gain understanding of their environments.
Level 2. 2 to 7 years	Preoperational	Children begin to use symbols. They respond to objects and events according to how they appear to be.
Level 3. 7 to 11 years	Concrete operations	Children begin to think logically. This stage is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume. Increasing intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking—mental actions that are reversible—develops.
Level 4. 11 years and beyond	Formal operations	Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, thinking logically about abstract propositions, testing hypotheses, and gaining and using higher-order knowledge and skills.

Figure 5. Piaget's Stages of Cognitive Development

Piaget's stages of cognitive development are not specific to any particular discipline. However, a math-oriented reader of Figure 5 might decide that Concrete Operations and Formal Operations seem to be somewhat math oriented. Later in this section I explore a still more math-oriented cognitive development scale.

Cognitive development is dependent on both nature and nurture. Roughly speaking—assuming decent food and home environment—a child's progress though the first two

Piagetian Developmental stages is mainly dependent on nature, while progress in the other two stages is very dependent on nurture. However, nature versus nurture is not that simple. Marcus (2004) argues that the two are so thoroughly intertwined that is hopeless to attempt to separate them. Moreover, his arguments provide strong support for the value of high quality informal and formal education.

Although the Piagetian scale has only four labeled parts, it is a continuous scale. It is a common mistake to think of a person either being at Formal Operations or not being at Formal Operations. Rather, a person moves toward Formal Operations, and this movement may be at different rates in different disciplines.

There are a variety of instruments used to measure cognitive development, and with such an instrument one can define a specific score as being the minimum score to be labeled "Formal Operations." When that is done, researchers find that only about 35% of children in industrialized societies have achieved Formal Operations by the time they finish high school (MacDonald, n.d.).

> However, data from similar cross-sectional studies of adolescents do not support the assertion that all individuals will automatically move to the next cognitive stage as they biologically mature. Data from adult populations provides essentially the same result: Between 30 to 35% of adults attain the cognitive development stage of formal operations (Kuhn, Langer, Kohlberg & Haan, 1977). For formal operations, it appears that maturation establishes the basis, but a special environment is required for most adolescents and adults to attain this stage. Huitt, W., & Hummel, J. (2003). Piaget's theory of cognitive development. *Educational Psychology Interactive*. Valdosta, GA: Valdosta State University. Retrieved 9/16/04 from http://chiron.valdosta.edu/whuitt/col/cogsys/piaget.html.

These findings suggest that we need to take a careful look at the cognitive expectations in courses in all disciplines and at all grade levels. For example, the study of causality and the generating and testing of hypotheses are key ideas in the discipline of History and in the sciences. A ninth grade history or science course is apt to have a significant emphasis on these ideas. But, these ideas are part of Formal Operations. Unless they are presented and explored in a careful and appropriate Concrete Operations manner, they will be well over the heads of most of the ninth graders. Needless to say, this difficulty grows as one attempts to teach such ideas to still less cognitively developmentally mature students.

The same sort of analysis is applicable to our math curriculum. About 50 years ago, the Dutch educators Dina and Pierre van Hiele focused some of their research efforts on defining a Piagetian-type developmental scale for Geometry (van Hiele, n.d.). Their five-level scale is shown in Figure 6. Notice that the van Hieles, being mathematicians, labeled their first stage Level 0.

Stage	**Description**
Level 0 (Visualization)	Students recognize figures as total entities (triangles, squares), but do not recognize properties of these figures (right angles in a square).
Level 1 (Analysis)	Students analyze component parts of the figures (opposite angles of parallelograms are congruent), but interrelationships between figures and properties cannot be explained.
Level 2 (Informal Deduction)	Students can establish interrelationships of properties within figures (in a quadrilateral, opposite sides being parallel necessitates opposite angles being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle). Informal proofs can be followed but students do not

	see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.
Level 3 (Deduction)	At this level the significance of deduction as a way of establishing geometric theory within an axiom system is understood. The interrelationship and role of undefined terms, axioms, definitions, theorems and formal proof is seen. The possibility of developing a proof in more than one way is seen.
Level (Rigor)	Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples.

Figure 6. Van Hiele five-level developmental scale for geometry.

The van Hieles' scale is mainly a school math (as distinguished from folk math) scale. The van Hieles' work suggested that the typical high school geometry course was being taught at a developmental level considerably above that of the typical students taking such courses. Their scale also suggests that a Math Cognitive Developmental Scale needs to have one or more scale points above the Formal Operations point on a Piagetian scale.

Figure 7 is represents my current thinking on a six-level Piagetian-type scale for school mathematics (as distinguished from folk math). It s an amalgamation and extension of ideas of Piaget and the van Hieles.

Stage Name	Math Developments
Level 1. Piagetian and Math sensorimotor.	Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments. Research on very young infants suggests some innate ability to deal with small quantities such as 1, 2, and 3. As infants gain crawling or walking mobility, they can display innate spatial sense. For example, they can move to a target along a path requiring moving around obstacles, and can find their way back to a parent after having taken a turn into a room where they can no longer see the parent.
Level 2. Piagetian and Math preoperational.	During the preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. The children are making rapid progress in receptive and generative oral language. They accommodate to the language environments they spend a lot of time in, so can easily become bilingual or trilingual in such environments. During the preoperational stage, children develop an understanding of number line. They learn number words and the idea that to name the number of objects in a collection, one can "count" them, with the answer being the last number used in this counting process. A majority of children discover or learn "counting on" and counting on from the larger quantity as a way to speed up counting of two or more sets of objects. Children gain increasing fluency (speed, correctness, and understanding) in the types of counting activities mentioned above. In terms of nature and nurture in mathematical development, both are of considerable importance during the preoperational stage.
Level 3. Piagetian and Math concretel operations.	During the concrete operations stage, children begin to think logically. In this stage—characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume—intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops (mental actions that are reversible). While concrete objects are an important aspect of learning during this stage, children also begin to learn from words, language, and pictures/video, learning about objects that are not concretely available to them. For the average child, the time span of concrete operations is approximately the time span of elementary school (grades 1-5). During this time, learning math is somewhat linked to having

	previously developed some knowledge of math words (such as counting numbers) and concepts. However, the level of abstraction in the written and oral math language quickly surpasses a student's previous math experience. That is, math learning tends to proceed in an environment in which the new content materials and ideas are not strongly rooted in verbal, concrete, mental images and understanding of somewhat similar ideas that have already been acquired. There is a substantial difference between developing general ideas and understanding of conservation of number, length, liquid, mass, weight, area, volume, and learning the mathematics that corresponds to this. These tend to be relatively deep and abstract topics, although they can be taught in very concrete manners.
Level 4. Piagetian and Math formal operations. Van Hiele level 2: informal deduction.	Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, and gaining and using higher-order knowledge and skills. Understanding of and proficiency in math at the level of a strong high school level of math curriculum. Beginnings of understanding of math-type arguments and proof . Piagetian and Math formal operations includes being able to recognize math aspects of problem situations in both math and non-math disciplines, convert these aspects into math problems (math modeling), and solve the resulting math problems if they are within the range of the math that one has studied. Such transfer of learning is a core aspect of Level 4.
Level 5. Abstract mathematical operations. Van Hiele level 3: deduction.	Mathematical fluency at the of contemporary math texts used at the senior undergraduate level in strong programs, or first year graduate level in less strong programs. Good ability to learn math through some combination of reading required texts and other math literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level math problems posed by others (such as in the text books and course assignments). Pose and solve problems at the level of one's math reading skills and knowledge. Follow the logic and arguments in mathematical proofs. Fill in details of proofs when steps are left out in textbooks and other representations of such proofs.
Level 6. Mathematician. Van Hiele level 4: rigor.	A very high level of mathematical fluency and maturity. This includes speed, accuracy, and understanding in reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level mathematics. Pose and solve original math problems at the level of contemporary research frontiers.

Figure 7. Six-stage mathematical cognitive developmental scale.

Brain Science

Research using brain imaging is beginning to make significant contributions to our understanding of learning and using math. For example, by five years ago brain imaging showed different parts of the brain being used in exact calculations than being used in estimations or approximate calculations (Dehaene et al. 1999). Brain imaging has identified regions of the brain associated with different types of dyscalculia (Stanescu-Cosson et al., 2000).

Mind research on gF suggests that this component of g increases into early adulthood. A recently published longitudinal brain imaging study reports results that seem to be consistent with this gF result (Gogtay et al., 2004).

We now have theory and instrumentation that helps us gain increased understanding of the human brain. We have steadily increasing knowledge of the human genome, noninvasive tools for brain imaging, and tools and skills for manipulation of individual genes. This progress has raised the nature versus nurture discussion to an entirely new level. We are gaining increased understanding of nature, and we now have the ability to change nature. Here are to quotes from Marcus (2004):

> …in our world, nature's contribution to development comes not by providing a finely detailed sketch of a finished product, but by providing a complex system of self-regulating recipes. These recipes provide for many different things—from the construction of enzymes and structural proteins to the construction of motors, transporters, receptors, and regulatory proteins—and thus there is no single, easily characterizable genetic contribution to the mind. In the ongoing everyday functioning of the brain, genes supervise the construction of neurotransmitters, the metabolism of glucose, and the maintenance of synapses. In early development, they help to lay down a rough draft, guiding the specialization and migration of cells as well as the initial pattern of wiring. In synaptic strengthening, genes are a vital participant in a mechanism by which experience can alter the wiring of the brain (thereby influencing the way that an organism interprets and responds to the environment). (Marcus, 2004, p168.)

> In the coming decades, we will all collectively as a society need to decide what we think about biotechnology and what applications we are and are not willing to allow. The debates we have now, about cloning and stem cell research, pale in comparison to debates we are likely to encounter as the technology for manipulating genes advances. We are already at the point where it is possible to screen embryos for the predisposition to certain life-threatening illnesses; as we unravel more and more of the genome, we will be able to detect more and more disorders (or predispositions to disorders) well in advance of birth. Ultimately, if we so choose, we may be able to directly manipulate embryonic genomes—add a gene here, delete a gene there. The genes of a child might eventually be more a matter of choice than of chance (Marcus, 2004, p174).

Many years ago I memorized the statement: An electronic digital computer is a machine for the input, storage, manipulation, and output of information. Compare this definition with the following observation: A brain is an organ for the input, storage, manipulation, and output of information. In thinking about brain versus computer, I find it helpful to make use of the diagram given in Figure 8.

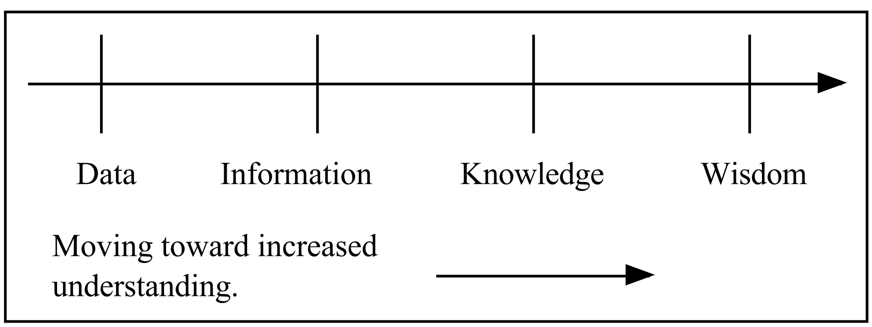

Figure 8. Data, information, knowledge, and wisdom.

You might enjoy listening to your students discuss and argue about what it might mean for a computer to have knowledge and how this knowledge might be the same as or different than the type of knowledge that a person can have. For more information on this and related topics, see Moursund (2004c). Many businesses now make use of ICT systems that they feel have reached the knowledge level on this scale.

The next three subsections of this document are *Big Ideas* that are quite important in education.

Brain Versus Computer

In the early days of computers, people often referred to such machines as *electronic brains*. Even now, more than 50 years later, many people still use this term. Certainly a human brain and a computer have some characteristics in common. However:

- Computers are very good at carrying out tasks in a mechanical, "non-thinking" manner. They are millions of times as fast as humans in tasks such as doing arithmetic calculations or searching through millions of pages of text to find occurrences of a certain set of words. Moreover, they can do such tasks without making any errors.
- Human brains are very good at doing the thinking and orchestrating the processes required in many different very complex tasks such as carrying on a conversation with a person, reading for understanding, posing problems, and solving complex problems. Humans have minds and consciousness. A human's brain/mind capability for "meaningful understanding" is far beyond the capabilities of the most advanced computers we currently have.

Big Idea # 1: There are many things that computers can do much better than human brains, and there are many things that human brains can do much better than computers. Our educational system can be significantly improved by building on the relative strengths of brains and computers, and decreasing the emphasis on attempting to "train" students to compete with computers. We need to increase the focus on students learning to solve problems using the strengths of their brains and the strengths of ICT.

Chunks and Chunking

Here are three different types of human memory:

- Sensory memory stores data from one's senses, and for only a short time. For example, visual sensory memory stores an image for less than a second, and auditory sensory memory stores aural information for less than four seconds.
- Working memory (short term memory) can store and actively process a small number of chunks. It retains these chunks for less than 20 seconds.
- Long-term memory has large capacity and stores information for a long time.

Research on working memory indicates that for most people the size of this memory is about 7 ± 2 chunks (Miller, 1956). This means, for example, that a typical person can read or hear a seven-digit telephone number and remember it long enough to key into a telephone keypad. The word *chunk* is very important. When I was a child, my home phone number was the first two letters of the word diamond, followed by five digits. Thus, to remember the number (which I still do, to this day) I needed to remember only six chunks. But, I had to be able to decipher the first chunk, the word "diamond."

Long-term memory has a very large capacity, but this does not work like computer memory. Input to computer memory can be very rapid (for example, the equivalent of an entire book in a second), and a computer can store such data letter perfect for a long period of time. The human brain can memorize large amount of music, poetry, or other text. But, this is a long and slow process for most people. By dint of hard and sustained effort, an ordinary person can memorize nearly letter perfect the equivalent of a few books. However, the typical person is not very good at this. At the current time, the Web contains the equivalent of tens of millions of books.

On the other hand, the human brain is very good at learning meaningful chunks of information. Think about some of your personal chunks such as constructivism, multiplication, democracy, transfer of learning, and Mozart. Undoubtedly these chunks have different meanings to me than they do for you. As an example, for me, the chunk "multiplication" covers multiplication of positive and negative integers, fractions, decimal fractions, irrational numbers, complex numbers, functions (such as trigonometric and polynomial), matrices, and so on. My breadth and depth of meaning and understanding was developed through years of undergraduate and graduate work in mathematics.

It is useful to think of a chunk as a label or representation (perhaps a word, phrase, visual image, sound, smell, taste, or touch) and a collection of pointers. A chunk has two important characteristics:

1. It can be used by short-term memory in a conscious, thinking, problem-solving process.
2. It can be used to retrieve more detailed information from long-term memory.

Big Idea # 2: Our education system can be substantially improved by taking advantage of our steadily increasing understanding of how the mind/brain learns and then uses its learning in problem solving. Chunking information to be learned and used is a powerful aid to learning and problem solving. However, even if two people receive the same education about a topic, and use the same label for a chunk that they form on that topic, their chunks will be quite different. (This is a key idea in constructivism.)

Rate of Learning

Howard Gardner has identified eight domains or types of intelligence, including math/logic and spatial. One aspect of having varying levels of intelligence in these various domains is that a person is likely to have some differences in rates of learning and in learning potentials in these various domains.

Differences in rates of learning are very evident for students with special needs. Our school system does not deal very well with varying rates of learning, even though it puts a lot of money into special education. A solid example of this is provided by students who are classified as learning disabled. On average, such students learn math at approximately half the rate of average students. Thus, in school math they fall behind about a half-year for each year of school. Moreover, research suggests that on average such students top out at someplace in the grades 4 to 5 range. That is, with our current methods of teaching math and current amounts of instructional time placed on this discipline, learning disabled students who stay in school up through the 12th grade are unlikely to move past grades 4 or 5 in their Math Content Knowledge and their Math Maturity.

On the other end of the cognitive ability scale, there are quite a few students in school who can learn math much faster than the average student. A typically elementary school class will likely have several students who learn math at least 50% faster than average. That is, these students are capable of making one and a half (or more) years of school math progress per school year.

Augmentation to Brain/Mind

Reading and writing provide an augmentation to short term and long-term memory for personal use and that can be shared with others. Data and information can be stored and retrieved with great fidelity. As Confucius noted about 2,500 years ago, "The strongest memory is not as strong as the weakest ink."

Writing onto paper provides a passive storage of data and information. The "using" of such data and information is done by a human's brain/mind.

Computers add a new dimension to the storage and retrieval of data and information. Computers can process (carry out operations on) data and information. Thus, one can think of a computer as a more powerful augmentation to brain/mind than is provided by static storage on paper or other hardcopy medium.

Big Idea # 3: ICT provides a type of augmentation to one's brain/mind. The power, capability, and value of this type of augmentation continues to grow rapidly. Certainly this is one of the most important ideas in education at the current time. At the current time our formal educational system has yet to understand the idea of ICT as an augmentation to the mind/brain.

Here is some of my current thinking about the ideas listed above. From my point of view, this is a wide open area, ripe for research progress.

> In thinking about chunks and learning, I see two approaches. In the first approach, a framework is provided. Think of the framework as scaffolding for a chunk, and a label for the chunk. One learns the framework and then fits new knowledge and experiences into the framework. In the second approach, one creates their own framework. This is less efficient initially, but more productive over the long run (perhaps). This relates to discovery-based earning.
>
> To illustrate, suppose I want to know a modest amount about something that others have carefully studied. Since part of a discipline is how to teach and learn it, I decide to take advantage of this accumulated knowledge. I have the discipline taught to me by an expert teacher.
>
> But now, suppose that I want to extend my knowledge to "my" world and to situations not covered in the standard curriculum. Now, I hope that I have learned to learn on my own. I hope that I have the creativity and skill to discover, invent, find, and so on, and fit the new into the old framework. I hope that I can restructure the old framework so that it better fits the new and my needs.
>
> There is one more important piece to this. Suppose that the area that I want to study is one in which ICT provides powerful aids to solving its problems. Then I want my chunk to include a link to the capabilities and limitations of computers as an aid to solving the problems. I want to have the knowledge and skills to make use of this ICT augmentation to my brain.

Information and Communication Technology

ICT has been mentioned a number of time in earlier parts of this document. This section provides additional ideas on ICT and math education. ICT and math share much in common. For example, problem solving both within the discipline and across disciplines lies at the core of both ICT and math. Listed below are some math-related ICT topics. More detail on a number of these topics is available in my Website http://darkwing.uoregon.edu/~moursund/Math/.

1. Mathematics is now commonly divided into three major components: pure math, applied math, and computational math. "Computational" is also a new aspect of many other disciplines. For example, one of the winners of the 1998 Nobel Prize in Chemistry received the award for his past 15 years of work in Computational Chemistry. Computer-based modeling and

simulation, based on computational mathematics, is now a common component of each discipline that makes use of mathematics. This suggests the possibility that we should reconsider the content of our math curriculum. Is it appropriately attuned to the growing importance of computational mathematics.

2. Computer algebra systems (CAS) provide very powerful tools to carry out a wide range of mathematical procedures. The idea that a handheld calculator can add, subtract, multiply, divide, and calculate square roots is easy to understand. This raises the issue of the extent to which we want our math education system to focus on students gaining speed and accuracy in carrying out paper and pencil-assisted arithmetic computational algorithms. Today's powerful calculators and computers raise this question for the full range of mathematics taught up though the first couple of years of college. It opens up the possibility of spending much less time teaching procedures and much more time teaching higher-order thinking and problem solving, and in increasing math maturity.

3. Computer-assisted instruction is gradually improving. We now have Highly Interactive Intelligent Computer-Assisted Learning (HIICAL) systems that are quite good. The meaning of "quite good" can be debated. Research in this area tends to compare test scores of students taught by conventional instructional methods versus test scores of students taught by HIICAL. There is now a significant amount of such software that, on average, leads to better test scores than does conventional instruction. See http://www.uoregon.edu/~moursund/dave/second_order.htm.

   Here, I also want to discuss the idea of training versus education. While there is no fine dividing line between the two, it is helpful to think about how much of the math curriculum in PK-12 is training, and how much is education. HIICAL is especially effective in training. Thus, one might expect that over time we will see much of the training aspects of the PK-12 math curriculum be taken over by computers.

   This might also be a good place to raise the issue of the math preparation of people who teach math. If math education at the PK-2 level is mainly training, then it can probably be done fairly well by teachers who are not at math formal operations and who have a low level of math maturity. But, as the curriculum comes to have an increasing focus on education, such teachers are ill prepared to move students upward in their quest for increasing expertise.

4. HIICAL software can be developed that integrates the power of computer-assisted instruction with the power of CAS systems. That is, we are gradually seeing a merger of powerful computer tools and powerful aids to learning and using the tools. Such software has the potential to lead to major changes in math education. The goal might become to educate students so that they function well mathematically in a world in which such systems are readily available.

5. Artificial intelligence (AI) is a branch of the field of computer and information science. It focuses on developing hardware and software systems

that solve problems and accomplish tasks that—if accomplished by humans—would be considered to be a display of intelligence. If the focus is specifically on developing AI-using machines such as robots, automatic pilots, and "smart" military weapons, then the term machine intelligence is often used.

> What is artificial intelligence? It is often difficult to construct a definition of a discipline that is satisfying to all of its practitioners. AI research encompasses a spectrum of related topics. Broadly, AI is the computer-based exploration of methods for solving challenging tasks that have traditionally depended on people for solution. Such tasks include complex logical inference, diagnosis, visual recognition, comprehension of natural language, game playing, explanation, and planning (Horvitz, 1990).

6. Math education now makes considerable use of physical manipulatives. Doug Clements (1999) provides an excellent analysis of physical manipulatives versus virtual (computer-based) manipulative. For more information on this topic see http://darkwing.uoregon.edu/~moursund/Math/virtual_manipulatives.htm.
7. The NCTM first came out strongly in support of calculators in 1980. Slow progress has been occurring, and calculators are now reasonably well accepted in a variety of state and national testing situations. How long will it take before computer use is commonly accepted in testing situations? As calculators and computers become more thoroughly integrated into the math curriculum, the basics of authentic assessment dictate that assessment be done in a hands on environment. See http://darkwing.uoregon.edu/~moursund/PBL/part_7.htm for more information about authentic assessment. The idea of "hands on" computers is related to open book tests.

Conclusions and Recommendations

Math education is a large, complex, and challenging discipline. The formal teaching of math began at the time of the first formal teaching of reading and writing, a little more than 5,000 years ago. During the past 5,000 years the collected mathematical knowledge of the human race has grown immensely. A number of ideas that challenged the mathematical geniuses of their time have trickled down into the precollege school math curriculum.

We have no evidence that people have become genetically more intelligent during the past 5,000 years. However, as the agriculture age has given way to the industrial age and now the information age, the math-related demands placed on people have grown. In information age societies such as the United States, there are now much higher math education expectations than there were in the industrial age or the agricultural age. As our society continues to raise its math education expectations, it is not achieving the gains that it would like.

This document supports the idea that with appropriate informal and formal teaching and support, students (on average) can gain greater Math Content Knowledge and greater Math Maturity than they are currently obtaining. However, such math education goals leave us with many challenging issues. Here are a few examples:

1. It is likely that the typical parent and well over half of elementary school teachers have not achieved Math Formal Operations. Their levels of School

Math Maturity and School Math Content Knowledge are low. Thus, on average, children growing up in our society tend gain their first 10 to 12 years (birth through grade school) of informal and formal math education in what I would call relatively poor math education environments. If we want to significantly improve our math education system, we will have to make significant progress toward addressing this problem.

2. The transfer of learning from School Math Content Knowledge and School Math Maturity is not very high. From a math education point of view, the School Math orientation is not particularly well suited to people who are better served by gaining folk math knowledge and folk math maturity.

3. ICT brings new dimensions to both School Math and Folk Math. We have yet to appropriately understand and implement a math education system that adequately takes into consideration the capabilities of ICT as aids to teaching, learning, and using math.

For example, consider computer tools that are routinely used by graphic artists. They are based on a very large amount of mathematics. However, very few graphic artists feel the need to have studied this underlying mathematics, and few people who teach graphic artist use of computers have appreciable insights into the underlying mathematics. The issue here is related to the issue of children using calculators rather than paper and pencil algorithms, or researchers using statistical packages of computer programs without having mastered the underlying mathematics.

But, the issue is also quite different. The goal of a graphic artist is to solve a graphic artist problem (complete a graphic artist task). The graphic artist has non-mathematical knowledge and skills that can provide feedback on progress toward solving the problem or accomplishing the task.

This one example suggests that we look for other examples. Perhaps the fundamental question is: In light of changing needs of our society and the greatly increasing power of ICT, what changes should be made in our school math curriculum, instruction, and assessment so that it better fits the wide range of interests and needs of students?

4. There are a variety of math topics that require a student to be at or near math Formal Operations in order to gain a significant understanding of the topic. Examples include probability, ratio and proportion, and algebra. Roughly speaking, if many of the students you are teaching "just don't seem to get it" for certain topics, then there is a good chance that they are not developmentally ready for the topic.

5. One of the most important ideas in math education is learning to build upon and make effective use of the accumulated knowledge in the discipline of math. ICT is a powerful aid to learning, a powerful aid to information retrieval, and a powerful aid to carrying out many of the types of procedures that are important in solving math problems. Our current math education system is not doing well in making effective use of this range of ICT uses.

References

Ackerman, Phillip L (November 2003). Cognitive ability and non-ability trait determinants of expertise. *Educational Researcher,* Vol. 32, No. 8, pp. 15-20. Accessed 7/23/04: http://www.aera.net/pubs/er/toc/er3208.htm.

Ball, Deborah Loewenberg, Chair (March 2002). *RAND Mathematics Study Panel. Mathematical Proficiency for All Students: Towards a Strategic Research and Development Program in Mathematics Education.* Accessed 5/9/03: http://www7.nationalacademies.org/mseb/Rand%20Report%20-%20PDF.pdf.

Barnett, S. M. & Ceci, S. J. (2002). When and where do we apply what we learn? A taxonomy for far transfer. *Psychological Bulletin,* Vol. 128(4), 612-637.

Battista, Michael T. (February 1999). Ignoring research and scientific study in education. *Phi Delta Kappan.* Accessed 2/5/04: http://www.pdkintl.org/kappan/kbat9902.htm.

Bloom, B.S. (1984). The 2 Sigma problem: The search for methods of group instruction as effective as one-to-one tutoring. *Educational Researcher.* v13, n6, pp4-16.

Bloom's Taxonomy (n.d.). *Major categories in the taxonomy of educational objectives.* Accessed 2/29/04: http://faculty.washington.edu/krumme/guides/bloom.html.

Bransford, J.D.; A. L. Brown; & R.R. Cocking: editors (1999). *How people learn: Brain, mind, experience, and school*. Washington, D.C.: National Academy Press. Accessed 5/14/03: http://books.nap.edu/catalog/6160.html.

Brown, J.S., Collins, A., and Duguid, P. (1989). *Situated cognition and the culture of learning.* Accessed 6/2/03: http://www.ilt.columbia.edu/ilt/papers/JohnBrown.html.

Bruer, John T. (May 1999). In search of . . . brain-based education. *Phi Delta Kappan.* Accessed 2/19/04: http://www.pdkintl.org/kappan/kbru9905.htm.

Clements, D. H. (1999). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood.* 1(1), 45-60. Accessed 3/22/01: http://www.gse.buffalo.edu/org/buildingblocks/NewsLetters/Concrete_Yelland.htm. The Web reference is for a slightly updated version of the original article.

Dehaene, S. et al.(May 7, 1999). Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science*: 970-974.

Gardner, Howard (1993). *Multiple intelligences: The theory in practice.* NY: Basic Books.

Gardner, Howard (n.d.). Accessed 9/14/04: http://www.infed.org/thinkers/gardner.htm .

Gogtay, Nitin et al. (May 25, 2004). Dynamic mapping of human cortical development during childhood through early adulthood. *PNAS*, vol. 101 no. 21, 8174-8179.

Halber, Deborah (1999). *Different kinds of math use different parts of brain, research finds.* Accessed 2/14/04: http://web.mit.edu/newsoffice/tt/1999/may12/math.html.

Halmos, Paul R., Moise, Edwin E. and Piranian, George. The Problem of Learning to Teach. *American Mathematical Monthly* 82, no. 5 (1975) 466-476.

Healy, A.F., & McNamara, D.S. (1996). Verbal Learning and memory: Does the modal model still work? Annual *Review of Psychology,* 47, 143-172.

History of I.Q. (n.d.). Accessed 2/13/04: http://www.testcafe.com/iqtest/history.html.

Horvitz, Eric (September 1990). Automated reasoning for biology and medicine. Accessed 7/21/03: http://research.microsoft.com/~horvitz/AIBIO.HTM.

Kulik, J. A. (1994). Meta-analytic studies of findings on computer-based instruction. In *Technology assessment in education and training (pp. 9-33*). Hillsdale, NJ: Lawrence Erlbaum Associates. A summary of this chapter and additional information is available in *"The Costs and Effectiveness of Educational Technology - November 1995."* Accessed 7/21/03: http://www.ed.gov/Technology/Plan/RAND/Costs/costs4.html.

Ma, Liping (1999). *Knowing and teaching elementary mathematics.* NJ and London: Lawrence Erlbaum Associates, Publishers.

MacDonald, Kevin (n.d.). *Psychology 346IC: Evolution, IQ, and domain general mechanisms*. Accessed 3/1/04: http://www.csulb.edu/~kmacd/346-IQ.htm.

Maier, Gene (n.d.). Gene's Corner. http://www.mlc.pdx.edu/GC.html.

Marcus, Gary (2004). *The birth of the mind: How a tiny number of genes creates the complexities of human thought.* NY: Basic Books.

McArdle, John et al. (2002). Comparative longitudinal structural analyses of the growth and decline of multiple intellectual abilities over the life span. *Developmental Psychology* Vol. 38, No. 1, 115–142. Accessed 9/6/04: http://66.102.7.104/search?q=cache:0JRtLVQbhvEJ:psychology.ucdavis.edu/labs/ferrer/pubs/dp_2002.pdf+gF+gC+growth&hl=en.

McKenzie, Walter. *The one and only Surfaquarium.* Accessed 7/21/03: http://surfaquarium.com/theory/index.htm.

Moursund, D.G. (2004a). *Introduction to information and communication technology in education.* Accessed 2/14/04: http://darkwing.uoregon.edu/~moursund/Intro%20ICT/index.htm.

Moursund, D.G. (2004b). *Planning, forecasting, and inventing your computers-in-education future.* Accessed 2/4/04: http://darkwing.uoregon.edu/~moursund/InventingFutures/index.htm.

Moursund, D.G. (2004c). *Brief introduction to educational implications of Artificial Intelligence.* Accessed 9/6/04: http://darkwing.uoregon.edu/%7emoursund/AIBook/.

Moursund, D.G. (n.d.). Math website. Accessed 9/16/04: http://darkwing.uoregon.edu/~moursund/Math/learning-theories.htm.

NCTM (n.d.). NCTM Standards. Accessed 9/16/04: http://www.nctm.org/standards/.

Nutrition (n.d.). Nutrition—the brainpower advantage: An average difference of 20 IQ points. Accessed 9/14/04: http://www.healingwithnutrition.com/adisease/add-adhd/harrellstudy.html.

Padraig, M & McLoughlin, M.M. (3 August 2002). *The central role of proof in the mathematics canon: The efficacy of teaching students to do proofs using a fusion of modified Moore, traditional, and reform methods.* Accessed 7/22/04: http://facstaff.morehouse.edu/~pmclough/MAA Paper Proof 20-7-02.pdf.

Pearson, Helen (13 November 2003). Brain furrow may cause maths problem. Dyscalculia appears to cloud number images. Nature Science Update. Accessed 2/17/04: http://www.nature.com/nsu/031110/031110-13.html.

Perkins, David (1995). *Outsmarting IQ: The emerging science of learnable intelligence.* NY: The Free Press.

Polya, George (1957). *How to solve it.* 2nd ed. Princeton University Press. See also: Accessed 3/6/04: http://math.arizona.edu/~lkondek/302/handouts/Polya.htm.

Polya, George (1969). The Goals of Mathematical Education. *Mathematically Sane.* Accessed 3/6/04: http://mathematicallysane.com/analysis/polya.asp.

Shanchez, Robert (August 7, 2004). CSAP tests expose gap: Lower income, lower scores linked, state review shows *Rocky Mountain News*. Accessed 9/16/04: http://www.rockymountainnews.com/drmn/education/article/0,1299,DRMN_957_3094699,00.html.

Schoenfeld, Alan (1992.) Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. [Chapter 15, pp. 334-370, of the *Handbook for Research on Mathematics Teaching and Learning* (D. Grouws, Ed.). New York: MacMillan, 1992.] Accessed 2/21/04: http://www-gse.berkeley.edu/faculty/aschoenfeld/LearningToThink/Learning_to_think_Math.html.

Stanescu-Cosson, Ruxandra et al. (November 2000). Understanding dissociations in dyscalculia: A brain imaging study of the impact of number size on the cerebral networks for exact and approximate calculation. *Brain*, Vol. 123, No. 11, 2240-2255. Accessed 8/5/04: http://brain.oupjournals.org/cgi/content/full/123/11/2240.

Stephens, C.F. (n.d.). *Developing mathematical maturity and understanding.* Accessed 7/22/04: http://www.potsdam.edu/MATH/HTMLPublications/DevMathMatur&Under.html.

Sternberg, Robert J. (November, 2003). What is an "expert student?" *Educational Researcher,* Vol. 32, No. 8, pp. 5-9. Accessed 7/23/04: http://www.aera.net/pubs/er/toc/er3208.htm.

Sternberg, Robert (1988). *The triarchic mind: A new theory of human intelligence.* NY: Penguin Books. Information about Sternberg and his writings is available at (Accessed 7/21/03): http://www.psy.pdx.edu/PsiCafe/KeyTheorists/Sternberg.htm.

van Hiele, Dina and Pierre (n.d.). *Model of geometric thought.* Accessed 2/6/04: http://euler.slu.edu/teachmaterial/Van_Hiele_Model_of_Geometr.html.

Yekovich, Frank R. (1994). Current issues in research on intelligence. *ERIC/AE Digest.* Accessed 7/21/03: http://ericae.net/edo/ed385605.htm.

The following quote from Michael Battista captures the essence of the problems being addressed in this document. His 1999 article provides valuable information to all people interested in math education.

> For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them. Though the same topics are taught and retaught year after year, the students do not learn them. Numerous scientific studies have shown that traditional methods of teaching mathematics not only are ineffective but also seriously stunt the growth of students' mathematical reasoning and problem-solving skills. Traditional methods ignore recommendations by professional organizations in mathematics education, and they ignore modern scientific research on how children learn mathematics (Battista, 1999).