**Positivity (to appear)**

# Duality of Weighted Anisotropic Besov and Triebel-Lizorkin Spaces

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**Abstract.** Let $A$ be an expansive dilation on $\mathbb{R}^n$ and $w$ a Muckenhoupt $\mathcal{A}_\infty(A)$ weight. In this paper, for all parameters $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty)$, the authors identify the dual spaces of weighted anisotropic Besov spaces $\dot{B}^{\alpha}_{p,\,q}(A;\, w)$ and Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,\,q}(A;\, w)$ with some new weighted Besov-type and Triebel-Lizorkin-type spaces. The corresponding results on anisotropic Besov spaces $\dot{B}^{\alpha}_{p,\,q}(A;\, \mu)$ and Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,\,q}(A;\, \mu)$ associated with $\rho_A$-doubling measure $\mu$ are also established. All results are new even for the classical weighted Besov and Triebel-Lizorkin spaces in the isotropic setting. In particular, the authors also obtain the $\varphi$-transform characterization of the dual spaces of the classical weighted Hardy spaces on $\mathbb{R}^n$.

## 1 Introduction

There were many efforts on generalizing various specific function spaces to applications of analysis such as PDEs, harmonic analysis and approximation theory; see, for example, [1, 2, 3, 11, 12, 13, 15, 16, 23, 24, 25, 30, 33, 34, 35, 38, 39]. This gave rise to the study of Besov and Triebel-Lizorkin spaces which form a unifying class of function spaces containing many well-known classical function spaces such as Lebesgue spaces, Hardy spaces and Hardy-Sobolev spaces.

In particular, there were also several efforts to extending the classical function spaces arising in harmonic analysis from Euclidean spaces and isotropic settings to other domains and anisotropic settings. Calderón and Torchinsky introduced and investigated Hardy spaces associated with anisotropic dilations [10, 11, 12]. A theory of anisotropic Hardy spaces and their weighted counterparts were recently developed by Bownik et al. in [1, 7]. Anisotropic Besov and Triebel-Lizorkin spaces including their weighted variants (more generally, associated with doubling measures) were also introduced and studied;

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see, for example, [2, 3, 4, 5, 15, 25]. In these papers, the discrete wavelet transform, the atomic and molecular decompositions of these spaces, and the dual spaces of anisotropic Triebel-Lizorkin spaces without weights were established. However, the duality of weighted anisotropic Triebel-Lizorkin spaces in [4, Theorem 4.10] was obtained under an *additional assumption* that the considered expansive dilations admit a Meyer-orthonormal wavelet.

In this paper, we introduce some new weighted anisotropic Besov-type and Triebel-Lizorkin-type spaces and we identify the dual spaces of weighted anisotropic Besov and Triebel-Lizorkin spaces with these new weighted spaces. We point out that our results are new even for the classical weighted Besov and Triebel-Lizorkin spaces in the isotropic setting. In particular, by relaxing the assumption that $w \in \mathcal{A}_p(\mathbb{R}^n)$ (the class of Muckenhoupt's weights) into $w \in \mathcal{A}_\infty(\mathbb{R}^n)$, our results also improve the results obtained by Bui in [9], Roudenko in [29] and Frazier and Roudenko in [19], which are respectively [9, Theorem 2.10] and the scalar versions of [29, Theorem A1(3)] and [19, Theorem 5.9] on the dual spaces of the matrix-weighted Besov spaces. As a special case of our results on the weighted Triebel-Lizorkin spaces in isotropic settings, we also obtain the $\varphi$-transform characterization of the dual spaces of the classical weighted Hardy spaces on $\mathbb{R}^n$, which also seems new. Recall that the classical weighted Hardy spaces on $\mathbb{R}^n$ and their dual spaces were first studied by García-Cuerva in [20]. The wavelet characterizations of the weighted Hardy spaces on $\mathbb{R}^n$ and their dual spaces were obtained in [21, 28, 41].

Let $A$ be an expansive dilation on $\mathbb{R}^n$ (see [1] or Definition 2.1 below) and $w$ a Muckenhoupt $\mathcal{A}_\infty(A)$ weight associated with $A$ (see [5] or Definition 2.2 below). In what follows, for any $p \in (0,\infty]$, $p'$ denotes the *conjugate index* of $p$, namely, $p' \equiv \infty$ when $p \in (0,1]$ and $p' \equiv 1/(1-1/p)$ when $p \in (1,\infty]$. In this paper, for all parameters $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$ and $p,\, q \in (0,\infty]$, we introduce new weighted anisotropic Besov-type spaces $\dot{B}^{\alpha,\tau}_{p,q}(A;\, w)$ and Triebel-Lizorkin-type spaces $\dot{F}^{\alpha,\tau}_{p,q}(A;\, w)$; see Definitions 2.4 and 2.5 below. By establishing the duality results on their corresponding sequence spaces, we prove that, for all $\alpha \in \mathbb{R}$ and $p,\, q \in (0,\infty)$, the dual space of the weighted anisotropic Besov space $\dot{B}^{\alpha}_{p,q}(A;\, w)$ is $\dot{B}^{-\alpha,\max\{1/p,1\}}_{p',q'}(A;\, w)$ (see Theorem 2.2 below), and the dual space of the weighted anisotropic Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(A;\, w)$ is $\dot{F}^{-\alpha,\,1/p+1/q'-1}_{q',q'}(A;\, w)$ when $p \in (0,\,1]$ or $\dot{F}^{-\alpha,\,0}_{p',q'}(A;\, w)$ when $p \in (1,\,\infty)$ (see Theorem 2.1 below). These results are also true for those anisotropic Besov spaces $\dot{B}^{\alpha}_{p,q}(A;\, \mu)$ and Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,q}(A;\, \mu)$ associated with $\rho_A$-doubling measure $\mu$; see Theorem 4.1 below.

We remark that when $w \equiv 1$, for any $\alpha \in \mathbb{R}$, $p \in (0,\,1]$, $q \in (0,\,\infty)$ and $\tau_0 = 1/p+1/q'-1$, $\dot{F}^{-\alpha,\,\tau_0}_{q',q'}(A;\, w) = \dot{F}^{-\alpha+1/p-1}_{\infty,\infty}(A)$ with equivalent norms (see Corollary 2.1(ii) below), which further shows the coincidence of our results on duality with existing known results in [4] in unweighted case. Moreover, if $w \equiv 1$ and $A \equiv 2\, I_{n\times n}$, where $I_{n\times n}$ denotes the $n \times n$ *unit matrix*, then the Triebel-Lizorkin-type spaces $\dot{F}^{\alpha,\tau}_{p,q}(A;\, w)$ were introduced and studied in [42, 43], and it was proved in [42, 43, 31, 44] that they include several classical spaces such as Triebel-Lizorkin spaces (see [33]), $Q$ spaces (see [14]), Morrey spaces and part of Morrey-Campanato spaces; while, in this case, the Besov-type spaces $\dot{B}^{\alpha,\tau}_{p,q}(A;\, w)$ are just the Besov spaces $\dot{B}^{\alpha+\tau-1}_{p,q}(A)$. This reflects the difference between $\dot{B}^{\alpha,\tau}_{p,q}(A;\, w)$ and $\dot{F}^{\alpha,\tau}_{p,q}(A;\, w)$.

Two key ideas used in the proofs of Theorems 2.1 and 2.2 are that, differently from the proofs on the duality in [4, 17, 28, 40], we adopt the notion of the tents (see [3] or (2.1) below) for dilated cubes and also introduce the notion of the pseudo-maximal dilated cubes, which are used to subtly classify dilated cubes (see (3.12) below). In this sense, the proofs of Theorems 2.1 and 2.2 are quite geometrical.

The organization of this paper is as follows. In Section 2, we present some basic notions and the duality results on weighted anisotropic Besov and Triebel-Lizorkin spaces, whose proofs are given in Section 3. In Section 4, we prove that the duality results in Section 2 are also true for anisotropic Besov and Triebel-Lizorkin spaces associated with doubling measures. We point out that all results of this paper are also true for inhomogeneous spaces with slight modifications; see, for example, [33, 44]. We omit the details.

Finally, we make some conventions on *symbols*. Throughout the paper, we denote by $C$ a *positive constant* which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as $C_0$, do not change in different occurrences. The *symbol* $A \lesssim B$ means that $A \le CB$ and the *symbol* $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. Denote by $\sharp E$ the *cardinality of the set* $E$. We will use the convention that the *conjugate exponent* $q'$ *satisfies* $1/q + 1/q' = 1$ *if* $1 < q \le \infty$ *and* $q' = \infty$ *if* $0 < q \le 1$. We also set $\mathbb{N} \equiv \{1, 2, \cdots\}$, $\mathbb{Z}_+ \equiv \{0\} \cup \mathbb{N}$ and $\mathbb{Z}_+^n = (\mathbb{Z}_+)^n$. If $E$ is a subset of $\mathbb{R}^n$, we denote by $\chi_E$ the *characteristic function of* $E$.

## 2 Main Results

We begin with the notion of expansive dilations on $\mathbb{R}^n$; see [1].

**Definition 2.1.** A real $n \times n$ matrix $A$ is called an *expansive dilation*, shortly a *dilation*, if $\max_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ is the set of all *eigenvalues* of $A$. A *quasi-norm* associated with expansive matrix $A$ is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$, for simplicity, denoted as $\rho$, such that

(i) $\rho(x) > 0$ for $x \ne 0$;

(ii) $\rho(Ax) = b\rho(x)$ for $x \in \mathbb{R}^n$, where $b \equiv |\det A|$;

(iii) $\rho(x + y) \le H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \ge 1$ is a constant.

Throughout the whole paper, we always let $A$ be an expansive dilation on $\mathbb{R}^n$ and $b \equiv |\det A|$. The set $\mathcal{Q}$ of *dilated cubes* of $\mathbb{R}^n$ is defined by

$$\mathcal{Q} \equiv \{Q \equiv A^j([0, 1)^n + k) : j \in \mathbb{Z}, \ k \in \mathbb{Z}^n\}.$$

For any $Q \equiv A^j([0, 1)^n + k)$, let the *symbol* $\mathrm{scale}\,(Q) \equiv j$ and $x_Q \equiv A^j k$ be the "*lower-left corner*" of $Q$. We see that for any fixed $j \in \mathbb{Z}$, $\{Q \equiv A^j([0, 1)^n + k) : \ k \in \mathbb{Z}^n\}$ is a *partition* of $\mathbb{R}^n$. For any $P \in \mathcal{Q}$, let

$$\mathcal{T}(P) \equiv \{Q \in \mathcal{Q} : Q \cap P \ne \emptyset, \ \mathrm{scale}\,(Q) \le \mathrm{scale}\,(P)\} \tag{2.1}$$

be the *tent of* $P$; see [3, Definition 2.4].

We now recall the weight class of Muckenhoupt associated with $A$ introduced in [5].

**Definition 2.2.** Let $p \in [1, \infty)$, $A$ be a dilation and $w$ a non-negative and almost everywhere positive measurable function on $\mathbb{R}^n$. A function $w$ is said to belong to the *weight class* $\mathcal{A}_p(A) \equiv \mathcal{A}_p(\mathbb{R}^n;\, A)$ *of Muckenhoupt*, if there exists a positive constant $C$ such that when $p \in (1, \infty)$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{b^{-k}\int_{B_\rho(x,\,b^k)} w(y)\,dy\right\}\left\{b^{-k}\int_{B_\rho(x,\,b^k)} [w(y)]^{-1/(p-1)}\,dy\right\}^{p-1} \le C,$$

and when $p = 1$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{b^{-k}\int_{B_\rho(x,\,b^k)} w(y)\,dy\right\}\left\{\operatorname*{esssup}_{y\in B_\rho(x,\,b^k)} [w(y)]^{-1}\right\} \le C;$$

and the minimal constant $C$ as above is denoted by $C_{p,\,A,\,n}(w)$. Here, for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, $B_\rho(x,\, b^k) \equiv \{y \in \mathbb{R}^n :\ \rho(x - y) < b^k\}$.

Define $\mathcal{A}_\infty(A) \equiv \bigcup_{1\le p<\infty} \mathcal{A}_p(A)$.

For all $p \in (0,\, \infty)$ and $w \in \mathcal{A}_\infty(A)$, the *weighted Lebesgue space* $L^p_w(\mathbb{R}^n)$ is defined to be the space of all $w(x)\,dx$-measurable functions on $\mathbb{R}^n$ such that $\|f\|_{L^p_w(\mathbb{R}^n)} \equiv \{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\}^{1/p} < \infty$.

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *set of all Schwartz functions on* $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space*. As in [18], we set

$$\mathcal{S}_\infty(\mathbb{R}^n) \equiv \left\{\phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x)x^\alpha\ dx = 0,\ \alpha \in \mathbb{Z}^n_+\right\}.$$

We consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Thus, $\mathcal{S}_\infty(\mathbb{R}^n)$ is a *complete metric space* (see, for example, [32, p. 21, (3.7)]). Let $\mathcal{S}'_\infty(\mathbb{R}^n)$ be the *topological dual space* of $\mathcal{S}_\infty(\mathbb{R}^n)$ with the weak-$*$ topology.

**Definition 2.3.** Let $A$ be an expansive dilation and $A^*$ its *transpose*. Define $\mathscr{S}_\infty(\mathbb{R}^n)$ to be the *set of all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *such that*

*(i)* $\operatorname{supp}\widehat{\varphi} \subset [-\pi,\, \pi]^n \setminus \{0\}$,

*(ii)* $\sup_{j\in\mathbb{Z}} |\widehat{\varphi}((A^*)^j\xi)| > 0$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$.

Obviously, $\mathscr{S}_\infty(\mathbb{R}^n) \subset \mathcal{S}_\infty(\mathbb{R}^n)$.

Now let us first recall the notion of the weighted anisotropic Triebel-Lizorkin spaces in [5] and then introduce some new weighted anisotropic Triebel-Lizorkin-type spaces. *In what follows, for all* $Q \in \mathcal{Q}$, *let* $j_Q \equiv -\operatorname{scale}(Q)$ *and* $\widetilde{\chi}_Q \equiv \chi_Q|Q|^{-1/2}$.

**Definition 2.4.** Let $w \in \mathcal{A}_\infty(A)$, $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0,\, \infty]$ and $\tau \in [0,\, \infty)$.

(i) The *weighted anisotropic Triebel-Lizorkin space* $\dot{F}^\alpha_{p,\,q}(A;\, w)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^\alpha_{p,\,q}(A;\,w)} \equiv \left\|\left\{\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|\varphi_{j_Q} * f|\chi_Q)^q\right\}^{\frac{1}{q}}\right\|_{L^p_w(\mathbb{R}^n)} < \infty$$

where and in what follows, for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, $\varphi_j(x) \equiv b^j \varphi(A^j x)$. The corresponding *discrete weighted anisotropic Triebel-Lizorkin space* $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$ is defined to be the set of all complex-valued sequences $s \equiv \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{f}^{\alpha}_{p,\,q}(A;\,w)} \equiv \left\| \left\{ \sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\widetilde{\chi}_Q)^q \right\}^{\frac{1}{q}} \right\|_{L^p_w(\mathbb{R}^n)} < \infty.$$

(ii) The *weighted anisotropic Triebel-Lizorkin-type space* $\dot{F}^{\alpha,\,\tau}_{p,\,q}(A;\,w)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{\alpha,\,\tau}_{p,\,q}(A;\,w)} \equiv \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^\tau} \left\{ \int_P \left[ \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{-\alpha}|\varphi_{j_Q} * f|\chi_Q \frac{|Q|}{w(Q)} \right)^q \right]^{\frac{p}{q}} w(x)\,dx \right\}^{\frac{1}{p}} < \infty.$$

Its corresponding *discrete weighted anisotropic Triebel-Lizorkin-type space* $\dot{f}^{\alpha,\,\tau}_{p,\,q}(A;\,w)$ is defined to be the set of all complex-valued sequences $s \equiv \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{f}^{\alpha,\,\tau}_{p,\,q}(A;\,w)} \equiv \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^\tau} \left\{ \int_P \left[ \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{-\alpha}|s_Q|\widetilde{\chi}_Q \frac{|Q|}{w(Q)} \right)^q \right]^{\frac{p}{q}} w(x)\,dx \right\}^{\frac{1}{p}} < \infty.$$

It is understood that the above definitions need the usual modification when $q = \infty$.

**Remark 2.1.** (i) Integrating the norm of $\|\cdot\|_{\dot{F}^{\alpha}_{p,\,q}(A,\,w)}$ over cubes $Q$ with fixed scale $j$ yields a familiar equivalent form $\|f\|_{\dot{F}^{\alpha}_{p,\,q}(A,\,w)} \equiv \|\{\sum_{j\in\mathbb{Z}} b^{qj\alpha}|\varphi_j * f|^q\}^{1/q}\|_{L^p_w(\mathbb{R}^n)}$.

(ii) The weighted anisotropic Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,\,q}(A;\,w)$ and its discrete variant $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$ were first introduced in [5]. Moreover, when $p,\,q \in (0,\,\infty)$, by the $\varphi$-transform characterization of $\dot{F}^{\alpha}_{p,\,q}(A;\,w)$ (see [5, Theorem 3.5]) and the fact that sequences with finite support are dense in $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$ (see [5, p. 1452]), we know that $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\dot{F}^{\alpha}_{p,\,q}(A;\,w)$.

(iii) We point out that in Definition 2.4(ii), when $w \equiv 1$ and $A \equiv 2\,I_{n\times n}$, the space $\dot{F}^{\alpha,\,\tau}_{p,\,q}(\mathbb{R}^n)$ and its corresponding discrete sequences spaces were introduced in [31, 42, 43] (see also [44] for inhomogeneous versions).

The following is the main theorem of this paper.

**Theorem 2.1.** *Let $\alpha \in \mathbb{R}$, $p,\,q \in (0,\,\infty)$, $\tau_0 = 1/p + 1/q' - 1$ and $w \in \mathcal{A}_\infty(A)$. Then,*

*(i)*

$$(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^* = \begin{cases} \dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w), & p \in (0,\,1], \\ \dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w), & p \in (1,\,\infty). \end{cases}$$

*More precisely, $l$ is a bounded linear functional on $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$ if and only if $l$ is of the form*

$$l(\lambda) = \langle \lambda, t\rangle \equiv \sum_{Q\in\mathcal{Q}} \lambda_Q \overline{t_Q}, \qquad \textit{where } \lambda \equiv \{\lambda_Q\}_{Q\in\mathcal{Q}}, \tag{2.2}$$

*for some sequence $t \equiv \{t_Q\}_{Q\in\mathcal{Q}}$ such that*

$$||l||_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*} \sim \begin{cases} ||t||_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)}, & p \in (0,\,1], \\ ||t||_{\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)}, & p \in (1,\,\infty). \end{cases}$$

*(ii)*

$$(\dot{F}^{\alpha}_{p,\,q}(A;\,w))^* = \begin{cases} \dot{F}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w), & p \in (0,\,1], \\ \dot{F}^{-\alpha,\,0}_{p',\,q'}(A;\,w), & p \in (1,\,\infty) \end{cases}$$

*in the following sense. For each $g \in \dot{F}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)$ when $p \in (0,\,1]$ or $g \in \dot{F}^{-\alpha,\,0}_{p',\,q'}(A;\,w)$ when $p \in (1,\,\infty)$, the map*

$$l(f) = \langle f, g\rangle \equiv \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx, \tag{2.3}$$

*defined initially for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, has a bounded linear extension to $\dot{F}^{\alpha}_{p,\,q}(A;\,w)$. Conversely, any bounded linear functional $l$ on $\dot{F}^{\alpha}_{p,\,q}(A;\,w)$ is of the form* (2.3) *and*

$$||l||_{(\dot{F}^{\alpha}_{p,\,q}(A;\,w))^*} \sim \begin{cases} ||g||_{\dot{F}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)}, & p \in (0,\,1], \\ ||g||_{\dot{F}^{-\alpha,\,0}_{p',\,q'}(A;\,w)}, & p \in (1,\,\infty). \end{cases}$$

Observe that Theorem 2.1 when $w \equiv 1$ includes [4, Theorem 4.8] which briefly states as follows:

$$(\dot{F}^{\alpha}_{p,\,q}(A))^* = \begin{cases} \dot{F}^{-\alpha+\frac{1}{p}-1}_{\infty,\,\infty}(A), & p \in (0,\,1) \\ \dot{F}^{-\alpha}_{p',\,q'}(A), & p \in [1,\,\infty). \end{cases}$$

Indeed, let $w \equiv 1$, $p,\,q \in (0,\,\infty)$ and $\tau_0 = 1/p + 1/q' - 1$. By definitions of these spaces, we immediately have that when $p \in (1,\,\infty)$, $\dot{F}^{\alpha,0}_{p,q}(A) = \dot{F}^{\alpha}_{p,\,q}(A)$, and when $p = 1$, $\dot{F}^{-\alpha,\tau_0}_{q',\,q'}(A;\,w) = \dot{F}^{-\alpha}_{\infty,\,q'}(A)$. By the following Corollary 2.1(ii), when $p \in (0,\,1)$, we also have $\dot{F}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w) = \dot{F}^{-\alpha+1/p-1}_{\infty,\,\infty}(A)$. This shows the above claim. Moreover, Theorem 2.1 when $w \equiv 1$ and $A \equiv 2I_{n\times n}$ coincides with the corresponding classical results in [17, Section 5].

As a consequence of Theorem 2.1 and [4, Theorem 4.2] we have the following result.

**Corollary 2.1.** *Let $\alpha \in \mathbb{R}$, $q \in (1,\infty]$ and $\tau \in (1/q,\infty)$.*

*(i) If $w \in \mathcal{A}_\infty(A)$, then the space $\dot{f}^{\alpha}_{\infty,\infty}(A;\, w)$ is isomorphic with the space $\dot{f}^{\alpha,\tau}_{q,q}(A;\, w)$ via the map $\{s_Q\}_{Q\in\mathcal{Q}} \mapsto \{\frac{[w(Q)]^{\tau-1/q+1}}{|Q|} s_Q\}_{Q\in\mathcal{Q}}$. That is, for all $\{s_Q\}_{Q\in\mathcal{Q}} \in \dot{f}^{\alpha}_{\infty,\infty}(A;\, w)$,*

$$\|\{s_Q\}_{Q\in\mathcal{Q}}\|_{\dot{f}^{\alpha}_{\infty,\infty}(A;\,w)} \sim \left\| \left\{ \frac{[w(Q)]^{\tau-1/q+1}}{|Q|} s_Q \right\}_{Q\in\mathcal{Q}} \right\|_{\dot{f}^{\alpha,\tau}_{q,q}(A;\,w)}$$

*(ii) If $w \equiv 1$, then $\dot{f}^{\alpha,\tau}_{q,q}(A;\, w) = \dot{f}^{\alpha+\tau-1/q}_{\infty,\infty}(A;\, w)$ with equivalent norms.*
*The same conclusions are true for the spaces $\dot{F}^{\alpha,\tau}_{q,q}(A;\, w)$.*

The proof of Corollary 2.1 is given in Section 3. We point out that part (ii) of Corollary 2.1 may not be true if $w \not\equiv 1$. We give a counter-example on *1-dimensional Euclidean space* $\mathbb{R}$ as follows.

**Example 2.1.** Let $\alpha \in \mathbb{R}$, $q \in (1,\infty]$, $\tau \in (1/q,\infty)$, $A = 2$, $\ell \in (0,\,\infty)$ and $w(x) = |x|^\ell \in \mathcal{A}_\infty(A)$ (see [22, p. 286, Example 9.1.7]). In this case, we know that $\mathcal{Q}$ is the set of all classical dyadic cubes in $\mathbb{R}$. Now, we construct a sequence $s \equiv \{s_Q\}_{Q\in\mathcal{Q}}$ such that $\|s\|_{\dot{f}^{\alpha+\tau-1/q}_{\infty,\infty}(A;\,w)} = 1$ but $\|s\|_{\dot{f}^{\alpha,\tau}_{q,q}(A;\,w)} = \infty$. Since

$$\|s\|_{\dot{f}^{\alpha+\tau-1/q}_{\infty,\infty}(A;\,w)} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{-(\alpha+\tau-1/q)}|s_Q||Q|^{-1/2} \quad (\text{see} \quad [4, (2.17)]),$$

we set $s \equiv \{s_Q\}_{Q\in\mathcal{Q}}$ with $s_Q \equiv |Q|^{\alpha+\tau-1/q+1/2}$ for all $Q \in \mathcal{Q}$. Then $\|s\|_{\dot{f}^{\alpha+\tau-1/q}_{\infty,\infty}(A;\,w)} \equiv 1$.

On the other hand,

$$\begin{aligned}\|s\|_{\dot{f}^{\alpha,\tau}_{q,q}(A;\,w)} &= \sup_{P\in\mathcal{Q}} [w(P)]^\tau \left\{ \sum_{Q\in\mathcal{T}(P)} |Q|^{-\alpha q}|s_Q|^q |Q|^{-q/2} \frac{|Q|^q}{[w(Q)]^q} w(Q) \right\}^{1/q} \\ &= \sup_{P\in\mathcal{Q}} [w(P)]^\tau \left\{ \sum_{Q\in\mathcal{T}(P)} |Q|^{\tau q} \left[\frac{|Q|}{w(Q)}\right]^{q-1} \right\}^{1/q} \geq \sup_{P\in\mathcal{Q}} \left[\frac{|P|}{w(P)}\right]^{\tau+1-1/q}.\end{aligned}$$

Let $\widetilde{\mathcal{Q}} \equiv \{Q \equiv [2^k,\, 2^{k+1}) :\, k \in \mathbb{Z}\}$. Then, for any $P \in \widetilde{\mathcal{Q}}$ and $w(x) = |x|^\ell$, we have $w(P) = \int_{2^k}^{2^{k+1}} |x|^\ell dx \sim 2^{k(\ell+1)}$. Combining this, the above estimate, $\ell > 0$ and $\tau \in (1/q,\infty)$, we obtain that $\|s\|_{\dot{f}^{\alpha,\tau}_{q,q}(A;\,w)} \gtrsim 2^{-k\ell(\tau+1-1/q)}$. By letting $k \to -\infty$, we further obtain that $\|s\|_{\dot{f}^{\alpha,\tau}_{q,q}(A;\,w)} = \infty$, which implies that the spaces $\dot{f}^{\alpha+\tau-1/q}_{\infty,\infty}(A;\, w)$ and $\dot{f}^{\alpha,\tau}_{q,q}(A;\, w)$ are not the same spaces with equivalent norms.

We also have the corresponding duality theory for weighted anisotropic Besov spaces. Let us begin with recalling the notion of the weighted anisotropic Besov spaces introduced in [2] and then introduce some new weighted anisotropic Besov-type spaces.

**Definition 2.5.** Let $w \in \mathcal{A}_\infty(A)$, $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $p,\, q \in (0,\, \infty]$ and $\tau \in [0,\, \infty)$.

(i) The *weighted anisotropic Besov space* $\dot{B}^{\alpha}_{p,q}(A;\,w)$ is defined to be the set of all $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(A;\,w)}\equiv\left\{\sum_{j\in\mathbb{Z}}\left[\sum_{\substack{Q\in\mathcal{Q}\\ \mathrm{scale}\,(Q)=-j}}\int_{\mathbb{R}^n}|Q|^{-\alpha p}|\varphi_j*f(x)\chi_Q(x)|^p\,w(x)\,dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}<\infty.$$

The corresponding *discrete weighted anisotropic Besov space* $\dot{b}^{\alpha}_{p,q}(A;\,w)$ is defined to be the set of all complex-valued sequences $s\equiv\{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{b}^{\alpha}_{p,q}(A;\,w)}\equiv\left\{\sum_{j\in\mathbb{Z}}\left[\sum_{\substack{Q\in\mathcal{Q}\\ \mathrm{scale}\,(Q)=-j}}\int_{\mathbb{R}^n}(|Q|^{-\alpha}s_Q\widetilde{\chi}_Q(x))^p\,w(x)\,dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}<\infty.$$

(ii) The *weighted anisotropic Besov-type space* $\dot{B}^{\alpha,\tau}_{p,q}(A;\,w)$ is defined to the set of all $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(A;\,w)}\equiv\left\{\sum_{j\in\mathbb{Z}}\left[\int_{\mathbb{R}^n}\left(\sum_{\substack{Q\in\mathcal{Q}\\ \mathrm{scale}\,(Q)=-j}}|Q|^{-\alpha}|\varphi_j*f(x)|\frac{|Q|\chi_Q(x)}{[w(Q)]^\tau}\right)^p w(x)\,dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}<\infty.$$

The corresponding *discrete weighted anisotropic Besov-type space* $\dot{b}^{\alpha,\tau}_{p,q}(A;\,w)$ is defined to be the set of all complex-valued sequences $s\equiv\{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{b}^{\alpha,\tau}_{p,q}(A;\,w)}\equiv\left\{\sum_{j\in\mathbb{Z}}\left[\int_{\mathbb{R}^n}\left(\sum_{\substack{Q\in\mathcal{Q}\\ \mathrm{scale}\,(Q)=-j}}|Q|^{-\alpha}|s_Q|\frac{|Q|\widetilde{\chi}_Q(x)}{[w(Q)]^\tau}\right)^p w(x)\,dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}<\infty.$$

It is understood that the above definitions need the usual modifications when $q=\infty$ or $p=\infty$.

**Remark 2.2.** (i) Integrating the norm of $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}(A,\,w)}$ over cubes $Q$ with fixed scale $j$ yields a familiar equivalent form

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(A:\,w)}=\left\{\sum_{j\in\mathbb{Z}}\left[\int_{\mathbb{R}^n}b^{jp\alpha}|\varphi_j*f|^p w(x)\,dx\right]^{q/p}\right\}^{1/q}.$$

(ii) The weighted anisotropic Besov space $\dot{B}^{\alpha}_{p,q}(A;\,w)$ and its discrete counterpart were first introduced in [2]. Moreover, when $p,\,q\in(0,\,\infty)$, by the $\varphi$-transform characterization of $\dot{B}^{\alpha}_{p,q}(A;\,w)$ (see [2, Theorem 3.5]) and the fact that sequences with finite support are dense in $\dot{b}^{\alpha}_{p,q}(A;\,w)$ (see [2, p. 553]), we know that $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\dot{B}^{\alpha}_{p,q}(A;\,w)$.

(iii) Observe that when $w \equiv 1$, the Besov space $\dot{B}^{\alpha,\tau}_{p,q}(A;\, w)$ coincides with $\dot{B}^{\alpha+\tau-1}_{p,q}(A)$, see Proposition 2.1. This is in contrast with $\dot{F}^{\alpha,\tau}_{p,q}(A;\, w)$, where the parameter $\tau$ plays a significant role; see Remark 2.1(iii).

From Definition 2.5, we can immediately deduce the following result.

**Proposition 2.1.** *Let $w \equiv 1$, $\alpha \in \mathbb{R}$ and $p,\, q \in (0,\, \infty]$, $\tau \geq 0$. Then,*

$$\dot{b}^{\alpha,\tau}_{p,q}(A;\, w) = \dot{b}^{\alpha+\tau-1}_{p,q}(A)$$

*and $\dot{B}^{\alpha,\tau}_{p,q}(A;\, w) = \dot{B}^{\alpha+\tau-1}_{p,q}(A)$ with equivalent norms.*

**Example 2.2.** In general, Proposition 2.1 may not be true when $w \not\equiv 1$. For example, letting the dimension $n = 1$, $\alpha = 0$, $p = q = \infty$, $A = 2$ and $s \equiv \{s_Q\}_{Q\in\mathcal{Q}}$ with $s_Q \equiv |Q|^{\tau-1/2}$ for all $Q \in \mathcal{Q}$, we see that

$$\|s\|_{\dot{b}^{\tau-1}_{\infty,\infty}(A)} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{1/2-\tau}|s_Q| = 1$$

and

$$\|s\|_{\dot{b}^{0,\tau}_{\infty,\infty}(A;\, w)} \equiv \sup_{Q\in\mathcal{Q}} |s_Q||Q|^{1/2}/[w(Q)]^{\tau} = \sup_{Q\in\mathcal{Q}} [|Q|/w(Q)]^{\tau}.$$

Choose $w(x) \equiv |x|$ for all $x \in \mathbb{R}$. Then, for all $j \in \mathbb{Z}$,

$$\|s\|_{\dot{b}^{0,\tau}_{\infty,\infty}(A;\, w)} = \sup_{Q\in\mathcal{Q}} [|Q|/w(Q)]^{\tau} \geq \left[\frac{2^j}{\int_{2^j}^{2^{j+1}} x\, dx}\right]^{\tau} \gtrsim 2^{-j\tau}.$$

Letting $j \to -\infty$, we obtain $\|s\|_{\dot{b}^{0,\tau}_{\infty,\infty}(A;\, w)} = \infty$, which implies that the spaces $\dot{b}^{\tau-1}_{\infty,\infty}(A)$ and $\dot{b}^{0,\tau}_{\infty,\infty}(A;\, w)$ are not the same.

We have the following duality results on weighted anisotropic Besov spaces, which is another main theorem of this paper.

**Theorem 2.2.** *Let $\alpha \in \mathbb{R}$, $p,\, q \in (0,\, \infty)$ and $w \in \mathcal{A}_\infty(A)$. Then*

$$\left(\dot{b}^{\alpha}_{p,q}(A;\, w)\right)^* = \dot{b}^{-\alpha,\, \max\{1/p,1\}}_{p',q'}(A;\, w)$$

*in the sense of* (2.2), *and*

$$\left(\dot{B}^{\alpha}_{p,q}(A;\, w)\right)^* = \dot{B}^{-\alpha,\, \max\{1/p,1\}}_{p',q'}(A;\, w)$$

*in the sense of* (2.3).

**Remark 2.3.** (i) We point out that the duality results obtained in Theorem 2.2 when $w \equiv 1$ and $A \equiv 2I_{n\times n}$ generalize the classical results on Besov spaces in [33, 17].

(ii) Theorem 2.2 when $A \equiv 2I_{n\times n}$ and $w \in \mathcal{A}_{\max\{p,1\}}(\mathbb{R}^n)$ (the class of Muckenhoupt's weights) coincides with the scalar versions of [29, Theorem A1(3)] ($p \in [1,\, \infty)$) and [19, Theorem 5.9] ($p \in (0, 1)$).

We finish this section by giving a couple of equivalent descriptions of anisotropic weighted Besov-type spaces and Triebel-Lizorkin-type spaces for certain parameters.

**Definition 2.6.** Let $w \in \mathcal{A}_\infty(A)$, $\alpha \in \mathbb{R}$, $q \in (0, \infty)$, $\tau_0 \equiv 1/q + 1/q' - 1$ and $\tau_1 \equiv \max\{1/q, 1\}$.

(i) The space $\widetilde{F}^{\alpha,\tau_0}_{q',q'}(A; w)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{F}^{\alpha,\tau_0}_{q',q'}(A;w)} \equiv \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\tau_0}} \left\{ \int_P \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{-\alpha} |\varphi_{j_Q} * f| \frac{\chi_Q |Q|^{1-\frac{1}{q'}}}{[w(Q)]^{1-\frac{1}{q'}}} \right)^{q'} dx \right\}^{\frac{1}{q'}} < \infty$$

with the usual modification made when $q' = \infty$.

(ii) The space $\widetilde{B}^{\alpha,\tau_1}_{q',q'}(A; w)$ is defined to the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{B}^{\alpha,\tau_1}_{q',q'}(A;w)} \equiv \left\{ \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \sum_{\substack{Q\in\mathcal{Q} \\ \mathrm{scale}\,(Q)=-j}} \left( |Q|^{-\alpha} |\varphi_j * f(x)| \frac{|Q|^{1-\frac{1}{q'}} \chi_Q(x)}{[w(Q)]^{\tau_1-\frac{1}{q'}}} \right)^{q'} dx \right\}^{\frac{1}{q'}} < \infty$$

with the usual modification made when $q' = \infty$.

Comparing with the definitions of $\dot{F}^{\alpha,\tau_0}_{q',q'}(A; w)$ and $\dot{B}^{\alpha,\tau_1}_{q',q'}(A; w)$, we find that in the definitions of $\widetilde{F}^{\alpha,\tau_0}_{q',q'}(A; w)$ and $\widetilde{B}^{\alpha,\tau_1}_{q',q'}(A; w)$, the integrals are not weighted. However, the two couples of spaces are equivalent as follows.

**Corollary 2.2.** *Let $w \in \mathcal{A}_\infty(A)$, $\alpha \in \mathbb{R}$, $q \in (0, \infty)$, $\tau_0 \equiv 1/q + 1/q' - 1$ and $\tau_1 \equiv \max\{1/q, 1\}$. Then $\dot{F}^{\alpha,\tau_0}_{q',q'}(A; w) = \widetilde{F}^{\alpha,\tau_0}_{q',q'}(A; w)$ and $\dot{B}^{\alpha,\tau_1}_{q',q'}(A; w) = \widetilde{B}^{\alpha,\tau_1}_{q',q'}(A; w)$ with equivalent norms.*

By adapting the proof of Lemma 3.1 below, we show that Lemma 3.1 also holds with $\dot{F}^{\alpha,\tau_0}_{q',q'}(A; w)$ and $\dot{B}^{\alpha,\tau_1}_{q',q'}(A; w)$ replaced, respectively, by $\widetilde{F}^{\alpha,\tau_0}_{q',q'}(A; w)$ and $\widetilde{B}^{\alpha,\tau_1}_{q',q'}(A; w)$ albeit with the same sequence spaces $\dot{f}^{\alpha,\tau_0}_{q',q'}(A; w)$ and $\dot{b}^{\alpha,\tau_1}_{q',q'}(A; w)$. Once this is shown, Corollary 2.2 follows immediately. We omit the details.

## 3 Proofs of Theorems 2.1 and 2.2

Let us begin with recalling some notation. For all functions $\varphi$ on $\mathbb{R}^n$, $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $Q \equiv A^{-j}([0, 1)^n + k)$, let $\varphi_Q(x) \equiv |Q|^{\frac{1}{2}} \varphi_j(x - x_Q)$, where $|\cdot|$ means the Lebesgue measure on $\mathbb{R}^n$.

Let $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n)$. For all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, recall that the *$\varphi$-transform* $S_\varphi$ is defined by $S_\varphi(f) \equiv \{(S_\varphi(f))_Q\}_{Q\in\mathcal{Q}} \equiv \{\langle f, \varphi_Q\rangle\}_{Q\in\mathcal{Q}}$, and the *inverse $\varphi$-transform* $T_\varphi$ is defined by $T_\varphi(t) \equiv \sum_{Q\in\mathcal{Q}} t_Q \varphi_Q$ initially for finitely supported sequences $t \equiv \{t_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$; see [3].

In what follows, for simplicity, we use the *symbol* $\dot{A}^\alpha_{p,q}(A; w)$ to denote either the space $\dot{B}^\alpha_{p,q}(A; w)$ or the space $\dot{F}^\alpha_{p,q}(A; w)$, and use the *symbol* $\dot{a}^\alpha_{p,q}(A; w)$ to denote the corresponding sequence spaces. Likewise we introduce the *symbols* $\dot{A}^{\alpha,\tau}_{p,q}(A; w)$ and $\dot{a}^{\alpha,\tau}_{p,q}(A; w)$.

The $\varphi$-transform characterizations for weighted anisotropic Besov and Triebel-Lizorkin spaces in Definitions 2.4 and 2.5 are presented as follows.

**Lemma 3.1.** *Let $w \in \mathcal{A}_\infty(A)$, $\varphi,\ \psi \in \mathscr{S}_\infty(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $p,\ q \in (0,\ \infty)$ and $\tau_0 = 1/p + 1/q' - 1$. Then, the following hold.*

*(i) The $\varphi$-transform $S_\varphi$ is bounded, respectively, from the spaces $\dot{A}^{\alpha}_{p,q}(A;w)$, $\dot{F}^{\alpha,\tau_0}_{q',q'}(A;\ w)$, $\dot{F}^{\alpha,\,0}_{p',\,q'}(A;\ w)$ and $\dot{B}^{\alpha,\,\max\{1/p,1\}}_{p',\,q'}(A;\ w)$ to the corresponding discrete spaces with the same parameters.*

*(ii) The inverse $\varphi$-transform $T_\psi$ is bounded, respectively, from the spaces $\dot{a}^{\alpha}_{p,q}(A;w)$, $\dot{f}^{\alpha,\,\tau_0}_{q',\,q'}(A;\ w)$, $\dot{f}^{\alpha,\,0}_{p',\,q'}(A;\ w)$, and $\dot{b}^{\alpha,\,\max\{1/p,1\}}_{p',\,q'}(A;\ w)$ to the corresponding continuous spaces with the same parameters.*

*(iii) Assume that $\varphi$ and $\psi$ additionally satisfy $\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi}((A^*)^j\xi)}\widehat{\psi}((A^*)^j\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $A^*$ denotes the transpose of $A$. Then, $T_\psi \circ S_\varphi$ is the identity on $\dot{A}^{\alpha}_{p,\,q}(A;\ w)$, $\dot{F}^{\alpha,\,\tau_0}_{q',\,q'}(A;\ w)$, $\dot{F}^{\alpha,\,0}_{p',\,q'}(A;\ w)$ and $\dot{B}^{\alpha,\,\max\{1/p,1\}}_{p',\,q'}(A;\ w)$.*

We first point out that Lemma 3.1 may be true for $\dot{A}^{\alpha,\,\tau}_{p,\,q}(A;\ w)$ and their corresponding spaces of sequences with full indices. However, to limit the length of this paper, we only indicate how to show Lemma 3.1 in these special indices described therein, which is enough for applications of this paper.

The results in Lemma 3.1 associated with Besov spaces $\dot{B}^{\alpha}_{p,\,q}(A;\ w)$ and Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{p,\,q}(A;\ w)$ were, respectively, obtained in [2] and [3]. The results in Lemma 3.1 associated with the spaces $\dot{A}^{\alpha,\,\tau}_{p,\,q}(A;\ w)$ can be obtained by a modification of the proofs for [3, Theorem 3.12] with $p \in (0,\ \infty)$ and [2, Theorem 1.1]; see also [27, Lemma 3.9]. We give some details only on the space $\dot{F}^{\alpha,\,\tau_0}_{q',q'}(A;\ w)$ and its sequence space $\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\ w)$.

Using Lemma 3.1, by repeating the proofs of [5, Corollary 3.7] and [2, Corollary 3.7], we have the following conclusion. We omit the details.

**Corollary 3.1.** *Let $\alpha$, $w$, $p$, $q$ and $\tau_0$ be as in Lemma 3.1. Then the spaces $\dot{A}^{\alpha}_{p,\,q}(A;\ w)$, $\dot{F}^{\alpha,\tau_0}_{q',q'}(A;\ w)$, $\dot{F}^{\alpha,\,0}_{p',\,q'}(A;\ w)$ and $\dot{B}^{\alpha,\max\{1/p,1\}}_{p',q'}(A;\ w)$ are independent of the choices of $\varphi$.*

For any $w \in \mathcal{A}_\infty(A)$ with $q_w \equiv \inf\{\ell \in [1,\infty) :\ w \in \mathcal{A}_\ell(A)\}$, $\lambda,\ r \in (0,\ \infty)$ satisfying some additional conditions, the sequence $s \equiv \{s_Q\}_{Q\in\mathcal{Q}} \in \dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\ w)$ and its majorant sequence $s^*_{r,\,\lambda} \equiv \{(s^*_{r,\,\lambda})_Q\}_{Q\in\mathcal{Q}}$ defined by

$$(s^*_{r,\,\lambda})_Q \equiv \left\{ \sum_{\substack{P\in\mathcal{Q}\\ \mathrm{scale}\,(P)=\,\mathrm{scale}\,(Q)}} \frac{|s_P|^r}{[1+|Q|^{-1}\rho(x_Q - x_P)]^{\lambda r}} \right\}^{1/r}.$$

By following the proofs of [3, Lemma 3.10] and [27, Lemma 3.9], we see that the key of the proof of Lemma 3.1 in this case is to show $\|s\|_{\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)} \sim \|s^*_{r,\lambda}\|_{\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)}$; once this is done, the other details are similar to those of the proof of [3, Theorem 3.12]. Now let us show this conclusion.

**Lemma 3.2.** *Let $w \in \mathcal{A}_\infty(A)$, $\alpha \in \mathbb{R}$, $p,\, q \in (0,\,\infty)$, $\tau_0 \equiv 1/p + 1/q' - 1$, $r \in [q',\,\infty]$ and $\lambda \in (1/q' + q_w \max\{1/p,\, 1-1/q'\}, \infty)$. Then there exists a positive constant $C$ such that for all sequence $s \equiv \{s_Q\}_{Q\in\mathcal{Q}} \in \dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)$ and its majorant sequence $s^*_{r,\,\lambda} \equiv \{(s^*_{q',\,\lambda})_Q\}_{Q\in\mathcal{Q}}$,*

$$\|s\|_{\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)} \le \|s^*_{r,\lambda}\|_{\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)} \le C\|s\|_{\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)}.$$

*Proof.* The first inequality is obvious, and we only need to prove the second inequality. For all $\alpha \in \mathbb{R}$, $w \in \mathcal{A}_\infty(A)$, $p,\, q \in (0,\,\infty)$ and $\tau_0 \equiv 1/p + 1/q' - 1$, we have

$$\|s\|_{\dot{f}^{\alpha,\,\tau_0}_{q',q'}(A;\,w)} \sim \sup_{P\in\mathcal{Q}} \left\{ \frac{1}{[w(P)]^{q'(\frac{1}{p}-1)+1}} \sum_{Q\in\mathcal{T}(P)} \left(|Q|^{-\alpha+1/2}|s_Q|\right)^{q'} [w(Q)]^{-(q'-1)} \right\}^{\frac{1}{q'}}. \tag{3.1}$$

By similarity, we only give the proof for the case that $q' \in (1,\,\infty)$. For any $P \in \mathcal{Q}$, by [3, Lemma 2.9], there exists a *positive integer* $c_0$ such that

$$\bigcup_{Q\in\mathcal{T}(P)} Q \subset B_\rho(x_P,\, b^{c_0+\operatorname{scale}(P)}) \ \text{ and } \ B_\rho(c_P,\, b^{-c_0+\operatorname{scale}(P)}) \subset P, \tag{3.2}$$

where $c_P$ is the center of $P$. Then for any fixed $P \in \mathcal{Q}$, let

$$B_P \equiv B_\rho(x_P,\, 3H^2 b^{c_0+\operatorname{scale}(P)}),$$

where $H$ is as in Definition 2.1. Let

$$U_{B_P} \equiv \left\{P' \in \mathcal{Q}:\ \operatorname{scale}(P') = \operatorname{scale}(P),\ P' \cap B_P \ne \emptyset\right\}$$

and $\widetilde{U}_{B_P} \equiv \cup_{P'\in U_{B_P}} P'$. Thus, by the fact that $\{P' \in \mathcal{Q}:\ \operatorname{scale}(P') = \operatorname{scale}(P)\}$ is a partition of $\mathbb{R}^n$, we have that

$$\sum_{\substack{R\in\mathcal{Q}\\ \operatorname{scale}(R)=\operatorname{scale}(Q)}} = \sum_{P'\in U_{B_P}} \sum_{\substack{R\in\mathcal{T}(P')\\ \operatorname{scale}(R)=\operatorname{scale}(Q)}} + \sum_{\substack{P'\cap\widetilde{U}_{B_P}=\emptyset\\ \operatorname{scale}(P')=\operatorname{scale}(P)}} \sum_{\substack{R\in\mathcal{T}(P'),\, R\cap\widetilde{U}_{B_P}=\emptyset\\ \operatorname{scale}(R)=\operatorname{scale}(Q)}},$$

which, together with the well-known inequality that for all $\gamma \in (0,\,1)$ and $\{a_j\}_j \subset \mathbb{C}$,

$$\left(\sum_j |a_j|\right)^\gamma \le \sum_j |a_j|^\gamma \tag{3.3}$$

and $|Q| = |R|$ when $\operatorname{scale}(Q) = \operatorname{scale}(R)$, further implies that

$$\frac{1}{[w(P)]^{q'(\frac{1}{p}-1)+1}} \sum_{Q\in\mathcal{T}(P)} \left(|Q|^{-\alpha+\frac{1}{2}}|(s^*_{r,\,\lambda})_Q|\right)^{q'} [w(Q)]^{1-q'}$$

$$\begin{aligned}&\le \frac{1}{[w(P)]^{q'(\frac{1}{p}-1)+1}}\sum_{Q\in\mathcal{T}(P)}\sum_{P'\in U_{B_P}}\sum_{\substack{R\in\mathcal{T}(P')\\ \mathrm{scale}\,(R)=\mathrm{scale}\,(Q)}}\frac{|R|^{q'(\frac{1}{2}-\alpha)}|s_R|^{q'}[w(Q)]^{1-q'}}{[1+|Q|^{-1}\rho(x_Q-x_R)]^{\lambda q'}}\\ &\quad+\frac{1}{[w(P)]^{q'(\frac{1}{p}-1)+1}}\sum_{Q\in\mathcal{T}(P)}\sum_{\substack{P'\cap\widetilde{U}_{B_P}=\emptyset\\ \mathrm{scale}\,(P')=\mathrm{scale}\,(P)}}\sum_{\substack{R\in\mathcal{T}(P'),\,R\cap\widetilde{U}_{B_P}=\emptyset\\ \mathrm{scale}\,(R)=\mathrm{scale}\,(Q)}}\cdots\\ &\equiv \mathrm{I}+\mathrm{J}.\end{aligned}$$

**Step 1. Prove** $\mathrm{I}\lesssim\|s\|^{q'}_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;w)}$**.** For any $R\in\mathcal{Q}$, let

$$M_{R,0}\equiv\left\{Q\in\mathcal{T}(P):\ \mathrm{scale}\,(Q)=\mathrm{scale}\,(R),\ |Q|^{-1}\rho(x_Q-x_R)<b\right\},$$

and

$$M_{R,l}\equiv\left\{Q\in\mathcal{T}(P):\ \mathrm{scale}\,(Q)=\mathrm{scale}\,(R),\ b^l\le|Q|^{-1}\rho(x_Q-x_R)<b^{l+1}\right\}$$

for all $l\in\mathbb{N}$. Then we have

$$\mathrm{I}=\frac{1}{[w(P)]^{q'(\frac{1}{p}-1)+1}}\sum_{P'\in U_{B_P}}\sum_{R\in\mathcal{T}(P')}\sum_{l\in\mathbb{Z}_+}\sum_{Q\in M_{R,l}}\frac{|R|^{q'(\frac{1}{2}-\alpha)}|s_R|^{q'}[w(Q)]^{1-q'}}{[1+|Q|^{-1}\rho(x_Q-x_R)]^{\lambda q'}}.$$

Since $\lambda>1/q'+q_w(1-1/q')$, we choose $\widetilde{q}\in(q_w,\infty)$ sufficiently close to $q_w$ such that $\lambda>1/q'+\widetilde{q}(1-1/q')$. For any $Q\in M_{R,l}$ and $P'\in U_{B_P}$, by [3, Lemma 2.9] and [7, Proposition 2.6(i)], we have $w(R)\lesssim b^{\widetilde{q}l}w(Q)$ and $w(P)\sim w(P')$. Moreover, by an elementary lattice counting lemma (see [6, Lemma 2.8]), $\sharp M_{R,l}\lesssim b^l$ and $\sharp U_{B_P}\lesssim 1$. From the above estimates, $\lambda>1/q'+\widetilde{q}(1-1/q')$ and (3.1), it follows that

$$\mathrm{I}\lesssim\sum_{P'\in U_{B_P}}\frac{1}{[w(P')]^{q'(\frac{1}{p}-1)+1}}\sum_{R\in\mathcal{T}(P')}\frac{(|R|^{\frac{1}{2}-\alpha}|s_R|)^{q'}}{[w(R)]^{q'-1}}\sum_{l\in\mathbb{Z}_+}b^{l[\widetilde{q}(q'-1)+1-\lambda q']}\lesssim\|s\|^{q'}_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;w)},$$

which is the desired inequality.

**Step 2. Prove** $\mathrm{J}\lesssim\|s\|^{q'}_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;w)}$**.** For any fixed $P\in\mathcal{Q}$, $Q\in\mathcal{T}(P)$, $P'\in\mathcal{Q}$ with $P'\cap\widetilde{U}_{B_P}=\emptyset$ and $\mathrm{scale}\,(P')=\mathrm{scale}\,(P)$, and any $R\in\mathcal{T}(P')$ with $R\cap\widetilde{U}_{B_P}=\emptyset$ and $\mathrm{scale}\,(R)=\mathrm{scale}\,(Q)$, by (3.2) and $B_P\equiv B_\rho(x_P,3H^2b^{c_0+\mathrm{scale}\,(P)})\subset U_{B_P}$, we obtain

$$\begin{aligned}\rho(x_P-x_{P'})&\le H^2[\rho(x_P-x_Q)+\rho(x_Q-x_R)+\rho(x_R-x_{P'})]\\ &\le H^2[2b^{c_0+\mathrm{scale}\,(P)}+\rho(x_Q-x_R)],\end{aligned}$$

which, together with

$$\rho(x_R-x_Q)\ge\frac{\rho(x_R-x_P)}{H}-\rho(x_Q-x_P)\ge 3Hb^{c_0+\mathrm{scale}\,(P)}-b^{c_0+\mathrm{scale}\,(P)}\ge 2Hb^{c_0+\mathrm{scale}\,(P)},$$

implies that

$$\rho(x_P-x_{P'})\le 2H^2\rho(x_Q-x_R). \tag{3.4}$$

Moreover, by $P'\cap \widetilde{U}_{B_P}=\emptyset$ and $\operatorname{scale}(P')=\operatorname{scale}(P)$, we have

$$\begin{aligned}
(3.5)\quad &\left\{P'\in\mathcal{Q}:\ P'\cap \widetilde{U}_{B_P}=\emptyset,\ \operatorname{scale}(P')=\operatorname{scale}(P)\right\}\\
&\subset \left\{P'\in\mathcal{Q}:\ \rho(x_P-x_{P'})\ge 3H^2b^{c_0+\operatorname{scale}(P)},\ \operatorname{scale}(P')=\operatorname{scale}(P)\right\}\\
&=\bigcup_{j\in\mathbb{Z}_+}\left\{P'\in\mathcal{Q}:\ 3H^2b^{c_0+\operatorname{scale}(P)+j}\le\rho(x_P-x_{P'})<3H^2b^{c_0+\operatorname{scale}(P)+j+1},\right.\\
&\qquad\qquad \left.\operatorname{scale}(P')=\operatorname{scale}(P)\right\}\\
&\equiv\bigcup_{j\in\mathbb{Z}_+}V_{P,j}.
\end{aligned}$$

Since $\lambda>1/q'+q_w\max\{1/p,\,1-1/q'\}$, we choose $\widetilde{q}\in(q_w,\,\infty)$ sufficiently close to $q_w$ such that $\lambda>1/q'+\widetilde{q}\max\{1/p,\,1-1/q'\}$. Notice that for any $j\in\mathbb{Z}_+$ and $P'\in V_{P,j}$, by $P'\subset B_\rho(x_P,\,4H^3b^{c_0+\operatorname{scale}(P)+j+1})$, $w(B_\rho(x_P,\,b^{-c_0+\operatorname{scale}(P)}))\sim w(B_\rho(c_P,\,b^{-c_0+\operatorname{scale}(P)}))$, $B_\rho(c_P,\,b^{-c_0+\operatorname{scale}(P)})\subset P$ and [7, Proposition 2.6(i)], we have

$$\begin{aligned}
(3.6)\quad w(P')&\le w\left(B_\rho(x_P,\,4H^3b^{c_0+\operatorname{scale}(P)+j+1})\right)\lesssim b^{j\widetilde{q}}w\left(B_\rho(x_P,\,b^{-c_0+\operatorname{scale}(P)})\right)\\
&\lesssim b^{j\widetilde{q}}w\left(B_\rho(c_P,\,b^{-c_0+\operatorname{scale}(P)})\right)\lesssim b^{j\widetilde{q}}w(P).
\end{aligned}$$

Symmetrically, we also have

$$(3.7)\qquad w(P)\lesssim b^{j\widetilde{q}}w(P').$$

Moreover, for any $j\in\mathbb{Z}_+$, $k\in\mathbb{Z}_+$, $Q\in\mathcal{T}(P)$ with $\operatorname{scale}(P)=\operatorname{scale}(Q)+k$, $P'\in V_{P,j}$, $R\in\mathcal{T}(P')$ with $R\cap U_{B_P}=\emptyset$ and $\operatorname{scale}(R)=\operatorname{scale}(Q)$, by (3.4), we obtain

$$(3.8)\qquad b^{j+k+\operatorname{scale}(R)}\sim\rho(x_P-x_{P'})\lesssim\rho(x_Q-x_R).$$

Furthermore, for any $j\in\mathbb{Z}_+$, $k\in\mathbb{Z}_+$ $P\in V_{P,j}$, $Q\in\mathcal{T}(P)$, $R\in\mathcal{T}(P')$ with $R\cap\widetilde{U}_{B_P}=\emptyset$ and $\operatorname{scale}(R)=\operatorname{scale}(Q)=\operatorname{scale}(P)-k$, by

$$R\subset B_\rho(x_P,\,4H^3b^{c_0+\operatorname{scale}(P)+j+1})\subset B_\rho(c_Q,5H^4b^{c_0+\operatorname{scale}(P)+j+1}),$$

$B_\rho(c_Q,\,b^{-c_0+\operatorname{scale}(Q)})\subset Q$ and [7, Proposition 2.6(i)], we have

$$\begin{aligned}
(3.9)\quad w(R)&\le w\left(B_\rho(c_Q,5H^4b^{c_0+\operatorname{scale}(P)+j+1})\right)\\
&\lesssim b^{\widetilde{q}(j+k)}w\left(B_\rho(c_Q,\,b^{-c_0+\operatorname{scale}(Q)})\right)\lesssim b^{\widetilde{q}(j+k)}w(Q).
\end{aligned}$$

Thus, for any $p\in(0,\,\infty)$, $q'\in(1,\,\infty)$ and $P\in\mathcal{Q}$, using (3.5) through (3.9),

$$\sharp\{Q\in\mathcal{T}(P):\ \operatorname{scale}(Q)+k=\operatorname{scale}(P)\}\lesssim b^k$$

and $\sharp V_{P,j}\lesssim b^j$ (see [6, Lemma 2.8]), (3.2) and $\lambda>1/q'+\widetilde{q}\max\{1/p,\,1-1/q'\}$, we obtain

$$\mathrm{J}\lesssim\frac{1}{[w(P)]^{q'(\frac{1}{p}-1)+1}}\sum_{k\in\mathbb{Z}_+}\sum_{\substack{Q\in\mathcal{T}(P)\\ \operatorname{scale}(Q)+k=\operatorname{scale}(P)}}\sum_{j\in\mathbb{Z}_+}\sum_{P'\in V_{P,j}}$$

$$\times \sum_{\substack{R\in\mathcal{T}(P'),\, R\cap U_{B_P}=\emptyset \\ \text{scale}\,(R)=\text{scale}\,(Q)}} \frac{|R|^{q'(\frac{1}{2}-\alpha)}|s_R|^{q'}[w(Q)]^{1-q'}}{[1+|Q|^{-1}\rho(x_Q-x_R)]^{\lambda q'}}$$

$$\lesssim \sum_{k\in\mathbb{Z}_+} b^k \sum_{j\in\mathbb{Z}_+}\sum_{P'\in V_{P,j}} \frac{b^{j\widetilde{q}[q'(\frac{1}{p}-1)+1]}}{[w(P')]^{q'(\frac{1}{p}-1)+1}} \sum_{R\in\mathcal{T}(P')} \frac{|R|^{q'(\frac{1}{2}-\alpha)}|s_R|^{q'}[w(R)]^{1-q'}b^{\widetilde{q}(j+k)(q'-1)}}{b^{\lambda q'(j+k)}}$$

$$\lesssim \|s\|^{q'}_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;\,w)} \sum_{k\in\mathbb{Z}_+} b^{k[\widetilde{q}(q'-1)+1-\lambda q']} \sum_{j\in\mathbb{Z}_+} b^{j(1+\frac{\widetilde{q}q'}{p}-\lambda q')}$$

$$\lesssim \|s\|^{q'}_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;\,w)},$$

which is also the desired inequality.

Combining the estimates of I and J, by the arbitrariness of $P\in\mathcal{Q}$, we have

$$\|s^*_{r,\lambda}\|_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;\,w)} \lesssim \|s\|_{\dot{f}^{\alpha,\tau_0}_{q',q'}(A;\,w)},$$

which completes the proof of Lemma 3.2. □

Now we are ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* Let $\tau_0 = 1/p+1/q'-1$ and $w\in\mathcal{A}_\infty(A)$. We prove Theorem 2.1 in three steps.

**Step 1. Proof of** $(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*\equiv \dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)$ **with** $(p,\,q)\in(1,\,\infty)\times(0,\,\infty)$**.**

We first prove $\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)\subset(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*$. For any $t\equiv\{t_Q\}_{Q\in\mathcal{Q}}\in\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)$, define a linear functional $\ell_t$ on $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$ by $\ell_t(s)\equiv\sum_{Q\in\mathcal{Q}} s_Q\bar{t}_Q$ for all $s\in\dot{f}^{\alpha}_{p,\,q}(A;\,w)$. By applying Hölder's inequality twice when $q\in(1,\,\infty)$, or by the imbedding $\dot{f}^{\alpha}_{p,\,q}(A;\,w)\to\dot{f}^{\alpha}_{p,\,1}(A;\,w)$ when $q\in(0,\,1]$, we have

$$\begin{aligned}|\ell_t(s)| &\le \int_{\mathbb{R}^n}\sum_{Q\in\mathcal{Q}}|Q|^{-\alpha}|s_Q|\widetilde{\chi}_Q(x)|Q|^{\alpha}|t_Q|\frac{|Q|}{w(Q)}\widetilde{\chi}_Q(x)w(x)\,dx\\ &\le \|s\|_{\dot{f}^{\alpha}_{p,\,q}(A;\,w)}\|t\|_{\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)},\end{aligned}$$

which yields $\|\ell_t\|_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*}\le\|t\|_{\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)}$, and hence $\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,w)\subset(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*$.

Let us prove the converse by referring some ideas from [17, p. 78]. Since sequences with finite support are dense in $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$, each bounded linear functional $\ell\in(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*$ must be of the form $\ell(s)\equiv\sum_{Q\in\mathcal{Q}} s_Q\bar{t}_Q$ for some $t\equiv\{t_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{C}$. It suffices to show that $\|t\|_{f^{-\alpha,\,0}_{p',\,q'}(A;\,w)}\lesssim\|l\|_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*}$.

For all $p,\,q\in(0,\,\infty]$, let $L^p_w(\ell^q)$ be the *space of all sequences* $\{f_j\}_{j\in\mathbb{Z}}$ *of functions on* $\mathbb{R}^n$ such that

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{L^p_w(\ell^q)}\equiv\left\|\left\{\sum_{j\in\mathbb{Z}}|f_j|^q\right\}^{1/q}\right\|_{L^p_w(\mathbb{R}^n)}<\infty.$$

By [4, Proposition 4.3], we know that $(L^p_w(\ell^q))^* = L^{p'}_w(\ell^{q'})$ for all $p \in (1, \infty)$ and $q \in (0, \infty)$. Notice that the map $\mathrm{In} : \dot{f}^{\alpha}_{p,q}(A; w) \to L^p_w(\ell^q)$ defined by setting, for all $s \in \dot{f}^{\alpha}_{p,q}(A; w)$, $\mathrm{In}(s) \equiv \{f_j\}_j$, where $f_j \equiv \sum_{\mathrm{scale}\,(Q)=-j} |Q|^{-\alpha} s_Q \widetilde{\chi}_Q$ for all $j \in \mathbb{Z}$, is a linear isometry onto a subspace of $L^p_w(\ell^q)$.

When $p \in (1,\infty)$ and $q \in [1,\infty)$, by the Hahn-Banach theorem, there exists an $\widetilde{\ell} \in (L^p_w(\ell^q))^*$ with $\|\widetilde{\ell}\|_{(\dot{f}^{\alpha}_{p,q}(A;w))^*} = \|\ell\|_{(\dot{f}^{\alpha}_{p,q}(A;w))^*}$ such that $\widetilde{\ell} \circ \mathrm{In} = \ell$.

In other words, there exists $g \equiv \{g_j\}_{j\in\mathbb{Z}} \in L^{p'}_w(\ell^{q'})$ with $\|g\|_{L^{p'}_w(\ell^{q'})} \le \|\ell\|_{(\dot{f}^{\alpha}_{p,q}(A;w))^*}$ such that for all $s \in \dot{f}^{\alpha}_{p,q}(A; w)$,

$$\sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} f_j(x)\bar{g}_j(x) w(x)\,dx.$$

By taking $s_Q = 0$ for all but one dilated cube, we obtain

$$t_Q = \int_Q |Q|^{-\alpha-1/2} g_j(x) w(x)\,dx \tag{3.10}$$

for all cubes $Q$ with $\mathrm{scale}\,(Q) = -j$.

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; w)$, which denotes the *space of all locally integrable functions on the measure* $w(x)\,dx$, define the weighted anisotropic Hardy-Littlewood maximal function of $f$ by $M_w(f)(x) \equiv \sup_{x\in Q\in\mathcal{Q}} \frac{1}{w(Q)} \int_Q |f(y)|w(y)\,dy$. Then by [3, Lemma 2.9 and Theorem 2.8] and the fact that $w(x)\,dx$ is a $\rho_A$-doubling measure (see Section 4 below), we have the vector-valued maximal inequality that for all $p' \in (1, \infty)$, $q' \in (1, \infty]$ and functions $\{f_i\}_i \subset L^{p'}_w(\ell^{q'})$,

$$\left\| \left( \sum_i |M_w f_i|^{q'} \right)^{1/q'} \right\|_{L^{p'}_w(\mathbb{R}^n)} \lesssim \left\| \left( \sum_i |f_i|^{q'} \right)^{1/q'} \right\|_{L^{p'}_w(\mathbb{R}^n)},$$

which together with (3.10) yields that

$$\|t\|_{\dot{f}^{-\alpha,0}_{p',q'}(A;w)} \le \|\{M_w(g_j)\}_{j\in\mathbb{Z}}\|_{L^{p'}_w(\ell^{q'})} \lesssim \|g\|_{L^{p'}_w(\ell^{q'})} \lesssim \|\ell\|_{(\dot{f}^{\alpha}_{p,q}(A;w))^*},$$

and hence $(\dot{f}^{\alpha}_{p,q}(A; w))^* \subset \dot{f}^{-\alpha,0}_{p',q'}(A; w)$. This finishes the proof of Step 1 when $p \in (1, \infty)$ and $q \in [1, \infty)$.

To complete the proof of Step 1, it suffices to prove $(\dot{f}^{\alpha}_{p,q}(A; w))^* \subset \dot{f}^{-\alpha,0}_{p',q'}(A; w)$ when $p \in (1, \infty)$ and $q \in (0, 1)$. We need to use Verbitsky's method from [36, 37]; see also [4]. In fact, since $l \in (\dot{f}^{\alpha}_{p,q}(A; w))^*$ is of the form $\ell(s) \equiv \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q$ for some $t \equiv \{t_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$, we know that there exists a positive constant $C$ such that for all $s \in \dot{f}^{\alpha}_{p,q}(A; w)$,

$$|\ell(s)| = \left| \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q \right| \le C\|s\|_{\dot{f}^{\alpha}_{p,q}(A;w)}.$$

Define $\mathcal{Q}' \equiv \{Q \in \mathcal{Q} :\ t_Q \neq 0\}$, $u_Q \equiv s_Q \bar{t}_Q$ for all $Q \in \mathcal{Q}$ and $c_Q \equiv |Q|^{-\alpha-1/2}|t_Q|^{-1}$ for all $Q \in \mathcal{Q}'$. Then the above inequality can be rewritten as

$$\|\{u_Q\}_{Q\in\mathcal{Q}}\|_{\ell^1} \le C \left\| \left\{ \sum_{Q\in\mathcal{Q}} |u_Q|^q (c_Q)^q \chi_Q \right\}^{1/q} \right\|_{L^p_w(\mathbb{R}^n)}.$$

Then applying [4, Theorem 4.4(ii)] with $0 < q < r = 1 < p < \infty$, we obtain that

$$\begin{aligned} \|t\|^{p'}_{\dot{f}^{-\alpha,0}_{p',q'}(A;w)} &= \int_{\mathbb{R}^n} \sup_{Q\in\mathcal{Q}',\ x\in Q} \left[ |Q|^{\alpha-1/2}|t_Q| \frac{|Q|}{w(Q)} \right]^{p'} w(x)\,dx \\ &= \int_{\mathbb{R}^n} \sup_{Q\in\mathcal{Q}',\ x\in Q} [c_Q w(Q)]^{-p'} w(x)\,dx < \infty, \end{aligned}$$

which implies that $(\dot{f}^{\alpha}_{p,q}(A;\,w))^* \subset \dot{f}^{-\alpha,0}_{p',q'}(A;\,w)$, and hence completes the proof of Step 1.

**Step 2. Proof of $(\dot{f}^{\alpha}_{p,q}(A;\,w))^* = \dot{f}^{-\alpha,\tau_0}_{q',q'}(A;\,w)$ for $(p,\,q) \in (0,\,1] \times (1,\,\infty)$.**

For any $t \in \dot{f}^{-\alpha,\tau_0}_{q',q'}(A;\,w)$, observe that

$$\begin{aligned} &\|t\|_{\dot{f}^{-\alpha,\tau_0}_{q',q'}(A;w)} \\ &\equiv \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\tau_0}} \left\{ \int_P \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{\alpha}|t_Q| \frac{|Q|}{w(Q)} \widetilde{\chi}_Q(x) \right)^{q'} w(x)\,dx \right\}^{\frac{1}{q'}} \\ &\sim \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\tau_0}} \left\{ \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{\alpha-\frac{1}{2}}|t_Q| \frac{|Q|}{w(Q)} \right)^{q'} w(Q) \right\}^{\frac{1}{q'}}, \end{aligned} \tag{3.11}$$

where $\mathcal{T}(P)$ is the tent of $P$ defined in (2.1).

For any $s \equiv \{s_Q\}_{Q\in\mathcal{Q}} \in \dot{f}^{\alpha}_{p,q}(A;\,w)$, $q \in (1,\,\infty)$, $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$, let

$$\Omega_k \equiv \left\{ x \in \mathbb{R}^n :\ \left\{ \sum_{Q\in\mathcal{Q}} \left[ |Q|^{-\alpha} s_Q \widetilde{\chi}_Q(x) \right]^q \right\}^{1/q} > 2^k \right\},$$

$\widetilde{\Omega}_k \equiv \{x \in \mathbb{R}^n :\ M_w(\chi_{\Omega_k})(x) > 1/2\}$ and

$$\mathcal{R}_k \equiv \{Q \in \mathcal{Q} :\ w(Q \cap \Omega_k) > w(Q)/2,\ w(Q \cap \Omega_{k+1}) \le w(Q)/2\}.$$

Then, we see that

(i) for all $k \in \mathbb{Z}$, $\Omega_{k+1} \subset \Omega_k$ and $\Omega_k \subset \widetilde{\Omega}_k$;

(ii) for any $k,\,j \in \mathbb{Z}$ with $k \neq j$, $\mathcal{R}_k \cap \mathcal{R}_j = \emptyset$;

(iii) $\mathbb{R}^n = \cup_{k\in\mathbb{Z}} \cup_{Q\in\mathcal{R}_k} Q$;

(iv) $\cup_{Q\in\mathcal{R}_k} Q \subset \widetilde{\Omega}_k$;

(v) $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$.

We point out that (v) holds by the $L^2_w(\mathbb{R}^n)$-boundedness of $M_w$. Moreover, we say that $\widetilde{Q} \in \mathcal{Q}$ is *pseudo-maximal* in $\mathcal{R}_k$ if there is no other $P \in \mathcal{R}_k$ such that $\mathrm{scale}\,(\widetilde{Q}) < \mathrm{scale}\,(P)$ and $\widetilde{Q}\cap P \neq \emptyset$. Notice that the pseudo-maximal cubes in $\mathcal{R}_k$ are disjoint with each other. Then, we obtain a classification for $\mathcal{R}_k$ associated with pseudo-maximal cubes in $\mathcal{R}_k$ such that any $Q \in \mathcal{R}_k$ belongs to one and only one tent of pseudo-maximal cubes. Precisely, picking any pseudo-maximal cube $\widetilde{Q}$ in $\mathcal{R}_k$, denoted by $\widetilde{Q}^{(1)}$, and set

$$\widetilde{\mathcal{T}}_k(\widetilde{Q}^{(1)}) \equiv \left\{Q \in \mathcal{R}_k :\ Q \cap \widetilde{Q}^{(1)} \neq \emptyset,\ \mathrm{scale}\,(Q) \leq \mathrm{scale}\,(\widetilde{Q}^{(1)})\right\}.$$

Then, we pick another pseudo-maximal cube $\widetilde{P}$ in $\mathcal{R}_k$, denoted by $\widetilde{Q}^{(2)}$, and set

$$\widetilde{\mathcal{T}}_k(\widetilde{Q}^{(2)}) \equiv \left\{Q \in \mathcal{R}_k \setminus \widetilde{\mathcal{T}}_k(\widetilde{Q}^{(1)}) :\ Q \cap \widetilde{Q}^{(2)} \neq \emptyset,\ \mathrm{scale}\,(Q) \leq \mathrm{scale}\,(\widetilde{Q}^{(2)})\right\}.$$

Inductively, for any $j \in \mathbb{N}$, picking pseudo-maximal cube $\widetilde{R}$ in $\mathcal{R}_k$, denoted by $\widetilde{Q}^{(j+1)}$, and set

$$\widetilde{\mathcal{T}}_k(\widetilde{Q}^{(j+1)}) \equiv \left\{Q \in \mathcal{R}_k \setminus \cup_{\ell=1}^{j} \widetilde{\mathcal{T}}_k(\widetilde{Q}^{(\ell)}) :\ Q \cap \widetilde{Q}^{(j+1)} \neq \emptyset,\ \mathrm{scale}\,(Q) \leq \mathrm{scale}\,(\widetilde{Q}^{(j+1)})\right\}.$$

Thus, $\mathcal{R}_k = \cup_{j\in\mathbb{N}} \widetilde{\mathcal{T}}_k(\widetilde{Q}^{(j)})$. For simplicity, let $\widetilde{\mathcal{R}}_k$ be the set of all pseudo-maximal cubes in $\mathcal{R}$ chosen as above. Then

$$\mathcal{R}_k = \bigcup_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k} \widetilde{\mathcal{T}}_k(\widetilde{Q}). \tag{3.12}$$

Furthermore, by the fact that the pseudo-maximal cubes in $\mathcal{R}_k$ are disjoint with each other, (iv) and (v), we have

$$\sum_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k} w(\widetilde{Q}) \leq w(\widetilde{\Omega}_k) \lesssim w(\Omega_k). \tag{3.13}$$

Now let us first prove that $\dot{f}^{-\alpha,\tau_0}_{q',q'}(A;\,w) \subset (\dot{f}^{\alpha}_{p,q}(A;\,w))^*$. For any $t \in \dot{f}^{-\alpha,\tau_0}_{q',q'}(A;\,w)$, define a linear functional $\ell_t$ on $\dot{f}^{\alpha}_{p,q}(A;\,w)$ by $\ell_t(s) \equiv \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q$ for any $s \in \dot{f}^{\alpha}_{p,q}(A;\,w)$. Then for all $\alpha \in \mathbb{R}$, $q \in (1,\,\infty)$, $p \in (0,\,1]$, by (iii), (3.12), Hölder's inequality with $q$, (3.11), (3.3), Hölder's inequality for $q/p$ and (3.13), we have

$$\begin{aligned}
|\ell_t(s)| &\leq \sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k} \sum_{Q\in\widetilde{\mathcal{T}}_k(\widetilde{Q})} |Q|^{-\alpha-\frac12}|s_Q|[w(Q)]^{\frac1q}|Q|^{\alpha}|t_Q||Q|^{-\frac12}\frac{|Q|}{[w(Q)]^{\frac{q'-1}{q'}}} \\
&\leq \sum_{k\in\mathbb{Z}} \sum_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k} \left[\sum_{Q\in\widetilde{\mathcal{T}}_k(\widetilde{Q})} \left(|Q|^{-\alpha-\frac12}|s_Q|\right)^q w(Q)\right]^{\frac1q}
\end{aligned} \tag{3.14}$$

$$\times \left[\sum_{Q\in\widetilde{\mathcal{T}}_k(\widetilde{Q})} \left(|Q|^{\alpha}|t_Q||Q|^{-\frac{1}{2}}\frac{|Q|}{w(Q)}\right)^{q'} w(Q)\right]^{\frac{1}{q'}}$$

$$\lesssim \|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)} \left\{\sum_{k\in\mathbb{Z}}\sum_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k} [w(\widetilde{Q})]^{1-\frac{p}{q}} \right.$$

$$\left.\times \left[\sum_{Q\in\widetilde{\mathcal{T}}_k(\widetilde{Q})} \left(|Q|^{-\alpha-\frac{1}{2}}|s_Q|\right)^{q} w(Q)\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$

$$\lesssim \|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)} \left\{\sum_{k\in\mathbb{Z}}\left[\sum_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k} w(\widetilde{Q})\right]^{1-\frac{p}{q}}\right.$$

$$\left.\times \left[\sum_{\widetilde{Q}\in\widetilde{\mathcal{R}}_k}\sum_{Q\in\widetilde{\mathcal{T}}_k(\widetilde{Q})} \left(|Q|^{-\alpha-\frac{1}{2}}|s_Q|\right)^{q} w(Q)\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$

$$\lesssim \|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)} \left\{\sum_{k\in\mathbb{Z}}[w(\Omega_k)]^{1-\frac{p}{q}}\left[\sum_{Q\in\mathcal{R}_k} \left(|Q|^{-\alpha-\frac{1}{2}}|s_Q|\right)^{q} w(Q)\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}}.$$

Moreover, notice that for any $k\in\mathbb{Z}$ and $Q\in\mathcal{R}_k$, we have $w(Q\cap\Omega_{k+1})\le w(Q)/2$, which implies that $w(Q\cap(\Omega_{k+1})^{\complement})>w(Q)/2$. By this and $Q\subset\widetilde{\Omega}_k$, we obtain that $w(Q\cap(\widetilde{\Omega}_k\setminus\Omega_{k+1}))\ge w(Q)/2$, which, together with (iii) and the definition of $\Omega_{k+1}$, yields that

$$\left\{\sum_{Q\in\mathcal{R}_k}(|Q|^{-\alpha-\frac{1}{2}}|s_Q|)^q w(Q)\right\}^{\frac{1}{q}}$$

$$\lesssim \left\{\int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}}\sum_{Q\in\mathcal{R}_k}(|Q|^{-\alpha}|s_Q|\widetilde{\chi}_Q(x))^q w(x)\,dx\right\}^{\frac{1}{q}} \lesssim 2^k[w(\Omega_k)]^{\frac{1}{q}}.$$

Combining this and (3.14) yields that $|\ell_t(s)|\lesssim\|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)}\|s\|_{\dot{f}^{\alpha}_{p,\,q}(A;\,w)}$, which further implies that $\|\ell_t\|_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*}\lesssim\|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)}$, and hence $\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)\subset(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*$.

Conversely, since sequences with finite support are dense in $\dot{f}^{\alpha}_{p,\,q}(A;\,w)$, each $\ell\in(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*$ must be of the form $\ell(s)\equiv\sum_{Q\in\mathcal{Q}}s_Q\bar{t}_Q$ for some $t\equiv\{t_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{C}$. It suffices to show that $\|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)}\lesssim\|\ell\|_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*}$.

For any $P\in\mathcal{Q}$, define a measure $\nu$ by $\nu(Q)\equiv w(Q)/w(P)$ if $Q\cap P\neq\emptyset$ and $\mathrm{scale}\,(Q)\le\mathrm{scale}\,(P)$ or else $\nu(Q)\equiv0$. Then, for any $(p,\,q)\in(0,\,1]\times(1,\,\infty)$, by (3.11), we have

$$\|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)}$$

$$\sim \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\tau_0}} \left\{ \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{\alpha-\frac{1}{2}} |t_Q| \frac{|Q|}{w(Q)} \right)^{q'} w(Q) \right\}^{\frac{1}{q'}}$$

$$\sim \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\frac{1}{p}-1}} \left[ \sum_{Q\in\mathcal{T}(P)} \left( |Q|^{\alpha-\frac{1}{2}} |t_Q| \frac{|Q|}{w(Q)} \right)^{q'} \frac{w(Q)}{w(P)} \right]^{\frac{1}{q'}}$$

$$\sim \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\frac{1}{p}-1}} \left\| \left\{ |Q|^{\alpha-\frac{1}{2}} t_Q \frac{|Q|}{w(Q)} \right\}_{Q\in\mathcal{Q}} \right\|_{\ell^{q'}(d\nu)}$$

$$\sim \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\frac{1}{p}-1}} \sup_{\|s\|_{\ell^q(d\nu)}\le 1} \left| \sum_{Q\in\mathcal{T}(P)} |Q|^{\alpha-\frac{1}{2}} t_Q \frac{|Q|}{w(Q)} s_Q \frac{w(Q)}{w(P)} \right|$$

$$\lesssim \|\ell\|_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*} \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^{\frac{1}{p}-1}} \sup_{\|s\|_{\ell^q(d\nu)}\le 1} \left\| \left\{ \frac{|Q|^{\alpha+\frac{1}{2}}}{w(P)} s_Q \right\}_{Q\in\mathcal{T}(P)} \right\|_{\dot{f}^{\alpha}_{p,\,q}(A;\,w)}.$$

Notice that by [3, Lemma 2.9], there exist positive constants $c_0$ and $c_1$ such that

$$\bigcup_{Q\in\mathcal{T}(P)} Q \subset B_\rho(c_P,\, b^{c_0}|P|)$$

and $B_\rho(c_P,\, b^{-c_1}|P|) \subset P$, where $c_P$ is the center of $P$. Thus, for any fixed $q > q_w$, by [8, Proposition 2.5] with $w \in \mathcal{A}_q(A)$, we have

$$w\left( \bigcup_{Q\in\mathcal{T}(P)} Q \right) \le w(B_\rho(c_P,\, b^{c_0}|P|)) \lesssim b^{q(c_0+c_1)} w(B_\rho(c_P,\, b^{-c_1}|P|)) \lesssim w(P), \tag{3.15}$$

which, together with Hölder's inequality, yields that

$$\left\| \left\{ \frac{|Q|^{\alpha+\frac{1}{2}}}{w(P)} s_Q \right\}_{Q\in\mathcal{T}(P)} \right\|_{\dot{f}^{\alpha}_{p,\,q}(A;\,w)}$$

$$\equiv \frac{1}{w(P)} \left\{ \int_{\bigcup_{Q\in\mathcal{T}(P)} Q} \left[ \sum_{Q\in\mathcal{T}(P)} (|s_Q|\chi_Q)^q \right]^{\frac{p}{q}} w(x)\, dx \right\}^{\frac{1}{p}}$$

$$\le \frac{1}{w(P)} \left\{ \left[ \sum_{Q\in\mathcal{T}(P)} |s_Q|^q w(Q) \right]^{\frac{p}{q}} \left[ w\left( \bigcup_{Q\in\mathcal{T}(P)} Q \right) \right]^{1-\frac{p}{q}} \right\}^{\frac{1}{p}} \lesssim [w(P)]^{\frac{1}{p}-1} \|s\|_{\ell^q(d\nu)}.$$

Combining these estimates yields that $\|t\|_{\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)} \lesssim \|\ell\|_{(\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*}$, which further implies that $\dot{f}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w) \supset (\dot{f}^{\alpha}_{p,\,q}(A;\,w))^*$. This finishes the proof of Step 2.

**Step 3. Proof of** $(\dot f^{\alpha}_{p,q}(A;w))^* = \dot f^{-\alpha,\tau_0}_{\infty,\infty}(A;w)$ **for** $(p,q)\in(0,1]\times(0,1]$**.**

For any $(p,q)\in(0,1]\times(0,1]$ and $\alpha\in\mathbb{R}$, by (3.3), we obtain that $\dot f^{\alpha}_{p,q}(A;w)\subset \dot f^{\alpha}_{p,1}(A;w)$, and hence $(\dot f^{\alpha}_{p,1}(A;w))^*\subset(\dot f^{\alpha}_{p,q}(A;w))^*$. Thus, to prove that $(\dot f^{\alpha}_{p,q}(A;w))^*\supset \dot f^{-\alpha,\tau_0}_{\infty,\infty}(A;w)$, we only need to show $(\dot f^{\alpha}_{p,1}(A;w))^*\supset \dot f^{-\alpha,\tau_0}_{\infty,\infty}(A;w)$. For any $k\in\mathbb{Z}$ and $\alpha\in\mathbb{R}$, set

$$\Omega_k \equiv \left\{x\in\mathbb{R}^n:\ \sum_{Q\in\mathcal{Q}}|Q|^{-\alpha}s_Q\widetilde{\chi}_Q(x)>2^k\right\},$$

and $\widetilde{\Omega}_k$, $\mathcal{R}_k$ and $\widetilde{Q}$ as in Step 2. Then, by an argument similar to that of Step 2, we obtain $(\dot f^{\alpha}_{p,1}(A;w))^*\supset \dot f^{-\alpha,\tau_0}_{\infty,\infty}(A;w)$.

Conversely, notice that for any $\ell\in(\dot f^{\alpha}_{p,q}(A;w))^*$, $\ell$ must be of the form $\ell(s)\equiv\sum_{Q\in\mathcal{Q}}s_Q\bar t_Q$ for some $t\equiv\{t_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{C}$. Then, it suffices to prove that $\|t\|_{\dot f^{-\alpha,1/p-1}_{\infty,\infty}(A;w)}\lesssim\|\ell\|_{(\dot f^{\alpha}_{p,q}(A;w))^*}$. For any fixed $Q\in\mathcal{Q}$, define a sequence $s^Q\equiv\{(s^Q)_R\}_{R\in\mathcal{Q}}$ by $(s^Q)_R\equiv 1$ if $R=Q$ or else $(s^Q)_R\equiv 0$. Then, we have

$$\begin{aligned}
&\|t\|_{\dot f^{-\alpha,\frac1p-1}_{\infty,\infty}(A;w)}\\
&=\sup_{P\in\mathcal{Q}}\frac{1}{[w(P)]^{\frac1p-1}}\sup_{Q\in\mathcal{T}(P)}|Q|^{\alpha-\frac12}|t_Q|\frac{|Q|}{w(Q)}\\
&=\sup_{P\in\mathcal{Q}}\frac{1}{[w(P)]^{\frac1p-1}}\sup_{Q\in\mathcal{T}(P)}\left|\sum_{R\in\mathcal{Q}}|R|^{\alpha-\frac12}|t_R|(s^Q)_R\frac{|R|}{w(R)}\right|\\
&\le\|\ell\|_{(\dot f^{\alpha}_{p,q}(A;w))^*}\sup_{P\in\mathcal{Q}}\frac{1}{[w(P)]^{\frac1p-1}}\sup_{Q\in\mathcal{T}(P)}\left\|\left\{|R|^{\alpha+\frac12}(s^Q)_R[w(R)]^{-1}\right\}_{R\in\mathcal{Q}}\right\|_{\dot f^{\alpha}_{p,q}(A;w)}.
\end{aligned}$$

By (3.15), we know that for any $p\in(0,1]$ and fixed $P\in\mathcal{Q}$,

$$\begin{aligned}
&\sup_{Q\in\mathcal{T}(P)}\left\|\left\{|R|^{\alpha+\frac12}(s^Q)_R[w(R)]^{-1}\right\}_{R\in\mathcal{Q}}\right\|_{\dot f^{\alpha}_{p,q}(A;w)}\\
&\quad=\sup_{Q\in\mathcal{T}(P)}[w(Q)]^{\frac1p-1}\le\left[w\left(\bigcup_{Q\in\mathcal{T}(P)}Q\right)\right]^{\frac1p-1}\lesssim[w(P)]^{\frac1p-1}.
\end{aligned}$$

By this, we finally obtain $\|t\|_{\dot f^{-\alpha,1/p-1}_{\infty,\infty}(A;w)}\lesssim\|\ell\|_{(\dot f^{\alpha}_{p,q}(A;w))^*}$.

Combining Step 1 through Step 3, we obtain that the desired duality results for the discrete Triebel-Lizorkin spaces $\dot f^{\alpha}_{p,q}(A;w)$.

Applying Lemma 3.1 and similarly to the proof of [17, Theorem 5.13], we also obtain the corresponding duality results for Triebel-Lizorkin spaces, which completes the proof of Theorem 2.1. □

Next we give the proof of Corollary 2.1.

*Proof of Corollary 2.1.* Let $\alpha \in \mathbb{R}$, $q \in (1, \infty]$, $\tau > 1/q$ and $w \in \mathcal{A}_\infty(A)$. Define $p \in (0,1)$ such that $\tau = 1/p - 1$.

By [4, Theorem 4.2] we have that the dual of the space $\dot{f}^{-\alpha}_{p,\,q'}(A;\,w)$ can be identified with $\dot{f}^{\alpha}_{\infty,\,\infty}(A;\,w)$, albeit with a different pairing than the standard scalar product pairing (2.2). That is,

$$\lambda \equiv \{\lambda_Q\}_{Q\in\mathcal{Q}} \mapsto \langle \lambda, t\rangle_{w,p} \equiv \sum_{Q\in\mathcal{Q}} \lambda_Q \overline{t_Q} \frac{[w(Q)]^{\max(1,1/p)}}{|Q|}.$$

On the other hand, Theorem 2.1(i) states that the dual of $\dot{f}^{-\alpha}_{p,\,q'}(A;\,w)$ is $\dot{f}^{\alpha,\tau_0}_{q,q}(A;\,w)$, where $\tau_0 = 1/p + 1/q - 1 = \tau + 1/q$. Therefore, the spaces $\dot{f}^{\alpha}_{\infty,\,\infty}(A;\,w)$ and $\dot{f}^{\alpha,\tau+1/q}_{q,q}(A;\,w)$ are isomorphic. The isomorphism map is given by the multiplier operator $\{s_Q\}_{Q\in\mathcal{Q}} \mapsto \{\frac{[w(Q)]^{1/p}}{|Q|} s_Q\}_{Q\in\mathcal{Q}}$. Consequently, in the unweighted case $w \equiv 1$ we have the identification $\dot{f}^{\alpha+\tau}_{\infty,\,\infty}(A;\,w) = \dot{f}^{\alpha,\tau+1/q}_{q,q}(A;\,w)$. The same conclusions for the continuous spaces $\dot{F}^{\alpha,\tau}_{q,q}(A;\,w)$ follow as a consequence of Theorem 2.1. This finishes the proof of Corollary 2.1. □

We finally give the proof of Theorem 2.2.

*Proof of Theorem 2.2.* Let $\alpha \in \mathbb{R}$, $\tau_0 = 1/p + 1/q' - 1$ and $w \in \mathcal{A}_\infty(A)$.

Since the proof is similar to that of Theorem 2.1, we only prove Theorem 2.2 under the cases of $(p,\,q) \in (0,\,1] \times (1,\,\infty)$ and $(p,\,q) \in (1,\,\infty) \times (0,\,\infty)$.

**Step 1. Proof of** $(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^* = \dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)$ **for** $(p,\,q) \in (0,\,1] \times (1,\,\infty)$**.**

We first prove $(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^* \supset \dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)$. For any $t \in \dot{b}^{-\alpha,\,\tau_0}_{q',\,q'}(A;\,w)$, define a linear functional $\ell_t$ on $\dot{b}^{\alpha}_{p,\,q}(A;\,w)$ by $\ell_t(s) \equiv \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q$ for all $s \in \dot{b}^{\alpha}_{p,\,q}(A;\,w)$. We only need to show that $\|\ell_t\|_{(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*} \le \|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)}$.

For any $(p,\,q) \in (0,\,1] \times (1,\,\infty)$, by Hölder's inequality and (3.3), we have

$$\begin{aligned}
|\ell_t(s)| &\le \sum_{j\in\mathbb{Z}} \sum_{\mathrm{scale}\,(Q)=-j} |Q|^{-\alpha-\frac{1}{2}} |s_Q| [w(Q)]^{\frac{1}{p}} \frac{1}{[w(Q)]^{\frac{1}{p}-1}} |Q|^{\alpha-\frac{1}{2}} |t_Q| \frac{|Q|}{w(Q)} \\
&\le \sum_{j\in\mathbb{Z}} \sum_{\mathrm{scale}\,(Q)=-j} |Q|^{-\alpha-\frac{1}{2}} |s_Q| [w(Q)]^{\frac{1}{p}} \\
&\quad \times \left[\sup_{\mathrm{scale}\,(Q)=-j} |Q|^{\alpha-\frac{1}{2}} |s_Q| |Q| [w(Q)]^{-\frac{1}{p}}\right] \\
&\le \|s\|_{\dot{b}^{\alpha}_{p,\,q}(A;\,w)} \left\{\sum_{j\in\mathbb{Z}} \left[\sup_{\mathrm{scale}\,(Q)=-j} |Q|^{\alpha-\frac{1}{2}} |s_Q| |Q| [w(Q)]^{-\frac{1}{p}}\right]^{q'}\right\}^{\frac{1}{q'}} \\
&= \|s\|_{\dot{b}^{\alpha}_{p,\,q}(A;\,w)} \|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;w)},
\end{aligned}$$

which implies that $\|\ell_t\|_{(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*} \le \|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)}$.

Conversely, for any $\ell \in (\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*$, $\ell$ must be of the form $\ell(s) \equiv \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q$ for some $t \equiv \{t_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$. Then, it is left to show $\|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)} \lesssim \|\ell\|_{(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*}$. Notice that

$$\|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)} = \sup_{P\in\mathcal{Q}} \left\{ \sum_{j=-\operatorname{scale}(P)}^{\infty} \left[ \sup_{\operatorname{scale}(Q)=-j,\, Q\in\mathcal{T}(P)} |t_Q| \frac{|Q|^{\alpha+\frac12}}{[w(Q)]^{\frac1p}} \right]^{q'} \right\}^{\frac{1}{q'}}.$$

Since for any $P \in \mathcal{Q}$, there are finitely many cubes $Q$ in $\mathcal{T}(P)$ with $\operatorname{scale}(Q) = -j$, then for each $j \geq -\operatorname{scale}(P)$, there exists a cube $Q_j$ satisfying that $\operatorname{scale}(Q_j) = -j$ and $Q_j \in \mathcal{T}(P)$ such that

$$\sup_{\operatorname{scale}(Q)=-j,\, Q\in\mathcal{T}(P)} |t_Q| \frac{|Q|^{\alpha+1/2}}{[w(Q)]^{1/p}} = |t_{Q_j}| \frac{|Q_j|^{\alpha+1/2}}{[w(Q_j)]^{1/p}},$$

and hence

$$\begin{aligned}
\|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)} &= \sup_{P\in\mathcal{Q}} \left\{ \sum_{j=-\operatorname{scale}(P)}^{\infty} \left[ |t_{Q_j}| \frac{|Q_j|^{\alpha+\frac12}}{[w(Q_j)]^{\frac1p}} \right]^{q'} \right\}^{\frac{1}{q'}} \\
&= \sup_{P\in\mathcal{Q}} \sup_{\|\{s_{Q_j}\}_{j\geq-\operatorname{scale}(P)}\|_{\ell^q}\leq 1} \left| t_{Q_j} \frac{|Q_j|^{\alpha+\frac12}}{[w(Q_j)]^{\frac1p}} s_{Q_j} \right| \\
&\leq \|\ell\|_{(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*} \\
&\quad \times \sup_{P\in\mathcal{Q}} \sup_{\|\{s_{Q_j}\}_{j\geq-\operatorname{scale}(P)}\|_{\ell^q}\leq 1} \left\| \left\{ s_Q \frac{|Q|^{\alpha+\frac12}}{[w(Q)]^{\frac1p}} \right\}_{Q\in\{Q_j:\, j\geq-\operatorname{scale}(P)\}} \right\|_{\dot{b}^{\alpha}_{p,\,q}(A;\,w)}.
\end{aligned}$$

However,

$$\left\| \left\{ s_Q |Q|^{\alpha+1/2} [w(Q)]^{-1/p} \right\}_{Q\in\{Q_j:\, j\geq-\operatorname{scale}(P)\}} \right\|_{\dot{b}^{\alpha}_{p,\,q}(A;\,w)} \leq \|\{s_{Q_j}\}_{j\geq-\operatorname{scale}(P)}\|_{\ell^q}.$$

Thus, we obtain $\|t\|_{\dot{b}^{-\alpha,\,1/p}_{\infty,\,q'}(A;\,w)} \leq \|\ell\|_{(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*}$. This finishes the proof of Step 1.

**Step 2. Proof of $(\dot{b}^{\alpha}_{p,q}(A;\,w))^* = \dot{b}^{-\alpha,1}_{p',\infty}(A;\,w)$ for $(p,\,q) \in (1,\,\infty) \times (0,\,1]$.**

To show $\dot{b}^{-\alpha,1}_{p',\infty}(A;\,w) \subset (\dot{b}^{\alpha}_{p,q}(A;\,w))^*$, for any $t \in \dot{b}^{-\alpha,1}_{p',\infty}(A;\,w)$, we define a linear functional $\ell_t$ on $\dot{b}^{\alpha}_{p,q}(A;\,w)$ by $\ell_t(s) \equiv \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q$ for any $s \in \dot{b}^{\alpha}_{p,q}(A;\,w)$. We only need to prove that $\|\ell_t\|_{(\dot{b}^{\alpha}_{p,\,q}(A;\,w))^*} \leq \|t\|_{\dot{b}^{-\alpha,\,1}_{p',\,\infty}(A;\,w)}$. Indeed, for any $p \in (1,\,\infty)$ and $q \in (0,\,1]$, by Hölder's inequality and (3.3), we have

$$|\ell_t(s)| \leq \sum_{j\in\mathbb{Z}} \sum_{\operatorname{scale}(Q)=-j} |Q|^{-\alpha-\frac12} |s_Q| [w(Q)]^{\frac1p} |Q|^{\alpha-\frac12} |t_Q| \frac{|Q|}{w(Q)} [w(Q)]^{1-\frac1p}$$

$$\begin{aligned}
&\le \sum_{j\in\mathbb{Z}}\left[\sum_{\mathrm{scale}\,(Q)=-j}(|Q|^{-\alpha-\frac12}|s_Q|)^p w(Q)\right]^{\frac1p}\\
&\quad\times \sup_{j\in\mathbb{Z}}\left[\sum_{\mathrm{scale}\,(Q)=-j}\left(|Q|^{\alpha-\frac12}|t_Q|\frac{|Q|}{w(Q)}\right)^{p'} w(Q)\right]^{\frac1{p'}}\\
&\le \|s\|_{\dot b^{\alpha}_{p,\,q}(A;\,w)}\|t\|_{\dot b^{-\alpha,\,1}_{p',\,\infty}(A;\,w)},
\end{aligned}$$

which implies that $\|\ell_t\|_{(\dot b^{\alpha}_{p,\,q}(A;\,w))^*}\le \|t\|_{\dot b^{-\alpha,\,1}_{p',\,\infty}(A;\,w)}$.

Conversely, for any $\ell\in(\dot b^{\alpha}_{p,\,q}(A;\,w))^*$, $\ell$ must be of the form $\ell(s)\equiv\sum_{Q\in\mathcal{Q}}s_Q\bar t_Q$ for some $t\equiv\{t_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{C}$. Then, to complete the proof of Theorem 2.2, it suffices to show that

$$\|t\|_{\dot b^{-\alpha,\,1}_{p',\,\infty}(A;\,w)}\lesssim\|\ell\|_{(\dot b^{\alpha}_{p,\,q}(A;\,w))^*}.$$

Indeed, we have

$$\begin{aligned}
&\|t\|_{\dot b^{-\alpha,\,1}_{p',\,\infty}(A;\,w)}\\
&\quad= \sup_{j\in\mathbb{Z}}\left[\sum_{\mathrm{scale}\,(Q)=-j}\left(|Q|^{\alpha-\frac12}|t_Q|\frac{|Q|}{[w(Q)]^{\frac1p}}\right)^{p'}\right]^{\frac1{p'}}\\
&\quad= \sup_{j\in\mathbb{Z}}\sup_{\|\{(s_j)_Q\}_{Q\in\{Q\in\mathcal{Q}:\ \mathrm{scale}\,(Q)=-j\}}\|_{\ell^p}\le1}\left|\sum_{\mathrm{scale}\,(Q)=-j}|Q|^{\alpha-\frac12}\bar t_Q(s_j)_Q\frac{|Q|}{[w(Q)]^{\frac1p}}\right|\\
&\quad\le \|\ell\|_{(\dot b^{\alpha}_{p,\,q}(A;\,w))^*}\sup_{j\in\mathbb{Z}}\sup_{\|\{(s_j)_Q\}_{Q\in\{Q\in\mathcal{Q}:\ \mathrm{scale}\,(Q)=-j\}}\|_{\ell^p}\le1}\\
&\qquad\times\left\|\left\{|Q|^{\alpha-\frac12}(s_j)_Q\frac{|Q|}{[w(Q)]^{\frac1p}}\right\}_{Q\in\{Q\in\mathcal{Q}:\ \mathrm{scale}\,(Q)=-j\}}\right\|_{\dot b^{\alpha}_{p,\,q}(A;\,w)}\\
&\quad\le \|\ell\|_{(\dot b^{\alpha}_{p,\,q}(A;\,w))^*},
\end{aligned}$$

where

$$\left\|\{(s_j)_Q\}_{Q\in\{Q\in\mathcal{Q}:\ \mathrm{scale}\,(Q)=-j\}}\right\|_{\ell^p}\equiv\left[\sum_{\mathrm{scale}\,(Q)=-j}|(s_j)_Q|^p\right]^{1/p}.$$

From this, we deduce that $\|t\|_{\dot b^{-\alpha,\,1}_{p',\,\infty}(A;\,w)}\le\|\ell\|_{(\dot b^{\alpha}_{p,\,q}(A;\,w))^*}$ and then complete the proof for the case that $(p,q)\in(1,\infty]\times(0,1]$.

Now combining the existing proved results for discrete Besov spaces and Lemma 3.1, by a similar proof to that of [17, Theorem 5.13], we obtain the desired result for Besov spaces, which completes the proof of Theorem 2.2. □

## 4 Duality of Besov and Triebel-Lizorkin Spaces Associated with Doubling Measures

This section focuses on a more general setting involving anisotropic Besov and Triebel-Lizorkin spaces associated with $\rho_A$-doubling measures. We show that Theorems 2.1 and 2.2 are still true with $\mathcal{A}_\infty(A)$ weights replaced by $\rho_A$-doubling measures. Recall that $\rho_A$-doubling measures are first introduced in [2].

**Definition 4.1.** A non-negative Borel measure $\mu$ on $\mathbb{R}^n$ is called a *$\rho_A$-doubling measure* if there exists a nonnegative constant $\beta \equiv \beta(\mu)$ such that for all $x \in \mathbb{R}^n$ and $r > 0$, $\mu(B_{\rho_A}(x, br)) \le b^\beta \mu(B_{\rho_A}(x, r))$.

We point out that for any $w \in \mathcal{A}_\infty(A)$, $d\mu(x) \equiv w(x)\,dx$ (with respect to a quasi-distance $\rho_A$) also defines a $\rho_A$-doubling measure albeit with a positive constant $C$ (see [7, Proposition 2.6(i)]).

**Definition 4.2.** Let $\alpha \in \mathbb{R}$, $p, q \in (0,\infty)$ and $\mu$ be a $\rho_A$-doubling measure. The *spaces* $\dot{A}^{\alpha}_{p,q}(A;\mu)$ and $\dot{A}^{\alpha,\tau}_{p,q}(A;\mu)$, and their corresponding sequence spaces are defined as in Definitions 2.4 and 2.5 with $w(x)\,dx$, $w(P)$ and $w(Q)$ replaced, respectively, by $d\mu(x)$, $\mu(P)$ and $\mu(Q)$.

The spaces $\dot{F}^{\alpha}_{p,\,q}(A;\,\mu)$, $\dot{B}^{\alpha}_{p,\,q}(A;\,\mu)$ and their corresponding sequences spaces mentioned above were introduced, respectively, in [3] and [2].

Similarly to the proof of Theorems 2.1 and 2.2, we have the following conclusion.

**Theorem 4.1.** *Let $\mu$ be a $\rho_A$-doubling measure, $p$, $q$, $\tau_0$ and $\alpha$ the same as in Theorems 2.1 and 2.2. Then Theorems 2.1 and 2.2 still hold with those mentioned spaces associated $\mathcal{A}_\infty$ weight replaced by the corresponding spaces associated with $\rho_A$-doubling measures as in Definition 4.2.*

To prove Theorem 4.1, we first point out that, with a similar proof, Lemma 3.1 is also true for the spaces associated with $\rho_A$-doubling measures. With this, the proof of Theorem 4.1 is nearly the same as those of Theorems 2.1 and 2.2. We give some details for the special cases for the reader's convenience to show their differences.

*Proof of Theorem 4.1.* Since the proof is nearly verbatim repetition of the proofs of Theorems 2.1 and 2.2, we only give details for $(\dot{f}^{\alpha}_{p,\,q}(A;\,\mu))^* = \dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,\mu)$ in the case $(p,\,q) \in (1,\,\infty) \times (0,\,\infty)$.

We first prove that $\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,\mu) \subset (\dot{f}^{\alpha}_{p,\,q}(A;\,\mu))^*$. For any $t \equiv \{t_Q\}_{Q\in\mathcal{Q}} \in \dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,\mu)$, define a linear functional $\ell_t$ on $\dot{f}^{\alpha}_{p,\,q}(A;\,\mu)$ by $\ell_t(s) \equiv \sum_{Q\in\mathcal{Q}} s_Q \bar{t}_Q$ for all $s \in \dot{f}^{\alpha}_{p,\,q}(A;\,\mu)$. By applying Hölder's inequality twice for $q \ge 1$, while for $q \in (0,\,1)$, using the imbedding $\dot{f}^{\alpha}_{p,\,q}(A;\,\mu) \to \dot{f}^{\alpha}_{p,\,1}(A;\,\mu)$, we have

$$\begin{aligned}|\ell_t(s)| &\le \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} |Q|^{-\alpha}|s_Q|\widetilde{\chi}_Q(x)|Q|^{\alpha}|t_Q|\frac{|Q|}{\mu(Q)}\widetilde{\chi}_Q(x)\,d\mu(x)\\ &\le \|s\|_{\dot{f}^{\alpha}_{p,\,q}(A;\,\mu)}\|t\|_{\dot{f}^{-\alpha,\,0}_{p',\,q'}(A;\,\mu)},\end{aligned}$$

which yields $\|\ell_t\|_{(\dot f^{\alpha}_{p,q}(A;\,\mu))^*} \le \|t\|_{\dot f^{-\alpha,0}_{p',q'}(A;\,\mu)}$, and hence $\dot f^{-\alpha,0}_{p',q'}(A;\,\mu) \subset (\dot f^{\alpha}_{p,q}(A;\,\mu))^*$.

Conversely, since sequences with finite support are dense in $\dot f^{\alpha}_{p,q}(A;\,\mu)$, each bounded linear functional $\ell \in (\dot f^{\alpha}_{p,q}(A;\,\mu))^*$ must be of the form $\ell(s) \equiv s_Q\bar t_Q$ for some $t \equiv \{t_Q\}_{Q\in\mathcal{Q}} \subset \mathbb{C}$. It suffices to show that $\|t\|_{\dot f^{-\alpha,0}_{p',q'}(A;\,\mu)} \lesssim \|l\|_{(\dot f^{\alpha}_{p,q}(A;\,\mu))^*}$.

Let $L^p_\mu(\ell^q)$ be the *space of all sequences* $\{f_j\}_{j\in\mathbb{Z}}$ *of functions on* $\mathbb{R}^n$ such that

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{L^p_\mu(\ell^q)} \equiv \left\|\left\{\sum_{j\in\mathbb{Z}} |f_j|^q\right\}^{1/q}\right\|_{L^p_\mu(\mathbb{R}^n)} < \infty.$$

We know that $(L^p_\mu(\ell^q))^* = L^{p'}_\mu(\ell^{q'})$ for $p \in (1,\infty)$ and $q \in (0,\infty)$. Notice that the map $\mathrm{In} : \dot f^{\alpha}_{p,q}(A;\,w) \to L^p_\mu(\ell^q)$ defined by $\mathrm{In}(s) \equiv \{f_j\}_j$, where $f_j \equiv \sum_{\mathrm{scale}\,(Q)=-j} |Q|^{-\alpha} s_Q \widetilde{\chi}_Q$ for all $j \in \mathbb{Z}$, is a linear isometry onto a subspace of $L^p_\mu(\ell^q)$.

When $p \in (1,\infty)$ and $q \in [1,\infty)$, by the Hahn-Banach theorem, there exists an $\widetilde{\ell} \in (L^p_\mu(\ell^q))^*$ with $\|\widetilde{\ell}\|_{(\dot f^{\alpha}_{p,q}(A;\,\mu))^*} = \|\ell\|_{(\dot f^{\alpha}_{p,q}(A;\,\mu))^*}$ such that $\widetilde{\ell} \circ \mathrm{In} = \ell$. In other words, there exists $g \equiv \{g_j\}_{j\in\mathbb{Z}} \in L^{p'}_\mu(\ell^{q'})$ with $\|g\|_{L^{p'}_\mu(\ell^{q'})} \le \|\ell\|_{(\dot f^{\alpha}_{p,q}(A;\,\mu))^*}$ such that for all $s \in \dot f^{\alpha}_{p,q}(A;\,\mu)$,

$$\sum_{Q\in\mathcal{Q}} s_Q\bar t_Q = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} f_j(s)\overline{g_j(x)}\,d\mu(x). \tag{4.1}$$

By taking $s_Q = 0$ for all but one dilated cube, we obtain $t_Q = \int_Q |Q|^{-\alpha-1/2} g_j(x)\,d\mu(x)$ for all cubes $Q$ with $\mathrm{scale}\,(Q) = -j$.

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\,\mu)$, define the anisotropic Hardy-Littlewood maximal function of $f$ by $M_\mu(f)(x) \equiv \sup_{x\in Q\in\mathcal{Q}} \frac{1}{\mu(Q)}\int_Q |f(y)|\,d\mu(y)$. Then by [4, Proposition 4.3], we know that the vector-valued maximal inequality holds for $M_\mu$. Then, by (4.1) and the vector-valued maximal inequality for $M_\mu$ with $p' \in (1,\infty)$ and $q' \in (1,\infty]$, we obtain that

$$\|t\|_{\dot f^{-\alpha,0}_{p',q'}(A;\,\mu)} \le \|\{M_\mu(g_j)\}_{j\in\mathbb{Z}}\|_{L^{p'}_\mu(\ell^{q'})} \lesssim \|g\|_{L^{p'}_\mu(\ell^{q'})} \lesssim \|\ell\|_{(\dot f^{\alpha}_{p,q}(A;\,\mu))^*},$$

which implies that $(\dot f^{\alpha}_{p,q}(A;\,\mu))^* \subset \dot f^{-\alpha,0}_{p',q'}(A;\,\mu)$.

When $p \in (1,\infty)$ and $q \in (0,1)$, similarly to Step 1 of the proof for Theorem 2.1, we also obtain $(\dot f^{\alpha}_{p,q}(A;\,\mu))^* \subset \dot f^{-\alpha,0}_{p',q'}(A;\,\mu)$, and hence complete the proof of Theorem 4.1. □

We point out that when dilation $A$ admits a *Meyer-type wavelet* $\Psi$, Bownik [4, Theorem 4.10] determined the dual spaces of Triebel-Lizorkin spaces with $\rho_A$ doubling measures under the pairing

$$\langle f, g\rangle \equiv \sum_{\psi\in\Psi}\sum_{Q\in\mathcal{Q}} \langle f,\,\psi_Q\rangle\langle\psi_Q,\,g\rangle \frac{[\mu(Q)]^{\max\{1,\,\frac{1}{p}\}}}{|Q|}. \tag{4.2}$$

Theorem 4.1 identifies the dual spaces of $\dot{F}^{\alpha}_{p,q}(A;\,\mu)$ for arbitrary dilation $A$ under the pairing $\langle f,\, g\rangle \equiv \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$. Since the pairing used in [4] and here are different, the dual spaces also appears differently. A question posed in [4, p. 155] asks whether the duality (4.2) holds without the assumption on the existence of Meyer-type wavelets. While our paper does not answer this question it provides the duality result without this extra assumption. The duality in Theorem 4.1 for anisotropic Besov spaces with doubling measures are new.

As an application of Theorem 4.1, let us discuss a particular class of Triebel-Lizorkin spaces associated with Hardy spaces. There are several equivalent definitions of Hardy spaces. The weighted Hardy spaces associated with $\mathcal{A}_\infty(A)$ defined via maximal functions or atomic decomposition were studied in [7]. We define two kinds of Hardy spaces with $\rho_A$-doubling measure $\mu$ via Littlewood-Paley $g$-function spaces and via square function.

**Definition 4.3.** Let $p\in(0,\,\infty)$, $\varphi\in\mathscr{S}_\infty(\mathbb{R}^n)$ and $\mu$ be a $\rho_A$-doubling measure.

(i) Define the *anisotropic Hardy space* $\widetilde{H}^p(A;\,\mu)$ with a $\rho_A$-doubling measure $\mu$ via Littlewood-Paley $g$-function by

$$\widetilde{H}^p(A;\,\mu)\equiv\left\{f\in\mathcal{S}'(\mathbb{R}^n):\,\|f\|_{\widetilde{H}^p(A;\,\mu)}\equiv\|g_\varphi(f)\|_{L^p_\mu(\mathbb{R}^n)}<\infty\right\},$$

where the *anisotropic Littlewood-Paley g-function* of $f$ is defined by

$$g_\varphi(f)\equiv\left(\sum_{j\in\mathbb{Z}}|\varphi_j*f|^2\right)^{1/2};$$

(ii) Define the *anisotropic Hardy space* $H^p(A;\,\mu)$ with a $\rho_A$-doubling measure $\mu$ via the square function by

$$H^p(A;\,\mu)\equiv\left\{f\in\mathcal{S}'(\mathbb{R}^n):\,\|f\|_{H^p(A;\,\mu)}\equiv\|S_\varphi(f)\|_{L^p_\mu(\mathbb{R}^n)}<\infty\right\},$$

where for any $x\in\mathbb{R}^n$, the *anisotropic square function* of $f$ is defined by

$$S_\varphi(f)(x)\equiv\left\{\sum_{k\in\mathbb{Z}}b^k\int_{B_\rho(x,b^{-k})}|f*\varphi_k(y)|^2\,dy\right\}^{1/2}.$$

**Corollary 4.1.** Let $p\in(0,\,\infty)$ and $\mu$ be a $\rho_A$-doubling measure. Then,

(i) $H^p(A;\,\mu)=\widetilde{H}^p_w(A;\,\mu)=\dot{F}^0_{p,\,2}(A;\,\mu)$ with equivalent norms;

(ii) $(H^p(A;\,\mu))^*=\dot{F}^{0,\,1/p-1/2}_{2,\,2}(A;\,\mu)$ in the sense of (2.3).

A weighted anisotropic product version of the first equality in Corollary 4.1(i) has been obtained in [27, Theorem 2.2]. With an obvious modification on its proof therein, namely, via replacing the weighted product measure in the proof of [27, Theorem 2.2] by $d\mu(x)$ here and then repeating the proof therein, we have $H^p(A;\,\mu)=\widetilde{H}^p_w(A;\,\mu)$ with equivalent norms. The spaces $\widetilde{H}^p(A;\,\mu)=\dot{F}^0_{p,\,2}(A;\,\mu)$ with equivalent norms follows directly from their definitions, which completes the proof of Corollary 4.1(i). Corollary 4.1(ii) is a simple corollary of Corollary 4.1(i) and Theorem 4.1. We omit the details.

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## References

[1] M. Bownik, Anisotropic Hardy spaces and wavelets, Mem. Amer. Math. Soc. 164 (2003), no. 781, vi+122 pp.

[2] M. Bownik, Atomic and molecular decompositions of anisotropic Besov spaces, Math. Z. 250 (2005), 539-571.

[3] M. Bownik, Anisotropic Triebel-Lizorkin spaces with doubling measures, J. Geom. Anal. 17 (2007), 387-424.

[4] M. Bownik, Duality and interpolation of anisotropic Triebel-Lizorkin spaces, Math. Z. 259 (2008), 131-169.

[5] M. Bownik and K.-P. Ho, Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces, Tran. Amer. Math. Soc. 358 (2006), 1469-1510.

[6] M. Bownik and J. Lemvig, Affine and quasi-affine frames for rational dilations, Tran. Amer. Math. Soc. (to appear).

[7] M. Bownik, B. Li, D. Yang and Y. Zhou, Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators, Indiana Univ. Math. J. 57 (2008), 3065-3100.

[8] M. Bownik, B. Li, D. Yang and Y. Zhou, Weighted anisotropic product Hardy spaces and boundedness of sublinear operators, Math. Nachr. 283 (2010), 392-442.

[9] H. Q. Bui, Weighted Besov and Triebel spaces: interpolation by the real method, Hiroshima Math. J. 12 (1982), 581-605.

[10] A.-P. Calderón, An atomic decomposition of distributions in parabolic $H^p$ spaces, Adv. Math. 25 (1977), 216-225.

[11] A.-P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. Math. 16 (1975), 1-64.

[12] A.-P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution. II, Adv. Math. 24 (1977), 101-171.

[13] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569-645.

[14] M. Essén, S. Janson, L. Peng and J. Xiao, $Q$ spaces of several real variables, Indiana Univ. Math. J. 49 (2000), 575-615.

[15] W. Farkas, Atomic and subatomic decompositions in anisotropic function spaces, Math. Nachr. 209 (2000), 83-113.

[16] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Group, Mathematical Notes, 28. Princeton University Press, Princeton, N. J., 1982.

[17] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. 93 (1990), 34-170.

[18] M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley theory and the study of function spaces, CBMS Regional Conference Series in Mathematics, 79. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991.

[19] M. Frazier and S. Roudenko, Matrix-weighted Besov spaces and conditions of $A_p$ type for $0 < p \leq 1$, Indiana Univ. Math. J. 53 (2004), 1225-1254.

[20] J. García-Cuerva, Weighted $H^p$ spaces, Dissertationes Math. (Rozprawy Mat.) 162 (1979), 1-63.

[21] J. García-Cuerva and J. M. Martell, Wavelet characterization of weighted spaces, J. Geom. Anal. 11 (2001), 241-264.

[22] L. Grafakos, Modern Fourier Analysis, Second Edition, Graduate Texts in Math., 250, Springer Press, New York, 2008.

[23] D. D. Haroske and I. Piotrowska, Atomic decompositions of function spaces with Muckenhoupt weights, and some relation to fractal analysis, Math. Nachr. 281 (2008), 1476-1494.

[24] D. D. Haroske and L. Skrzypczak, Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights, I, Rev. Mat. Complut. 21 (2008), 135-177.

[25] D. D. Haroske and E. Tamási, Wavelet frames for distributions in anisotropic Besov spaces, Georgian Math. J. 12 (2005), 637-658.

[26] D. Krug, A weighted version of the atomic decomposition for $H^p$ (bi-halfspace), Indiana Univ. Math. J. 37 (1988), 277-300.

[27] B. Li, M. Bownik and D. Yang, Littlewood-Paley characterization and duality of weighted anisotropic product Hardy spaces, Submitted.

[28] M. Lee, C. Lin and Y. Lin, A wavelet characterization for the dual of weighted Hardy spaces, Proc. Amer. Math. Soc. 137 (2009), 4219-4225.

[29] S. Roudenko, Duality of matrix-weighted Besov spaces, Studia Math. 160 (2004), 129-156.

[30] H.-J. Schmeisser and H. Triebel, Topics in Fourier Analysis and Function Spaces, John Wiley & Sons Ltd., Chichester, 1987.

[31] Y. Sawano, D. Yang and W. Yuan, New applications of Besov-type spaces and Triebel-Lizorkin-type spaces, J. Math. Anal. Appl. 363 (2010), 73-85.

[32] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, N. J., 1971.

[33] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.

[34] H. Triebel, Theory of Function Spaces II, Birkhäuser Verlag, Basel, 1992.

[35] H. Triebel, Theory of Function Spaces III, Birkhäuser Verlag, Basel, 2006.

[36] I. Verbitsky, Weighted norm inequalities for maximal operators and Pisier's theorem on factorization through $L^{p\infty}$, Integr. Equ. Oper. Theory 15 (1992), 124-153.

[37] I. Verbitsky, Imbedding and multiplier theorems for discrete Littlewood-Paley spaces, Pacific J. Math. 176 (1996), 529-556.

[38] J. Vybíral, Function spaces with dominating mixed smoothness, Dissertationes Math. (Rozprawy Mat.) 436 (2006), 1-73.

[39] J. Vybíral and W. Sickel, Traces of functions with a dominating mixed derivative in $\mathbb{R}^3$, Czechoslovak Math. J. 57 (132) (2007), 1239-1273.

[40] W. Wang, A discrete transform and Triebel-Lizorkin spaces on the bidisc, Trans. Amer. Math. Soc. 347 (1995), 1351-1364.

[41] S. Wu, A wavelet characterization for weighted Hardy spaces, Rev. Mat. Iberoamericana 8 (1992), 329-349.

[42] D. Yang and W. Yuan, A new class of function spaces connecting Triebel-Lizorkin spaces and $Q$ spaces, J. Funct. Anal. 255 (2008), 2760-2809.

[43] D. Yang and W. Yuan, New Besov-type spaces and Triebel-Lizorkin-type spaces including $Q$ spaces, Math. Z. 265 (2010), 451-480.

[44] W. Yuan, W. Sickel and D. Yang, Morrey and Campanato Meet Besov, Lizorkin and Triebel, Lecture Notes in Mathematics 2005, Springer-Verlag, Berlin, 2010.

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