Littlewood-Paley Characterization and Duality of Weighted Anisotropic Product Hardy Spaces

Baode Li, Marcin Bownik and Dachun Yang *

Abstract. The authors study anisotropic product Hardy spaces $H^p_w(\vec{A})$ associated with a pair $\vec{A} \equiv (A_1, A_2)$ of expansive dilations and a class of product Muckenhoupt weights $\mathcal{A}_\infty(\vec{A})$ on $\mathbb{R}^n \times \mathbb{R}^m$. This paper is a continuation of their earlier work in [Math. Nachr. 283 (2010), 392-442]. The authors establish the Littlewood-Paley g-function characterization and φ-transform characterization of $H^p_w(\vec{A})$, $0 < p \le 1$. The authors also introduce the weighted anisotropic product Campanato space $\mathcal{L}_{p,\,w}(\vec{A})$ and establish its φ-transform characterization. As an application, the authors identify the dual space of $H^p_w(\vec{A})$ with $\mathcal{L}_{p,\,w}(\vec{A})$. The results of this paper improve the existing results for weighted product Hardy spaces on $\mathbb{R} \times \mathbb{R}$ and are new even in weighted isotropic setting.

1 Introduction

Due to the request in applications of analysis such as PDEs, harmonic analysis and approximation theory, there were several efforts of extending classical function spaces arising in harmonic analysis from Euclidean spaces to other domains and anisotropic settings; see, for example, [1, 2, 3, 10, 15, 11, 16, 20, 31, 35, 36, 37, 24, 38, 39]. Calderón and Torchinsky initiated the study of Hardy spaces associated with anisotropic dilations in [10, 11, 9]. Recently, a theory of anisotropic Hardy spaces and their weighted theory were developed by Bownik et al in [1, 7].

Another direction is the development of the theory of Hardy spaces on product domains initiated by Gundy and Stein [23]. In particular, Chang and Fefferman [12, 13] characterized the classical product Hardy spaces via atoms. Fefferman [19], Krug [26] and Zhu [43] established the weighted theory of the classical product Hardy spaces, and Sato [29, 30] established parabolic Hardy spaces on product domains. It was also proved that the classical product Hardy spaces are good substitutes of product Lebesgue spaces when $p \in (0, 1]$; see, for example, [17, 18, 19, 30, 32].

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* Corresponding author.

Let $\vec{A} \equiv (A_1, A_2)$ be a pair of expansive dilations and $\mathcal{A}_\infty(\vec{A})$ the corresponding class of product Muckenhoupt weights on $\mathbb{R}^n \times \mathbb{R}^m$ (see Definition 2.4 below). Recently, a theory of the weighted anisotropic product Hardy spaces $H^p_w(\vec{A})$ associated with expansive dilations and product Muckenhoupt weights was established in [8]. In particular, the Hardy spaces $H^p_w(\vec{A})$ were characterized in terms of the Lusin-area function and the atomic decompositions. Moreover, the boundedness on $H^p_w(\vec{A})$ was obtained in [28] for a class of anisotropic singular integrals on $\mathbb{R}^n \times \mathbb{R}^m$, whose kernels are adapted to $\vec{A}$ in the sense of Bownik (see [1]) and have vanishing moments defined via bump functions in the sense of Stein (see [33]).

In this paper we continue our study by establishing Littlewood-Paley characterization and duality of weighted anisotropic product Hardy spaces. Our first result (see Proposition 2.1 below) shows the equivalence of the Lusin-area function definition of the space $H^p_w(\vec{A})$ for tempered distributions in $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ with tempered distributions in $\mathcal{S}'_{\infty,w}(\mathbb{R}^n \times \mathbb{R}^m)$ vanishing weakly at infinity. Here, $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is the dual space of the set of all Schwartz functions with all vanishing moments (see Section 2 below). This seemingly inconsequential result enables us to establish the φ-transform characterization (see Theorem 2.1 below) and the Littlewood-Paley g-function characterization (see Theorem 2.2 below) of the Hardy space $H^p_w(\vec{A})$. We also introduce the weighted anisotropic product Campanato space $\mathcal{L}_{p,w}(\vec{A})$ (see Definition 2.8 below) and establish its φ-transform characterization (see also Theorem 2.1 below). In the final part of the paper, we identify the dual space of $H^p_w(\vec{A})$ with $\mathcal{L}_{p,w}(\vec{A})$ in Theorem 2.3 below. This improves the result of Krug and Torchinsky [27] which describes the duals of the classical weighted product Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ when the weights w satisfy Muckenhoupt's $A_r(\mathbb{R} \times \mathbb{R})$ condition on rectangles and $2/r < p \le 1$. Moreover, the dual spaces in [27] have quite different description from $\mathcal{L}_{p,w}(\vec{A})$, and the method employed by Krug and Torchinsky [27] is based on the atomic decomposition characterization of $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$.

To achieve our targets, one key tool is the discrete Calderón reproducing formulae (see Lemma 2.1 below), which is a discrete variant, via dilated cubes introduced by Bownik-Ho [5], of the Calderón reproducing formulae in [8, Proposition 2.16]. Motivated by Frazier-Jawerth [21], Bownik [2] and Bownik-Ho [5], to obtain the Littlewood-Paley g-function characterization of $H^p_w(\vec{A})$, we invoke a weighted anisotropic product variant of the Plancherel-Pôlya inequality (see Lemmas 3.9(i) and 3.11(i) below) and the boundedness of the almost diagonal operators on the discrete weighted anisotropic product Hardy space (see Lemma 3.13 below).

Notice that the φ-transform characterization of $\mathcal{L}_{p,w}(\vec{A})$ closely connects with dilated cubes of Bownik-Ho associated to A_1 and A_2. Although dilated cubes nicely reflect the properties of expansive dilations, they have a critical defect, that is, dilated cubes with different levels have no nested property, which makes it impossible to establish Journé's covering lemma for these dilated cubes. To overcome this difficulty, we invoke the dyadic cubes of Christ [14] for general spaces of homogeneous type in the sense of Coifman and Weiss [15]. To be precise, we establish some subtle relations in Lemma 3.7 below between dilated cubes and dyadic cubes, which further induce some important relations between the sequence spaces $\ell_{p,w}(\vec{A})$ defined via dilated cubes and $\dot{\ell}_{p,w}(\vec{A})$ defined via dyadic cubes (see Lemma 3.8 below). Applying the nested property of dyadic cubes of Christ (see also Lemma 3.5 below) and Journé's covering lemma in [8, Lemma 4.9], we establish a weighted anisotropic product variant of the Plancherel-Pôlya inequality on $\ell_{p,\,w}(\vec{A})$ (see Lemmas 3.9(ii) and 3.11(ii) below), which together with some standard argument (see, for example, the proof of Bownik [2, Theorem 3.12]) yields the φ-transform characterization of $\ell_{p,\,w}(\vec{A})$. Applying the φ-transform characterizations of $H^p_w(\vec{A})$ and $\mathcal{L}_{p,\,w}(\vec{A})$ together with some ideas from Wang [40] and Frazier-Jawerth [21], we then prove that the dual space of $H^p_w(\vec{A})$ is just $\mathcal{L}_{p,\,w}(\vec{A})$. We particularly point out that Lemma 3.9 below plays a key role, whose proof is quite geometrical in the sense that we prove this lemma via subtly classifying the dyadic cubes of Christ in [14] (see also Lemma 3.5 below) and its associated Journé's covering lemma in [8, Lemma 4.9].

The main results of this paper are stated in Section 2 and their proofs are given in Section 3 below.

Finally, we make some conventions on symbols. Throughout the paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. The *symbol* $A \lesssim B$ means that $A \le CB$ and the *symbol* $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. Denote by $\sharp E$ the *cardinality of the set* E. For any $p \in [1,\,\infty]$, we denote by p' its *conjugate index*, namely, $1/p + 1/p' = 1$. We also set $\mathbb{N} \equiv \{1,\,2,\,\cdots\}$, $\mathbb{Z}_+ \equiv \{0\} \cup \mathbb{N}$ and $\mathbb{Z}^n_+ = (\mathbb{Z}_+)^n$. For any $a,\,b \in \mathbb{R}$, we denote $\min\{a,\,b\}$ and $\max\{a,\,b\}$, respectively, by $a \wedge b$ and $a \vee b$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E the *characteristic function* of E. For any multi-index $\gamma \equiv (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n_+$, let $|\gamma| \equiv \gamma_1 + \cdots + \gamma_n$, and $\partial^\gamma \equiv (\frac{\partial}{\partial \xi_1})^{\gamma_1} \cdots (\frac{\partial}{\partial \xi_n})^{\gamma_n}$. We also denote $(\overbrace{0,\,\cdots,\,0}^{n \text{ times}})$ by the *symbol* $\vec{0}_n$.

2 Main Results

We begin with some notions. Let $m,\,n \in \mathbb{Z}$. In what follows, for convenience, we often let $n_1 \equiv n$ and $n_2 \equiv m$.

Definition 2.1. Let $i = 1,\,2$. A real $n_i \times n_i$ matrix A_i is called an *expansive dilation*, shortly a *dilation*, if $\max_{\lambda_i \in \sigma(A_i)} |\lambda_i| > 1$, where $\sigma(A_i)$ is the *set of all eigenvalues of* A_i. A *quasi-norm* associated with expansive matrix A_i is a Borel measurable mapping $\rho_{A_i} : \mathbb{R}^{n_i} \to [0, \infty)$, for simplicity, denoted as ρ_i, such that

(i) $\rho_i(x_i) > 0$ for $x_i \neq 0$;

(ii) $\rho_i(A_i x_i) = b_i \rho_i(x_i)$ for $x_i \in \mathbb{R}^{n_i}$, where $b_i \equiv |\det A_i|$;

(iii) $\rho_i(x_i + y_i) \le H_i\,[\rho_i(x_i) + \rho_i(y_i)]$ for all $x_i,\,y_i \in \mathbb{R}^{n_i}$, where $H_i \ge 1$ is a constant.

Throughout the whole paper, we always let A_1 and A_2 be expansive dilations, respectively, on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, and ρ_1 and ρ_2 the corresponding quasi-norms. Such ρ_1 and ρ_2 indeed exist; see [1, p. 8]. Let $i = 1,\,2$. The set $\mathcal{Q}_i$ of *dilated cubes* of $\mathbb{R}^{n_i}$ is defined by $\mathcal{Q}_i \equiv \{Q_i \equiv A_i^{j_i}([0,\,1)^{n_i} + k_i) :\ j_i \in \mathbb{Z},\ k_i \in \mathbb{Z}^{n_i}\}$. For any $Q_i \equiv A_i^{j_i}([0,\,1)^{n_i} + k_i)$, let $x_{Q_i} \equiv A_i^{j_i} k_i$ be the "*lower-left corner*" of Q_i. It is easy to see that for any fixed $j_i \in \mathbb{Z}$, $\{Q_i = A_i^{j_i}([0,\,1)^{n_i} + k_i) :\ k_i \in \mathbb{Z}^{n_i}\}$ is a partition of $\mathbb{R}^{n_i}$. Denoted by $\mathcal{R} \equiv \mathcal{Q}_1 \times \mathcal{Q}_2$ the *set of all dilated rectangles.*

For any function $\varphi^{(i)}$ on $\mathbb{R}^{n_i}$, φ on $\mathbb{R}^n \times \mathbb{R}^m$, $j_i \in \mathbb{Z}$, $k_i \in \mathbb{Z}^{n_i}$, $Q_i \equiv A_i^{-j_i}([0, 1)^{n_i} + k_i)$ and $Q \equiv Q_1 \times Q_2$, let $\varphi_{j_i}^{(i)}(x_i) \equiv b_i^{j_i} \varphi^{(i)}(A_i^{j_i} x_i)$ for all $x_i \in \mathbb{R}^{n_i}$,

$$\varphi_{j_1, j_2}(x) \equiv b_1^{j_1} b_2^{j_2} \varphi(A_1^{j_1} x_1, A_2^{j_2} x_2) \quad \text{for all} \quad x \equiv (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m,$$

and, correspondingly,

$$\varphi_{Q_i}^{(i)}(x_i) \equiv |Q_i|^{\frac{1}{2}} \varphi_{j_i}^{(i)}(x_i - x_{Q_i}), \ \varphi_Q(x) \equiv |Q|^{\frac{1}{2}} \varphi_{j_1, j_2}(x_1 - x_{Q_1}, x_2 - x_{Q_2}), \tag{2.1}$$

where $|\cdot|$ means the *Lebesgue measure* on $\mathbb{R}^{n_i}$ or $\mathbb{R}^n \times \mathbb{R}^m$, respectively.

Denote by $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ the *set of all Schwartz functions on* $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ its *topological dual space*. As in [22], we set

$$\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m) \equiv \left\{ \phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m) : \int_{\mathbb{R}^{n_i}} \phi(x_1, x_2) x_i^{\alpha_i} \, dx_i = 0, \ \alpha_i \in \mathbb{Z}_+^{n_i}, \ i = 1, 2 \right\}.$$

We consider $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ as a subspace of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, including the topology. Thus, $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is a *complete metric space* (see, for example, [34, p. 21, (3.7)]). Equivalently, $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ can be defined as a set of $\phi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ such that the *semi-norms*

$$\|\phi\|_M^* \equiv \sup_{|\gamma| \le M} \sup_{\xi \in \mathbb{R}^n \times \mathbb{R}^m} |\partial^\gamma \widehat{\phi}(\xi)| \prod_{i=1}^{2} (|\xi_i|^M + |\xi_i|^{-M}) < \infty$$

for all $M \in \mathbb{Z}_+$ (see [5, p. 1479]). The semi-norms $\{\|\cdot\|_M^*\}_{M \in \mathbb{Z}_+}$ generate a topology of a locally convex space on $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ which coincides with the topology of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ as a subspace of a locally convex space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Let $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ be the *topological dual space* of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ with the weak-$*$ topology.

For any $N \in \mathbb{Z}_+$, let $\mathcal{S}_N(\mathbb{R}^n)$ be the *set of all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying* $\int_{\mathbb{R}^n} \varphi(x) x^\alpha \, dx = 0$ *for any* $\alpha \in \mathbb{Z}_+^n$ *with* $|\alpha| \le N$. Given two functions $\phi^{(i)}$ on $\mathbb{R}^{n_i}$, $i = 1, 2$, define $\phi \equiv \phi^{(1)} \otimes \phi^{(2)}$ by $\phi(x_1, x_2) \equiv \phi^{(1)}(x_1)\phi^{(2)}(x_2)$ for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Recall $\vec{0}_{n_i} \equiv (\overbrace{0, \cdots, 0}^{n_i \text{ times}})$.

Definition 2.2. Let $\mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ be the *set of all functions of the form* $\varphi \equiv \varphi^{(1)} \otimes \varphi^{(2)}$ *with* $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$, *such that*

(i) $\operatorname{supp} \widehat{\varphi^{(i)}} \subset [-\pi, \pi]^{n_i} \setminus \{\vec{0}_{n_i}\}$, *and*

(ii) $\sup_{j \in \mathbb{Z}} |\widehat{\varphi^{(i)}}((A_i^*)^j \xi_i)| > 0$ for all $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$, where A_i^* is the *transpose* of A_i.

Suppose that $\varphi, \psi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. The pair (φ, ψ) is called an *admissible pair of dual frame wavelets* if, in addition to (i) and (ii),

(iii) $\sum_{j \in \mathbb{Z}} \overline{\widehat{\varphi^{(i)}}((A_i^*)^j \xi_i)} \widehat{\psi^{(i)}}((A_i^*)^j \xi_i) = 1$ for all $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$.

We should point out that such φ and ψ indeed exist. Indeed, by [5, Lemma 3.6], for any $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, there exists some $\psi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that (φ, ψ) is an admissible pair of dual frame wavelets.

The following Calderón reproducing formulae are product variants of [5, Lemmas 2.6 and 2.8], which play an important role in the whole paper.

Lemma 2.1. *(i) Let* $\phi \equiv \phi^{(1)} \otimes \phi^{(2)}$*, where, for* $i = 1, 2$, $\phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ *satisfies that* $\operatorname{supp} \widehat{\phi^{(i)}}$ *is compact and bounded away from the origin and for all* $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$,

$$\sum_{j_i \in \mathbb{Z}} \widehat{\phi^{(i)}}((A_i^*)^{j_i}\xi_i) = 1. \tag{2.2}$$

Then for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ *(resp.* $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$*),*

$$f = \sum_{j_1, j_2 \in \mathbb{Z}} f * \phi_{j_1, j_2} \tag{2.3}$$

holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ *(resp.* $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$*).*

(ii) Let (φ, ψ) *be an admissible pair of dual frame wavelets as in Definition 2.2. For any* $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ *(resp.* $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$*),*

$$f = \sum_{R \in \mathcal{R}} \langle f, \varphi_R \rangle \psi_R \tag{2.4}$$

holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ *(resp.* $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$*), where* φ_R *and* ψ_R *are as in* (2.1)*.*

The proof of Lemma 2.1 is given in Section 3. Based on the Calderón reproducing formulae, we can establish some new equivalent characterizations of weighted anisotropic product Hardy spaces in [8].

We first recall the weight class of Muckenhoupt associated with A introduced in [5].

Definition 2.3. Let $p \in [1, \infty)$, A be a dilation and w a non-negative measurable function on $\mathbb{R}^n$. Let $b \equiv |\det A|$. The function w is said to belong to the *weight class* $\mathcal{A}_p(\mathbb{R}^n; A)$ *of Muckenhoupt*, if there exists a positive constant C such that when $p \in (1, \infty)$,

$$\sup_{x \in \mathbb{R}^n} \sup_{k \in \mathbb{Z}} \left\{ b^{-k} \int_{B_\rho(x, b^k)} w(y)\, dy \right\} \left\{ b^{-k} \int_{B_\rho(x, b^k)} [w(y)]^{-1/(p-1)}\, dy \right\}^{p-1} \le C,$$

and when $p = 1$,

$$\sup_{x \in \mathbb{R}^n} \sup_{k \in \mathbb{Z}} \left\{ b^{-k} \int_{B_\rho(x, b^k)} w(y)\, dy \right\} \left\{ \operatorname*{esssup}_{y \in B_\rho(x, b^k)} [w(y)]^{-1} \right\} \le C;$$

and the minimal constant C as above is denoted by $C_{p, A, n}(w)$. Here, $B_\rho(x, b^k) \equiv \{y \in \mathbb{R}^n :\ \rho(x - y) < b^k\}$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$.

Define $\mathcal{A}_\infty(\mathbb{R}^n; A) \equiv \bigcup_{1 \le p < \infty} \mathcal{A}_p(\mathbb{R}^n; A)$.

Definition 2.4. For $i = 1, 2$, let A_i be a dilation on $\mathbb{R}^{n_i}$ and $\vec{A} \equiv (A_1, A_2)$. Let $p \in (1, \infty)$ and w be a non-negative measurable function on $\mathbb{R}^n \times \mathbb{R}^m$. The function w is said to be in the *weight class* $\mathcal{A}_p(\vec{A})$ *of Muckenhoupt* , if $w(x_1, \cdot) \in \mathcal{A}_p(A_2)$ for almost every $x_1 \in \mathbb{R}^n$ and $\operatorname{esssup}_{x_1 \in \mathbb{R}^n} C_{p, A_2, m}(w(x_1, \cdot)) < \infty$, and $w(\cdot, x_2) \in \mathcal{A}_p(A_1)$ for almost every $x_2 \in \mathbb{R}^m$ and $\operatorname{esssup}_{x_2 \in \mathbb{R}^m} C_{p, A_1, n}(w(\cdot, x_2)) < \infty$. In what follows, let

$$C_{q, \vec{A}, n, m}(w) \equiv \max \left\{ \operatorname*{esssup}_{x_1 \in \mathbb{R}^n} C_{p, A_2, m}(w(x_1, \cdot)),\ \operatorname*{esssup}_{x_2 \in \mathbb{R}^m} C_{p, A_1, n}(w(\cdot, x_2)) \right\}.$$

Define $\mathcal{A}_\infty(\vec{A}) \equiv \cup_{1 < p < \infty} \mathcal{A}_p(\vec{A})$.

The above product anisotropic weights also satisfy similar basic properties of the classical weights; see [8, Proposition 2.10] for more details.

Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ is said to *vanish weakly at infinity* if for any $\varphi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, $f * \varphi_{k_1, k_2} \to 0$ in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ as $k_1, k_2 \to -\infty$, where $\varphi \equiv \varphi^{(1)} \otimes \varphi^{(2)}$; see [8]. Denote by $\mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m)$ the *set of all* $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ *vanishing weakly at infinity.*

Let $\Phi \equiv \Phi^{(1)} \otimes \Phi^{(2)}$ with $\Phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ satisfying $\widehat{\Phi^{(i)}}(\vec{0}_{n_i}) = 0$, $i = 1, 2$. For any $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ and all $x \in \mathbb{R}^n \times \mathbb{R}^m$, the *anisotropic product Lusin-area function* of f is defined by

$$\vec{S}_\Phi(f)(x) \equiv \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} b_1^{k_1} b_2^{k_2} \int_{B_{\rho_1}(x_1, b_1^{-k_1}) \times B_{\rho_2}(x_2, b_2^{-k_2})} |f * \Phi_{k_1, k_2}(y)|^2 \, dy \right\}^{\frac{1}{2}}.$$

The weighted anisotropic product Hardy space $H^p_w(\vec{A})$ was defined via the anisotropic product Lusin-area function in [8] as follows. The class of allowable test functions in [8] was somewhat restricted; see the following Definition 2.5. Later, we shall deduce from Theorem 2.2 that this restriction can be relaxed to $\Phi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

Definition 2.5. Let $\Psi \equiv \Psi^{(1)} \otimes \Psi^{(2)}$ and $\Phi \equiv \Phi^{(1)} \otimes \Phi^{(2)}$ be such that $\Psi^{(i)}, \Phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$, satisfying that

(i) $\operatorname{supp} \Psi^{(i)} \subset B_{\rho_i}(\vec{0}_{n_i}, 1) \equiv \{x \in \mathbb{R}^{n_i} : \rho_i(x_i) < 1\}$, $\Psi^{(i)} \in \mathcal{S}_{N_i}(\mathbb{R}^{n_i})$, where N_i is some fixed nonnegative integer, and $\widehat{\Psi}^{(i)}(\xi) \geq C > 0$ for ξ in some fixed annulus;

(ii) $\operatorname{supp} \widehat{\Phi^{(i)}}$ is compact and bounded away from the origin;

(iii) $\sum_{j \in \mathbb{Z}} \widehat{\Psi^{(i)}}((A_i^*)^j \xi_i) \widehat{\Phi^{(i)}}((A_i^*)^j \xi_i) = 1$ for all $\xi_i \in \mathbb{R}^{n_i} \setminus \{\vec{0}_{n_i}\}$.

We should point out that such pairs (Ψ, Φ) indeed exist by the virtue of [8, Proposition 2.14].

Definition 2.6. Let $p \in (0, 1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and Φ be as in Definition 2.5.

(i) ([8]) The *weighted anisotropic product Hardy space* $H^p_w(\vec{A})$ is defined by

$$H^p_w(\vec{A}) \equiv \{f \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m) : \|f\|_{H^p_w(\vec{A})} \equiv \|\vec{S}_\Phi(f)\|_{L^p_w(\mathbb{R}^n \times \mathbb{R}^m)} < \infty\}.$$

(ii) The *weighted anisotropic product Hardy space* $\widetilde{H}^p_w(\vec{A})$ is defined by replacing $\mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m)$ with $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ in (i).

The following theorem shows that $H^p_w(\vec{A})$ and $\widetilde{H}^p_w(\vec{A})$ are equivalent in some sense.

Proposition 2.1. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0, 1]$. Then $H^p_w(\vec{A}) = \widetilde{H}^p_w(\vec{A})$ in the following sense: if $f \in H^p_w(\vec{A})$, then $f \in \widetilde{H}^p_w(\vec{A})$ and there exists a positive constant C, independent of f, such that $\|f\|_{\widetilde{H}^p_w(\vec{A})} \leq C\|f\|_{H^p_w(\vec{A})}$. Conversely, if $f \in \widetilde{H}^p_w(\vec{A})$, then there exists a unique extension $\breve{f} \in \mathcal{S}'_{\infty, w}(\mathbb{R}^n \times \mathbb{R}^m)$ such that for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $\langle \breve{f}, \varphi \rangle = \langle f, \varphi \rangle$ and there exists a positive constant C, independent of f, such that $\|\breve{f}\|_{H^p_w(\vec{A})} \leq C\|f\|_{\widetilde{H}^p_w(\vec{A})}$.*

The proof of Proposition 2.1 is given in Section 3.

Now let us introduce two kinds of weighted anisotropic product Hardy spaces defined, respectively, via the Littlewood-Paley g-function.

Definition 2.7. Let $p \in (0, \infty)$, $w \in \mathcal{A}_\infty(\vec{A})$ and $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

The *weighted anisotropic product Hardy space* $\ddot{H}^p_w(\vec{A})$ is defined, via the Littlewood-Paley g-function, to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|f\|_{\ddot{H}^p_w(\vec{A})} \equiv \left\| \left\{ \sum_{j_1,\, j_2 \in \mathbb{Z}} |\varphi_{j_1,\, j_2} * f|^2 \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^n \times \mathbb{R}^m)} < \infty,$$

and the corresponding *discrete weighted anisotropic product Hardy space* $\ddot{h}^p_w(\vec{A})$ is defined to be the set of all complex-valued sequences $s \equiv \{s_R\}_{R \in \mathcal{R}}$ such that

$$\|s\|_{\ddot{h}^p_w(\vec{A})} \equiv \left\| \left\{ \sum_{R \in \mathcal{R}} |s_R|^2 |R|^{-1} \chi_R \right\}^{\frac{1}{2}} \right\|_{L^p_w(\mathbb{R}^n \times \mathbb{R}^m)} < \infty.$$

For any $Q \in \mathcal{R}$ with $Q \equiv Q_1 \times Q_2 \equiv A_1^{j_1}([0,\,1)^n + k_1) \times A_2^{j_2}([0,\,1)^m + k_2)$, where $j_1,\, j_2 \in \mathbb{Z}$ and $k_1 \in \mathbb{Z}^n$, $k_2 \in \mathbb{Z}^m$, let the *symbol* $\mathrm{scale}\,(Q) \equiv (\mathrm{scale}\,(Q_1),\ \mathrm{scale}\,(Q_2)) \equiv (j_1,\, j_2)$.

The weighted anisotropic product Campanato spaces are defined as follows, which are weighted variants of anisotropic Campanato spaces on $\mathbb{R}^n$ in [1], and is proved to be the dual spaces of weighted anisotropic Hardy spaces in Section 3.

Definition 2.8. Let $p \in (0, 1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

(i) The *space* $\mathcal{L}_{p,\,w}(\vec{A})$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|f\|_{\mathcal{L}_{p,\,w}(\vec{A})} \equiv \left\{ \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{\substack{R \in \mathcal{R},\, R \subset \Omega \\ \mathrm{scale}\,(R) = (-j_1,\, -j_2)}} |\varphi_{j_1,\, j_2} * f(x)|^2 \right.$$
$$\left. \times \frac{|R|^2}{[w(R)]^2} \chi_R(x) w(x)\, dx \right\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$.

(ii) The corresponding *sequence space* $\ell_{p,\,w}(\vec{A})$ is defined to be the set of all complex-valued sequences $s \equiv \{s_R\}_{R \in \mathcal{R}}$ such that

$$\|s\|_{\ell_{p,\,w}(\vec{A})} \equiv \left\{ \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R \in \mathcal{R},\, R \subset \Omega} |s_R|^2 \frac{|R|}{w(R)} \right\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$.

Definition 2.9. Let $\varphi \equiv \varphi^{(1)} \otimes \varphi^{(2)}$ and $\psi \equiv \psi^{(1)} \otimes \psi^{(2)}$ with $\varphi^{(i)},\, \psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, for $i = 1,\, 2$, such that $\mathrm{supp}\,\widehat{\varphi^{(i)}}$ and $\mathrm{supp}\,\widehat{\psi^{(i)}}$ are compact and bounded away from the origin. The *φ-transform* S_φ is the map taking each $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ to the sequence $S_\varphi f \equiv \{(\mathcal{S}_\varphi f)_R\}_{R \in \mathcal{R}}$ defined by $(S_\varphi(f))_R = \langle f, \varphi_R \rangle$. The *inverse φ-transform* T_ψ is the map taking the sequence $s = \{s_R\}_{R \in \mathcal{R}}$ to $T_\psi s \equiv \sum_{R \in \mathcal{R}} s_R \psi_R$.

Theorem 2.1. *Let $p \in (0, \infty)$ and $w \in \mathcal{A}_\infty(\vec{A})$. The φ-transform $T_\psi : \ddot{h}^p_w(\vec{A}) \to \ddot{H}^p_w(\vec{A})$ and the inverse transforms $S_\varphi : \ddot{H}^p_w(\vec{A}) \to \ddot{h}^p_w(\vec{A})$ are bounded. Moreover, if (ψ, φ) is an admissible pair of dual frame wavelets as in Definition 2.2, then the map $T_\psi \circ S_\varphi$ is an identity on $\ddot{H}^p_w(\vec{A}; \varphi) = \ddot{H}^p_w(\vec{A}; \widetilde{\varphi})$.*

The above results also hold if $\ddot{H}^p_w(\vec{A})$ and $\ddot{h}^p_w(\vec{A})$ are replaced, respectively, by $\mathcal{L}_{p,w}(\vec{A})$ and $\ell_{p,w}(\vec{A})$ for $p \in (0, 1]$.

The proof of Theorem 2.1 is given in Section 3.

Then by Theorem 2.1, with proofs similar to those of [2, Corollaries 3.13 and 3.14], we can obtain that the space $\mathcal{L}_{p,w}(\vec{A})$, equipped with $\|\cdot\|_{\mathcal{L}_{p,w}(\vec{A})}$, is well defined and complete as follows. We omit the details.

Corollary 2.1. *Let $p \in (0, \infty)$ and $w \in \mathcal{A}_\infty(\vec{A})$. The space $\ddot{H}^p_w(\vec{A})$ is well-defined in the following sense that, for any $\varphi, \widetilde{\varphi} \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, their associated quasi-norms, respectively, in $\ddot{H}^p_w(\vec{A}; \varphi)$ and $\ddot{H}^p_w(\vec{A}; \widetilde{\varphi})$ are equivalent, namely, there exist positive constants C_1 and C_2 such that for all $f \in \ddot{H}^p_w(\vec{A})$,*

$$C_1\|f\|_{\ddot{H}^p_w(\vec{A};\widetilde{\varphi})} \le \|f\|_{\ddot{H}^p_w(\vec{A};\varphi)} \le C_2\|f\|_{\ddot{H}^p_w(\vec{A};\widetilde{\varphi})}.$$

When $p \in (0, 1]$, the space $\mathcal{L}_{p,w}(\vec{A})$ is also well-defined in the above sense. Moreover, the spaces $\ddot{H}^p_w(\vec{A})$ and $\mathcal{L}_{p,w}(\vec{A})$, equipped respectively with $\|\cdot\|_{\ddot{H}^p_w(\vec{A})}$ and $\|\cdot\|_{\mathcal{L}_{p,w}(\vec{A})}$, are also complete.

We have the following equivalences on the product Hardy spaces $\widetilde{H}^p_w(\vec{A})$ and $\ddot{H}^p_w(\vec{A})$.

Theorem 2.2. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0, 1]$. Then $f \in \widetilde{H}^p_w(\vec{A})$ if and only if $f \in \ddot{H}^p_w(\vec{A})$. Moreover, their corresponding norms are equivalent.*

The proof of Theorem 2.2 is given in Section 3.

To show that $\mathcal{L}_{p,w}(\vec{A})$ is the dual space of $\ddot{H}^p_w(\vec{A})$, we first establish the duality between their corresponding sequence spaces.

Proposition 2.2. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0, 1]$. Then $(\ddot{h}^p_w(\vec{A}))^* = \ell_{p,w}(\vec{A})$ in the following sense: for any $t \in \ell_{p,w}(\vec{A})$, the map $L_t(h) \equiv \langle t, h\rangle \equiv \sum_{R\in\mathcal{R}} t_R \bar{h}_R$ for any $h \in \ddot{h}^p_w(\vec{A})$ defines a continuous linear functional on $\ddot{h}^p_w(\vec{A})$ with norm $\|L_t\|_{(\ddot{h}^p_w(\vec{A}))^*} \le C\|t\|_{\ell_{p,w}(\vec{A})}$, where C is some positive constant, independent of t. Conversely, every $L \in (\ddot{h}^p_w(\vec{A}))^*$ is of this form for some $t \in \ell_{p,w}(\vec{A})$ with norm $\|t\|_{\ell_{p,w}(\vec{A})} \le C\|L\|_{(\ddot{h}^p_w(\vec{A}))^*}$, where C is some positive constant, independent of L.*

The proof of Proposition 2.2 is also given in Section 3. Applying Proposition 2.2 and Theorem 2.1, we can prove that $\mathcal{L}_{p,w}(\vec{A})$ is the dual space of $\ddot{H}^p_w(\vec{A})$ as follows.

Theorem 2.3. *Let $w \in \mathcal{A}_\infty(\vec{A})$ and $p \in (0, 1]$. Then $(\ddot{H}^p_w(\vec{A}))^* = \mathcal{L}_{p,w}(\vec{A})$ in the following sense: there exists a positive constant C such that for any $g \in \mathcal{L}_{p,w}(\vec{A})$, there exists a linear functional L_g initially defined on $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, which has a uniquely continuous extension to $\ddot{H}^p_w(\vec{A})$ and $\|L_g\|_{(\ddot{H}^p_w(\vec{A}))^*} \le C\|g\|_{\mathcal{L}_{p,w}(\vec{A})}$. Conversely, there exists a positive constant C such that every continuous linear functional L on $\ddot{H}^p_w(\vec{A})$ can be written as $L = L_g$ with some $g \in \mathcal{L}_{p,w}(\vec{A})$ and $\|g\|_{\mathcal{L}_{p,w}(\vec{A})} \le C\|L\|_{(\ddot{H}^p_w(\vec{A}))^*}$.*

The proof of Theorem 2.3 is given in Section 3.

We shall finish the paper by giving another equivalent description of duals of anisotropic weighted Hardy spaces, which itself is quite interesting.

Definition 2.10. Let $p \in (0, 1]$, $w \in \mathcal{A}_\infty(\vec{A})$ and $\varphi \in \mathscr{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. The *space* $\widetilde{\mathcal{L}}_{p,\,w}(\vec{A})$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$\|f\|_{\widetilde{\mathcal{L}}_{p,\,w}(\vec{A})} \equiv \Bigg\{ \sup_{w(\Omega)<\infty} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{\substack{R\in\mathcal{R},\, R\subset\Omega \\ \mathrm{scale}\,(R)=(-j_1,\,-j_2)}} |\varphi_{j_1,\,j_2} * f(x)|^2 \times \frac{|R|}{w(R)} \chi_R(x)\, dx \Bigg\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$.

Comparing with the definition of $\mathcal{L}_{p,\,w}(\vec{A})$, an interesting phenomena appears in the definition of $\widetilde{\mathcal{L}}_{p,w}(\vec{A})$ is that the integral in Definition 2.10 is not weighted. However, both spaces are equivalent as follows.

Corollary 2.2. *Let* $w \in \mathcal{A}_\infty(\vec{A})$ *and* $p \in (0, 1]$. *Then* $\mathcal{L}_{p,\,w}(\vec{A}) = \widetilde{\mathcal{L}}_{p,\,w}(\vec{A})$ *with equivalent norms.*

Finally, we shall comment about the proof of Corollary 2.2. By adapting the proof of Theorem 2.1, we show that Theorem 2.1 also holds with $\mathcal{L}_{p,\,w}(\vec{A})$ replaced by $\widetilde{\mathcal{L}}_{p,\,w}(\vec{A})$, $p \in (0, 1]$, albeit with the same sequence space $\ell_{p,\,w}(\vec{A})$. Once this is shown, Corollary 2.2 follows immediately. We omit the details.

3 Proofs of Main Results

We first introduce some notation associated to expansive dilations.

Definition 3.1. Let A be an expansive dilation on $\mathbb{R}^n$ and $\sigma(A)$ the *set of all eigenvalues of* A. If A is diagonalizable over $\mathbb{C}$, then take $\lambda_- \equiv \min_{\lambda\in\sigma(A)} |\lambda|$ and $\lambda_+ \equiv \max_{\lambda\in\sigma(A)} |\lambda|$. Otherwise, let λ_- and λ_+ be some *positive real numbers such that* $1 < \lambda_- < \min_{\lambda\in\sigma(A)} |\lambda|$ *and* $\lambda_+ > \max_{\lambda\in\sigma(A)} |\lambda|$. Set $\zeta_+ \equiv \frac{\ln\lambda_+}{\ln b}$ and $\zeta_- \equiv \frac{\ln\lambda_-}{\ln b}$.

The following inequalities concerning A, ρ and the Euclidean norm $|\cdot|$ established in [1, Section 2] are used in the whole paper:

$$[\rho(x)]^{\zeta_-} \lesssim |x| \lesssim [\rho(x)]^{\zeta_+} \quad \text{for all } \rho(x) \ge 1, \tag{3.1}$$

$$[\rho(x)]^{\zeta_+} \lesssim |x| \lesssim [\rho(x)]^{\zeta_-} \quad \text{for all } \rho(x) \le 1, \tag{3.2}$$

$$b^{j\zeta_-}|x| \lesssim |A^j x| \lesssim b^{j\zeta_+}|x| \quad \text{for all } \ j \ge 0, \text{ and} \tag{3.3}$$

$$b^{j\zeta_+}|x| \lesssim |A^j x| \lesssim b^{j\zeta_-}|x| \quad \text{for all } \ j \le 0. \tag{3.4}$$

Let $i = 1, 2$. For dilation A_i on $\mathbb{R}^{n_i}$, set $\lambda_{i,\,-,}$, $\lambda_{i,\,+}$, $\zeta_{i,\,-}$ and $\zeta_{i,\,+}$ to be associated to A_i as above.

Proof of Lemma 2.1. To prove this lemma, we borrow some ideas from the proofs of [5, Lemma 2.8], [41, Lemma 2.1] and [42, Lemma 2.1].

(i) For any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $M \in \mathbb{Z}_+$ and $j_1,\, j_2 \in \mathbb{Z}$, let us first show that $\mathrm{I} \equiv \sum_{j_1, j_2 \in \mathbb{Z}} \|f * \phi_{j_1,\,j_2}\|_M^* < \infty$ by considering the following four cases.

We first assume that $j_1 \geq 0$ and $j_2 < 0$. Let $\beta \equiv (\beta_1,\ \beta_2)$ with $\beta_1 \in \mathbb{Z}_+^n$ and $\beta_2 \in \mathbb{Z}_+^m$. Since $\operatorname{supp} \widehat{\phi^{(i)}}$ is compact and bounded away from the origin, then there exists a positive constant C such that $\operatorname{supp} \widehat{\phi^{(i)}} \subset \{\xi_i \in \mathbb{R}^{n_i} : 1/C \leq |\xi_i| \leq C\}$, $i = 1,\, 2$. Moreover, notice that $\widehat{\phi_{j_i}^{(i)}}(\xi_i) = \widehat{\phi^{(i)}}((A_i^*)^{-j_i}\xi_i)$, by [1, (3.13)] for $j_1 \geq 0$ and a similar proof for $j_2 < 0$, we have that for any $M \in \mathbb{Z}_+$,

$$\sup_{|\beta_1|=M} \|\partial^{\beta_1} \widehat{\phi_{j_1}^{(1)}}\|_{L^\infty(\mathbb{R}^n)} \lesssim 1 \quad \text{and} \tag{3.5}$$

$$\sup_{|\beta_2|=M} \|\partial^{\beta_2} \widehat{\phi_{j_2}^{(2)}}\|_{L^\infty(\mathbb{R}^m)} \lesssim (\lambda_{2,+})^{-j_2 M} \sup_{|\beta_2|=M} \|\partial^{\beta_2} \widehat{\phi^{(2)}}\|_{L^\infty(\mathbb{R}^m)}.$$

Therefore, by (3.5), we obtain that

$$\begin{aligned}
&\|f * \phi_{j_1,\,j_2}\|_M^* \\
&\quad = \sup_{\xi \in \mathbb{R}^n \times \mathbb{R}^m} \sup_{|\beta| \leq M} |\partial^\beta (\widehat{f * \phi_{j_1,\,j_2}})(\xi)| (|\xi_1|^M + |\xi_1|^{-M})(|\xi_2|^M + |\xi_2|^{-M}) \\
&\quad \leq \sup_{\xi \in \mathbb{R}^n \times \mathbb{R}^m} \sup_{|\beta| \leq M} |\partial^\beta \widehat{f}(\xi)| \sup_{|\beta_1| \leq M} \prod_{i=1}^2 |\partial^{\beta_i} \widehat{\phi_{j_i}^{(i)}}(\xi_i)| (|\xi_i|^M + |\xi_i|^{-M}) \\
&\quad \lesssim (\lambda_{2,+})^{-j_2 M} \sup_{\substack{1/C \leq |(A_1^*)^{-j_1}\xi_1| \leq C \\ 1/C \leq |(A_2^*)^{-j_2}\xi_2| \leq C}} \sup_{|\beta| \leq M} |\partial^\beta \widehat{f}(\xi)| (|\xi_1|^M + |\xi_1|^{-M})(|\xi_2|^M + |\xi_2|^{-M}) \\
&\quad \equiv \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,
\end{aligned}$$

where

$$\mathrm{I}_1 \sim (\lambda_{2,+})^{-j_2 M} \sup_{\substack{1/C \leq |(A_1^*)^{-j_1}\xi_1| \leq C \\ 1/C \leq |(A_2^*)^{-j_2}\xi_2| \leq C}} \sup_{|\beta| \leq M} |\partial^\beta \widehat{f}(\xi)| |\xi_1|^M |\xi_2|^{-M},$$

and, similarly, I_2, I_3 and I_4 are defined by replacing $|\xi_1|^M|\xi_2|^{-M}$ in the definition of I_1, respectively, by $|\xi_1|^M|\xi_2|^M$, $|\xi_1|^{-M}|\xi_2|^M$ and $|\xi_1|^{-M}|\xi_2|^{-M}$.

We shall only estimate I_1 since the other estimates are similar. Let d be the minimal positive integer such that $(\lambda_{2,-})^d/\lambda_{2,+} > 1$. Thus, by (3.3) and (3.4), we have

$$\begin{aligned}
\mathrm{I}_1 &\sim (\lambda_{2,+})^{-j_2 M} \sup_{\substack{1/C \leq |(A_1^*)^{-j_1}\xi_1| \leq C \\ 1/C \leq |(A_2^*)^{-j_2}\xi_2| \leq C}} \sup_{|\beta| \leq M} |\xi_1|^{-1-dM} |\xi_2|^{dM+1} |\partial^\beta \widehat{f}(\xi)| (|\xi_1|/|\xi_2|)^{(d+1)M+1} \\
&\lesssim (\lambda_{2,+})^{-j_2 M} \sup_{\substack{1/C \leq |\xi_1| \leq C \\ 1/C \leq |\xi_2| \leq C}} |(A_1^*)^{j_1}\xi_1|^{-dM-1} |(A_2^*)^{j_2}\xi_2|^{dM+1} \|f\|_{(d+1)M+1}^* \\
&\lesssim (\lambda_{1,-})^{-j_1(dM+1)} \left(\frac{(\lambda_{2,-})^d}{\lambda_{2,+}}\right)^{j_2 M} (\lambda_{2,-})^{j_2} \|f\|_{(d+1)M+1}^*
\end{aligned}$$

$$\lesssim (\lambda_{1,-})^{-j_1(dM+1)}(\lambda_{2,-})^{j_2}\|f\|^*_{(d+1)M+1}.$$

Hence, $\sum_{j_1\ge 0,\, j_2<0} \mathrm{I}_1 \lesssim \|f\|^*_{(d+1)M+1}$. Similarly, we have $\sum_{j_1\ge 0,\, j_2<0}(\mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4) \lesssim \|f\|^*_{(d+1)M+1}$. Thus, $\sum_{j_1\ge 0,\, j_2<0} \|f * \phi_{j_1,\, j_2}\|^*_M \lesssim \|f\|^*_{(d+1)M+1}$, which is the desired estimate for this case.

In the remaining cases when $j_1 \ge 0$ and $j_2 \ge 0$, or $j_1 < 0$ and $j_2 \ge 0$, or $j_1 < 0$ and $j_2 < 0$, we get similar estimates with d replaced by $\widetilde{d} \equiv \min\{\ell \in \mathbb{N} : (\lambda_{i,-})^\ell/\lambda_{i,+} > 1,\ i = 1,\, 2\}$. Thus, combining these estimates yields that

$$\sum_{j_1,\, j_2\in\mathbb{Z}} \|f * \phi_{j_1,\, j_2}\|^*_M \lesssim \|f\|^*_{(\widetilde{d}+1)M+1}. \tag{3.6}$$

This implies that the series $\sum_{j_1, j_2\in\mathbb{Z}} f * \phi_{j_1,\, j_2}$ converges unconditionally in the semi-norms of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

Let $f_0 \equiv \sum_{j_1, j_2\in\mathbb{Z}} \phi_{j_1,\, j_2} * f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. For all $\xi \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \{(\xi_1,\, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^m : \xi_1 = \vec{0}_n \text{ or } \xi_2 = \vec{0}_m\}$, by (2.2), we have $\sum_{j_1,\, j_2\in\mathbb{Z}} \widehat{\phi}((A_1^*)^{j_1}\xi_1,\, (A_2^*)^{j_2}\xi_2) = 1$. Thus, we obtain $\widehat{f_0} = \sum_{j_1,\, j_2\in\mathbb{Z}} \widehat{\phi_{j_1,\, j_2} * f} = \sum_{j_1,\, j_2\in\mathbb{Z}} \widehat{\phi}_{j_1,\, j_2}\widehat{f} = \widehat{f}$ in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Notice that the Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ onto itself (see, for example, [34]), we obtain that $f_0 = f$. Thus, (2.3) holds for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

A standard duality argument shows that (2.3) also holds for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Indeed, let $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Since $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is a dual, endowed with the weak-$*$ topology, of the locally convex space $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if there exist a constant $C_f > 0$ and $M \in \mathbb{Z}_+$ such that for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $|\langle f,\, \phi\rangle| \le C_f\|\phi\|^*_M$. This observation and (3.6) further imply that for all $\theta \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left|\left\langle f,\ \sum_{j_1,\, j_2\in\mathbb{Z}} \widetilde{\phi}_{j_1,\, j_2} * \theta\right\rangle\right| \le C_f \sum_{j_1,\, j_2\in\mathbb{Z}} \|\widetilde{\phi}_{j_1,\, j_2} * \theta\|^*_M \lesssim \|\phi\|^*_{(\widetilde{d}+1)M+1},$$

where $\widetilde{\phi}(\cdot) \equiv \overline{\phi(-\cdot)}$. From this and the completeness of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, it follows that $\sum_{j_1,\, j_2\in\mathbb{Z}} \phi_{j_1,\, j_2} * f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Thus, for all $\theta \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\left\langle \sum_{j_1,\, j_2\in\mathbb{Z}} \phi_{j_1,\, j_2} * f,\ \theta\right\rangle = \left\langle f,\ \sum_{j_1,\, j_2\in\mathbb{Z}} \widetilde{\phi}_{j_1,\, j_2} * \theta\right\rangle = \langle f,\, \theta\rangle.$$

This finishes the proof of part (i) of Lemma 2.1.

(ii) Let $(\varphi,\, \psi)$ be an admissible triplet of dual frame wavelets in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ as in Definition 2.2. For $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$), by (2.3) with $\phi \equiv \psi * \widetilde{\varphi}$, we obtain that

$$f = \sum_{j_1,\, j_2\in\mathbb{Z}} \psi_{j_1,\, j_2} * \widetilde{\varphi}_{j_1,\, j_2} * f \tag{3.7}$$

in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$).

For $\vec{A} \equiv (A_1,\, A_2)$, $j_1,\, j_2 \in \mathbb{Z}$ and $\vec{k} \equiv (k_1,\, k_2) \in \mathbb{Z}^n \times \mathbb{Z}^m$, set

$$\vec{A}_{j_1,\, j_2} \equiv \begin{pmatrix} A_1^{j_1} & 0_{n\times m} \\ 0_{m\times n} & A_2^{j_2}\end{pmatrix},\ \vec{A}^*_{j_1,\, j_2} \equiv \begin{pmatrix} (A_1^*)^{j_1} & 0_{n\times m} \\ 0_{m\times n} & (A_2^*)^{j_2}\end{pmatrix} \text{ and } \vec{k} \equiv \begin{pmatrix} k_1 \\ k_2\end{pmatrix},$$

where $0_{n\times m}$ denotes the $(n\times m)$-matrix with all entries 0, and $0_{m\times n}$ is similarly defined. Let $g\equiv \widetilde{\varphi}_{j_1,\,j_2}*f$. We first claim that for all $j_1,\,j_2\in\mathbb{Z}$ and $f\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ (resp. $f\in\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$),

$$g*\psi_{j_1,\,j_2}(\cdot)=\sum_{k_1\in\mathbb{Z}^n,\,k_2\in\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(\cdot-\vec{A}_{-j_1,\,-j_2}\vec{k}) \tag{3.8}$$

in $\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$). Assuming this claim for the moment, combining this with (3.7) yields that (2.4) holds in $\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ (resp. $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$).

Now let us first prove the claim (3.8) for $f\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$. By $\widehat{\varphi^{(i)}_{j_i}}(\cdot)=\widehat{\varphi^{(i)}}((A_i^*)^{-j_i}\cdot)$ and $\operatorname{supp}\widehat{\varphi^{(i)}}\subset([-\pi,\,\pi]^{n_i}\setminus\{\vec{0}_{n_i}\})$ with $i=1,\,2$, we obtain that $g\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ with $\operatorname{supp}\widehat{g}\subset(\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m}))$. Then by using the Fourier orthonormal basis

$$\left\{\frac{b_1^{-j_1/2}b_2^{-j_2/2}}{(2\pi)^{(n+m)/2}}e^{-i\langle\vec{A}_{-j_1,\,-j_2}\vec{k},\,\xi\rangle}\right\}_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}$$

of $L^2(\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m}))$, we have that for all $\xi\in\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m})$,

$$\widehat{g}(\xi)=\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}\frac{b_1^{-j_1}b_2^{-j_2}}{(2\pi)^{n+m}}\left[\int_{\vec{A}_{j_1,\,j_2}([-\pi,\,\pi]^{n+m})}\widehat{g}(y)e^{i\left\langle\vec{A}_{-j_1,\,-j_2}\vec{k},\,y\right\rangle}\,dy\right]e^{-i\left\langle\vec{A}_{-j_1,\,-j_2}\vec{k},\,\xi\right\rangle}$$

in $L^2(\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m}))$. Since $\operatorname{supp}\widehat{g}\subset\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m})$, $\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m})$ can be replaced by $\mathbb{R}^n\times\mathbb{R}^m$ in the above integral. Thus, by the Fourier inversion formula, we have that for any $\xi\in\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m})$,

$$\widehat{g}(\xi)=\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,\,-j_2}\vec{k})e^{-i\left\langle\vec{A}_{-j_1,\,-j_2}\vec{k},\,\xi\right\rangle}$$

in $L^2(\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m}))$. Notice that $\operatorname{supp}\widehat{\psi}_{j_1,\,j_2}\subset\vec{A}^*_{j_1,\,j_2}([-\pi,\,\pi]^{n+m})$, we can replace $\widehat{g}$ by its periodic extension without altering the product $\widehat{g}\widehat{\psi_{j_1,\,j_2}}$. Using $g*\psi_{j_1,\,j_2}=(\widehat{g}\widehat{\psi_{j_1,\,j_2}})^\vee$ with $f^\vee(\cdot)\equiv\widehat{f}(-\cdot)$, we obtain

$$\begin{aligned}(g*\psi_{j_1,\,j_2})(x)&=\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,\,-j_2}\vec{k})\left(e^{-i\left\langle\vec{A}_{-j_1,\,-j_2}\vec{k},\,\xi\right\rangle}\widehat{\psi_{j_1,\,j_2}}(\xi)\right)^\vee(x)\\&=\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}b_1^{-j_1}b_2^{-j_2}g(\vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(x-\vec{A}_{-j_1,\,-j_2}\vec{k})\end{aligned} \tag{3.9}$$

holds in $L^2(\mathbb{R}^n\times\mathbb{R}^m)$ and hence pointwise.

To prove that (3.9) holds in $\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, we claim that for any $M\in\mathbb{Z}_+$ and $\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m$,

$$\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m}b_1^{-j_1}b_2^{-j_2}|g(\vec{A}_{-j_1,\,-j_2}\vec{k})|\left\|\psi_{j_1,\,j_2}(\cdot-\vec{A}_{-j_1,\,-j_2}\vec{k})\right\|_M^*<\infty. \tag{3.10}$$

Assuming the claim (3.10) for the moment, combining this with the completeness of $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ yields that

$$\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1} b_2^{-j_2} g(\vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(\cdot - \vec{A}_{-j_1,\,-j_2}\vec{k}) \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m).$$

By this and (3.9), we obtain that (3.8) holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. This shows that (2.4) holds in $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$.

The proof of (3.10) is similar to the estimate of I in (i). For simplicity, we only prove the claim (3.10) when $j_1 \geq 0$ and $j_2 < 0$. In this case, for any $M \in \mathbb{Z}_+$ and $\vec{k} \in \mathbb{Z}^n \times \mathbb{Z}^m$, by the chain rule and [1, (3.13)] for $j_1 \geq 0$ and a similar proof for $j_2 < 0$, we have

$$\begin{aligned}
&\left\|\psi_{j_1,\,j_2}(\cdot - \vec{A}_{-j_1,\,-j_2}\vec{k})\right\|_M^* \\
&\quad= \sup_{\xi\in\mathbb{R}^n\times\mathbb{R}^m} \sup_{|\gamma|\leq M} \left|\partial^\gamma \left[e^{-i\langle \vec{A}_{-j_1,\,-j_2}\vec{k},\,\xi\rangle}\widehat{\psi}_{j_1,\,j_2}(\xi)\right]\right| \prod_{i=1}^{2}\left(|\xi_i|^M + |\xi_i|^{-M}\right) \\
&\quad\lesssim \sup_{\xi\in\mathbb{R}^n\times\mathbb{R}^m} \sup_{|\gamma|\leq M} \left|\partial^\gamma \left(e^{-i\langle \vec{A}_{-j_1,\,-j_2}\vec{k},\,\xi\rangle}\right)\right| \|\psi_{j_1,\,j_2}\|_M^* \\
&\quad\lesssim \|\psi_{j_1,\,j_2}\|_M^* \sup_{|\gamma_1|\leq M} [(\lambda_{1,\,-})^{-j_1}|k_1|]^{|\gamma_1|} |\sup_{|\gamma_2|\leq M} |[(\lambda_{2,\,+})^{-j_2}|k_2|]^{|\gamma_2|} \\
&\quad\lesssim \|\psi_{j_1,\,j_2}\|_M^* |k_1|^M [(\lambda_{2,\,+})^{-j_2}|k_2|]^M.
\end{aligned}$$

From this and $|g(\vec{A}_{-j_1,\,-j_2}y)| \lesssim \prod_{i=1}^2 [1+\rho_i(A_i^{-j_i}y_i)]^{-(2+M\zeta_{i,\,+})}$ with $\zeta_{i,\,+}$ as in Definition 3.1, it follows that

$$\begin{aligned}
&\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} \left\|b_1^{-j_1} b_2^{-j_2} g(\vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(\cdot - \vec{A}_{-j_1,\,-j_2}\vec{k})\right\|_M^* \\
&\quad\lesssim \|\psi_{j_1,\,j_2}\|_M^* \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} |g(\vec{A}_{-j_1,\,-j_2}\vec{k})||k_1|^M |[(\lambda_{2,\,+})^{-j_2}|k_2|]^M \\
&\quad\lesssim \|\psi_{j_1,\,j_2}\|_M^* (\lambda_{2,\,+})^{-j_2 M} \prod_{i=1}^{2} \int_{\mathbb{R}^{n_i}} \frac{|y_i|^M}{[1+\rho_i(A_i^{-j_i}y_i)]^{2+M\zeta_{i,\,+}}}\,dy_i \lesssim \|\psi_{j_1,\,j_2}\|_M^*,
\end{aligned}$$

which is a desired estimate and hence shows the above claim (3.10).

It remains to prove that (3.8) also holds for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Let $g \equiv \widetilde{\varphi}_{j_1,\,j_2} * f$. It is well known that g is a slowly increasing C^∞ function on $\mathbb{R}^n \times \mathbb{R}^m$ (see, for example, [34]).

For $\delta > 0$, let $g_\delta(\cdot) \equiv \gamma(\delta\cdot)g(\cdot)$, where $\gamma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ satisfies $\gamma(\vec{0}_n,\,\vec{0}_m) = 1$ and $\operatorname{supp}\widehat{\gamma}$ is compact. Then $g_\delta \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$. If $\delta > 0$ is sufficiently small, we further have $\operatorname{supp}\widehat{g_\delta} \subset (\vec{A}^*_{j_1,\,j_2}([-\pi,\pi]^{n+m}))$. By the already shown part of (3.8), we have

$$\psi_{j_1,\,j_2} * g_\delta(\cdot) = b_1^{-j_1} b_2^{-j_2} \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} g_\delta(\vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(\cdot - \vec{A}_{-j_1,\,-j_2}\vec{k}) \tag{3.11}$$

$$= b_1^{-j_1} b_2^{-j_2} \sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} g_\delta(\cdot - \vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(\vec{A}_{-j_1,\,-j_2}\vec{k})$$

holds in $\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$.

Assume that g is at most polynomially increasing with order $M\in\mathbb{Z}_+$. Since $\psi_{j_1,\,j_2}\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, then for any fixed $x\in\mathbb{R}^n\times\mathbb{R}^m$, we have

$$\begin{aligned}&|g_\delta(x-\vec{A}_{-j_1,\,-j_2}\vec{k})\psi_{j_1,\,j_2}(\vec{A}_{-j_1,\,-j_2}\vec{k})|\\&\quad\le C_{\gamma,\,j_1,\,j_2}|x-\vec{A}_{-j_1,\,-j_2}\vec{k}|^M(1+|\vec{A}_{-j_1,\,-j_2}\vec{k}|)^{-(M+n+m+1)},\end{aligned}$$

and

$$\begin{aligned}&\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}|x-\vec{A}_{-j_1,\,-j_2}\vec{k}|^M(1+|\vec{A}_{-j_1,\,-j_2}\vec{k}|)^{-(M+n+m+1)}\\&\quad\le C_{\gamma,\,j_1,\,j_2}\int_{\mathbb{R}^n\times\mathbb{R}^m}|x-y|^M(1+|y|)^{-(M+n+m+1)}\,dy<\infty.\end{aligned}$$

By applying the Lebesgue dominated convergence theorem and taking the limit as $\delta\to 0$ in (3.11), we obtain that (3.8) converges pointwise.

Notice that for any $\theta\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, since $\psi\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, by [8, Lemma 5.5], we have $|\langle\psi_{j_1,\,j_2}(\cdot-\vec{A}_{-j_1,\,-j_2}\vec{k}),\,\theta\rangle|\le C_{j_1,\,j_2}\prod_{i=1}^2[1+\rho_i(k_i)]^{-2-M\zeta_{i,\,+}}$. From this and $|g_\delta(\vec{A}_{-j_1,\,-j_2}\vec{k})|\le C_\gamma(1+|\vec{A}_{-j_1,\,-j_2}\vec{k})^M\le C_{\gamma,\,j_1,\,j_2}[1+\rho_i(k_i)]^{M\zeta_{i,\,+}}$, it follows that

$$\begin{aligned}&\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}|g_\delta(\vec{A}_{-j_1,\,-j_2}\vec{k})||\langle\psi_{j_1,\,j_2}(\cdot-\vec{A}_{-j_1,\,-j_2}),\,\theta\rangle|\\&\quad\le C_{\gamma,j_1,\,j_2}\prod_{i=1}^2\int_{\mathbb{R}^{n_i}}[1+\rho_i(y_i)]^{-2}\,dy_i<\infty.\end{aligned}$$

This observation together with (3.11) and the Lebesgue dominated convergence theorem implies that for any $\theta\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$,

$$\begin{aligned}\langle\psi_{j_1,\,j_2}*g,\,\theta\rangle&=\lim_{\delta\to0}\langle\psi_{j_1,\,j_2}*g_\delta,\,\theta\rangle\\&=\lim_{\delta\to0}\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b^{-j}g_\delta(\vec{A}_{-j_1,\,-j_2}\vec{k})\langle\psi_{j_1,\,j_2}(\cdot-\vec{A}_{-j_1,\,-j_2}\vec{k}),\,\theta\rangle\\&=\sum_{\vec{k}\in\mathbb{Z}^n\times\mathbb{Z}^m} b_1^{-j_1}b_2^{-j_2}g(\vec{A}^{-j_1,\,-j_2}\vec{k})\langle\psi_{j_1,\,j_2}(\cdot-\vec{A}_{-j_1,\,-j_2}\vec{k}),\,\theta\rangle.\end{aligned}$$

Thus, (3.8) holds in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, which completes the proof of Lemma 2.1. □

Proof of Proposition 2.1. Noticing that $\mathcal{S}'_{\infty,\,w}(\mathbb{R}^n\times\mathbb{R}^m)\subset\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, we obviously have $H^p_w(\vec{A})\subset\widetilde{H}^p_w(\vec{A})$.

Conversely, let $(\Psi,\,\Phi)$ be as in Definition 2.5. For any $f\in\widetilde{H}^p_w(\vec{A})$, by Lemma 2.1(i) with $\phi\equiv\Psi*\Phi$, we obtain $f=\sum_{j_1,j_2\in\mathbb{Z}}\Psi_{j_1,j_2}*\Phi_{j_1,j_2}*f$ in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$. This result serves as a replacement of [8, Proposition 2.16] which is applicable only to elements in

$\mathcal{S}'_{\infty,w}(\mathbb{R}^n\times\mathbb{R}^m)$. By repeating the proof of [8, Lemma 4.6] with this modification, we obtain the atomic decomposition $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$ in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, where $\{a_j\}_{j\in\mathbb{N}}$ are $(p,\,q,\,\vec{s})_w$-atoms as in [8, Definition 4.2], $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ and $(\sum_{j\in\mathbb{N}}|\lambda_j|^p)^{1/p}\lesssim\|f\|_{\widetilde{H}^p_w(\vec{A})}$. Now, for any $\varphi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$, by the proof of [8, Lemma 4.10] (see [8, p. 424]), we have that $|\int_{\mathbb{R}^n\times\mathbb{R}^m}a_j(x)\varphi(x)\,dx|=|a_j*\widetilde{\varphi}(0)|\lesssim 1$, where $\widetilde{\varphi}(\cdot)\equiv\varphi(-\cdot)$.

Thus, if we define $\langle\breve{f},\,\varphi\rangle\equiv\sum_{j\in\mathbb{N}}\lambda_j\int_{\mathbb{R}^n\times\mathbb{R}^m}a_j(x)\varphi(x)\,dx$, then $|\langle\breve{f},\,\varphi\rangle|\lesssim\sum_{j\in\mathbb{N}}|\lambda_j|\lesssim\|f\|_{\widetilde{H}^p_w(\vec{A})}$. Therefore, $\breve{f}\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$, $\breve{f}=f$ in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ and $\breve{f}=\sum_{j\in\mathbb{N}}\lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$, which together with [8, Theorem 4.5] implies that $\breve{f}\in H^p_w(\vec{A})$ and $\|\breve{f}\|_{H^p_w(\vec{A})}\lesssim\|f\|_{\widetilde{H}^p_w(\vec{A})}$.

Now let us prove that the extension is unique. Assume that there are two extension $\breve{f}_1,\breve{f}_2\in H^p_w(\vec{A})$ with $\breve{f}_1=\breve{f}_2=f$ in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$. We need to check that $g\equiv\breve{f}_1-\breve{f}_2=0\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$.

Set $\mathscr{E}\equiv\{(x_1,\,x_2)\in\mathbb{R}^n\times\mathbb{R}^m:\,x_1=\vec{0}_n\text{ or }x_2=\vec{0}_m\}$. Let us first prove $\operatorname{supp}\widehat{g}\subset\mathscr{E}$. Take any $x\in(\mathbb{R}^n\times\mathbb{R}^m)\setminus\mathscr{E}$ and sufficiently small positive numbers δ_1 and δ_2 such that $(B_{\rho_1}(x_1,\,\delta_1)\times B_{\rho_2}(x_2,\,\delta_2))\cap\mathscr{E}=\emptyset$. Then, for any $\varphi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$ with $\operatorname{supp}\varphi\subset(B_{\rho_1}(x_1,\,\delta_1)\times B_{\rho_2}(x_2,\,\delta_2))$, we have $\langle\widehat{g},\,\varphi\rangle=0$. Indeed, for any $(x_1',\,x_2')\in\mathbb{R}^n\times\mathbb{R}^m$ and $\alpha_1\in\mathbb{Z}^n_+$, $\alpha_2\in\mathbb{Z}^m_+$, we have $\partial^{\alpha_1}\varphi(\vec{0}_n,\,x_2')=\partial^{\alpha_2}\varphi(x_1',\,\vec{0}_m)=0$, which implies that $\widehat{\varphi}\in\mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$. From this and $g=0$ in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, it follows that $\langle\widehat{g},\,\varphi\rangle=\langle g,\,\widehat{\varphi}\rangle=0$. Thus, $\operatorname{supp}\widehat{g}\subset\mathscr{E}$.

Finally, let us prove $g=0$ in $\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$. Since $\breve{f}_1,\,\breve{f}_2\in H^p_w(\vec{A})$, we have $g\in H^p_w(\vec{A})$. By this and Lemma 4.10 in [8], we obtain that $g\in\mathcal{S}'_{\infty,\,w}(\mathbb{R}^n\times\mathbb{R}^m)$. Let $\phi\equiv\phi^{(1)}\otimes\phi^{(2)}$ be as in Lemma 2.1(i). Then, there exist two integers k_i and ℓ_i such that $k_i\ge\ell_i$ and $\operatorname{supp}\widehat{\phi^i}\subset B_{\rho_i}(\vec{0}_{n_i},\,b_i^{k_i})\setminus B_{\rho_i}(\vec{0}_{n_i},\,b_i^{\ell_i})$, $i=1,\,2$. By the Calderón reproducing formula [8, Lemma 2.15], we have $g=\sum_{j_1,\,j_2\in\mathbb{Z}}g*\phi_{j_1,\,j_2}$ with convergence in $\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$. Therefore, for any $\varphi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$, we obtain

$$\langle g,\,\varphi\rangle=\sum_{j_1,\,j_2\in\mathbb{Z}}\langle g*\phi_{j_1,\,j_2},\,\varphi\rangle=\sum_{j_1,\,j_2\in\mathbb{Z}}\langle\widehat{g}(\cdot)\widehat{\phi}((A_1^*)^{-j_1}\cdot,\,(A_2^*)^{-j_2}\cdot),\,\widehat{\varphi}(-\cdot)\rangle. \tag{3.12}$$

Observe that for any $j_1,\,j_2\in\mathbb{Z}$,

$$\begin{aligned}&\operatorname{supp}\widehat{\phi}((A_1^*)^{-j_1}\cdot,\,(A_2^*)^{-j_2}\cdot)\\&\quad\subset\left[B_{\rho_1}(\vec{0}_n,\,b_1^{k_1+j_1})\setminus B_{\rho_1}(\vec{0}_n,\,b_1^{\ell_1+j_1})\right]\times\left[B_{\rho_2}(\vec{0}_m,\,b_2^{k_2+j_2})\setminus B_{\rho_2}(\vec{0}_m,\,b_2^{\ell_2+j_2})\right].\end{aligned}$$

From this and $\operatorname{supp}\widehat{g}\subset\mathscr{E}$, it follows that for any $j_1,\,j_2\in\mathbb{Z}$,

$$(\operatorname{supp}\widehat{g})\cap(\operatorname{supp}\widehat{\phi}((A_1^*)^{-j_1}\cdot,\,(A_2^*)^{-j_2}\cdot))=\emptyset.$$

Combining this with (3.12) yields $\langle g,\,\varphi\rangle=0$. This finishes the proof of Proposition 2.1. $\square$

The proof of Theorem 2.1 needs a series of lemmas. First, we need to show that the φ-transform T_ψ as in Definition 2.9 is well defined, respectively, on $\ddot{h}^p_w(\vec{A})$ and $\ell_{p,\,w}(\vec{A})$. Let us begin with the following technical lemma.

Lemma 3.1. *Let $\Phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $\Psi \equiv \psi^{(1)} \otimes \psi^{(2)}$, where $\psi^{(i)} \in \mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i = 1, 2$. For any positive constants L_1 and L_2, there exist positive integers N_1 and N_2 and positive constant C, depending only on L_1 and L_2, such that for all $P, Q \in \mathcal{R}$*

$$|\langle \Psi_Q, \Phi_P \rangle| \le C \|\Psi\|_{N_1, N_2} \|\Phi\|_{N_1, N_2} \prod_{i=1}^2 \left\{ \left[1 + \frac{\rho_i(x_{Q_i} - x_{P_i})}{|Q_i| \vee |P_i|} \right]^{-L_i} \left(\frac{|Q_i|}{|P_i|} \wedge \frac{|P_i|}{|Q_i|} \right)^{L_i} \right\},$$

where

$$\|\Psi\|_{N_1, N_2} \equiv \sup_{(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^m} \sup_{|\gamma_1| \le N_1, |\gamma_2| \le N_2} (1 + |x_1|)^{N_1} (1 + |x_2|)^{N_2} |\partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \Psi(x_1, x_2)|.$$

Proof. To prove this lemma, we need the estimate [2, (3.18)]. Precisely, for any $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and positive constant L, there exist a positive constant C and a positive integer N, depending only on L, such that for all $P, Q \in \mathcal{Q}_1$,

$$|\langle \varphi_Q, \phi_P \rangle| \le C \|\varphi\|_N \|\phi\|_N \left(1 + \frac{\rho(x_Q - x_P)}{|Q| \vee |P|} \right)^{-L} \left(\frac{|Q|}{|P|} \wedge \frac{|P|}{|Q|} \right)^L, \tag{3.13}$$

where $\|\varphi\|_N \equiv \sup_{x \in \mathbb{R}^n, |\gamma| \le N} [1 + |x|]^N |\partial^\gamma \varphi(x)|$.

For any $P, Q \in \mathcal{R}$, $\Phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $\Psi \equiv \psi^{(1)} \otimes \psi^{(2)}$ with $\psi^{(i)} \in \mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i = 1, 2$, let $\Psi_Q \equiv \psi^{(1)}_{Q_1} \otimes \psi^{(2)}_{Q_2}$ and Φ_P be as in (2.1). Moreover, for any $x \equiv (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$ and $P \equiv P_1 \times P_2 \in \mathcal{R}$ with $P_i \equiv A_i^{-j_i}([0, 1)^{n_i} + k_i)$, $k_i \in \mathbb{Z}_i^{n_i}$ $i = 1, 2$, we set $\Phi_{P_2}(x_1, x_2) \equiv |P_2|^{-1/2} \Phi(x_1, A_2^{j_2} x_2 - k_2)$. Then, it is easy to check that

$$\langle \psi^{(2)}_{Q_2}, \Phi_P \rangle \equiv \int_{\mathbb{R}^m} \psi^{(2)}_{Q_2}(x_2) \overline{\Phi_P(x_1, x_2)}\, dx_2 \equiv \langle \psi^{(2)}_{Q_2}, \Phi_{P_2} \rangle_{P_1} \in \mathcal{S}_\infty(\mathbb{R}^n).$$

Consequently, using the fact that $\Phi_{P_2}(x_1, \cdot) \in \mathcal{S}_\infty(\mathbb{R}^m)$ for all $x_1 \in \mathbb{R}^n$ and (3.13) twice (respectively with dilated cubes of $\mathbb{R}^n$ and $\mathbb{R}^m$), for any positive constants L_1 and L_2, there exist positive integers N_1 and N_2, depending only on L_1 and L_2, such that

$$\begin{aligned}
|\langle \Psi_Q, \Phi_P \rangle| &= |\langle \psi^{(1)}_{Q_1}, \langle \psi^{(2)}_{Q_2}, \Phi_{P_2} \rangle_{P_1} \rangle| \\
&\lesssim \|\psi^{(1)}\|_{N_1} \|\langle \psi^{(2)}_{Q_2}, \Phi_{P_2} \rangle\|_{N_1} \left(1 + \frac{\rho_1(x_{Q_1} - x_{P_1})}{|Q_1| \vee |P_1|} \right)^{-L_1} \left(\frac{|Q_1|}{|P_1|} \wedge \frac{|P_1|}{|Q_1|} \right)^{L_1} \\
&\lesssim \|\psi^{(1)}\|_{N_1} \sup_{x_1 \in \mathbb{R}^n} \sup_{|\gamma_1| \le N_1} (1 + |x_1|)^{N_1} \|\psi^{(2)}\|_{N_2} \left\| \partial_{x_1}^{\gamma_1} \Phi(x_1, \cdot) \right\|_{N_2} \\
&\quad \times \prod_{i=1}^2 \left\{ \left[1 + \frac{\rho_i(x_{Q_i} - x_{P_i})}{|Q_i| \vee |P_i|} \right]^{-L_i} \left(\frac{|Q_i|}{|P_i|} \wedge \frac{|P_i|}{|Q_i|} \right)^{L_i} \right\} \\
&\sim \|\Psi\|_{N_1, N_2} \|\Phi\|_{N_1, N_2} \prod_{i=1}^2 \left\{ \left[1 + \frac{\rho_i(x_{Q_i} - x_{P_i})}{|Q_i| \vee |P_i|} \right]^{-L_i} \left(\frac{|Q_i|}{|P_i|} \wedge \frac{|P_i|}{|Q_i|} \right)^{L_i} \right\},
\end{aligned}$$

which completes the proof of Lemma 3.1. □

The following technical lemma is just [8, Proposition 2.10(i)].

Lemma 3.2 ([8]). *Let $q \in (1, \infty)$ and $w \in \mathcal{A}_q(\vec{A})$. Then there exists a positive constant C such that for all $x \in \mathbb{R}^n \times \mathbb{R}^m$ and $k_i \in \mathbb{Z}_+$, $\ell_i \in \mathbb{Z}$ with $i = 1, 2$,*

$$\frac{w(B_{\rho_1}(x_1, b_1^{k_1+\ell_1}) \times B_{\rho_2}(x_2, b_2^{k_2+\ell_2}))}{w(B_{\rho_1}(x_1, b_1^{\ell_1}) \times B_{\rho_2}(x_2, b_2^{\ell_2}))} \le C\frac{|B_{\rho_1}(x_1, b_1^{k_1+\ell_1}) \times B_{\rho_2}(x_2, b_2^{k_2+\ell_2})|^q}{|B_{\rho_1}(x_1, b_1^{\ell_1}) \times B_{\rho_2}(x_2, b_2^{\ell_2})|^q} \sim [b_1^{k_1} b_2^{k_2}]^q.$$

For any $w \in \mathcal{A}_\infty(\vec{A})$, the *critical index* of w is defined by

$$q_w \equiv \inf\{q \in (1, \infty) : w \in \mathcal{A}_q(\vec{A})\}. \tag{3.14}$$

Obviously, $q_w \in [1, \infty)$ and if $q_w \in (1, \infty)$, then $w \notin \mathcal{A}_{q_w}(\vec{A})$. Moreover, Johnson and Neugebauer [25, p. 254] gave an example of $w \notin \mathcal{A}_1(2I_{n\times n})$ such that $q_w = 1$; see also [8].

Lemma 3.3. *Let $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14), $q \in (q_w, \infty)$ and $\delta \in \mathbb{R}$. Then, there exist positive constants L_1, L_2 and C, depending on δ, such that for all $j_1, j_2 \in \mathbb{Z}$,*

$$\sum_{R\in\mathcal{R},\ \mathrm{scale}\,(R)=(j_1, j_2)} \frac{[w(R)]^\delta}{\prod_{i=1}^2 [1 + \rho_i(x_{R_i})/(1 \vee |R_i|)]^{L_i}} \le C \prod_{i=1}^2 b_i^{(2q|\delta|+1)|j_i|}.$$

Proof. The proof follows along the lines of its one parameter variant [2, Lemma 2.11]. The key observation is that the measure $w(x)\,dx$ is doubling with respect to the action of 2-parameter group of dilations $\vec{A}_{j_1,j_2}$. Indeed, by Lemma 3.2, there exists a positive constant $C > 0$ such that for all $x \in \mathbb{R}^n \times \mathbb{R}^m$, $r_1, r_2 > 0$, and $k_1, k_2 \in \mathbb{Z}_+$,

$$w\left(\prod_{i=1}^2 B_{\rho_i}(x_i, (b_i)^{k_i} r_i)\right) \le C b_1^{k_1 q} b_2^{k_2 q} w\left(\prod_{i=1}^2 B_{\rho_i}(x_i, r_i)\right).$$

This practically means that w is a product q-doubling measure albeit with a positive constant C. In particular, for any dilated rectangles $P \equiv P_1 \times P_2, R \equiv R_1 \times R_2 \in \mathcal{R}$ of the form $P_i \equiv A_i^{j_i}([0,1)^{n_i} + k_i)$ and $R_i = A_i^{j_i}([0,1)^{n_i} + l_i)$, we have the following analogue of [2, (2.6)], namely,

$$w(R) \lesssim \prod_{i=1}^2 (1 + \rho_i(k_i - l_i))^q w(P). \tag{3.15}$$

Mimicking the proof of [2, Lemma 2.11] by considering four cases that $j_1, j_2 \ge 0$, $j_1, j_2 < 0$, $j_1 \ge 0$, $j_2 < 0$ and $j_1 < 0$, $j_2 \ge 0$, yields the desired estimate in Lemma 3.3. □

Lemma 3.4. *Suppose that $w \in \mathcal{A}_\infty(\vec{A})$, $p \in (0, \infty)$ and $\psi \equiv \psi^{(1)} \otimes \psi^{(2)}$, where $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$, satisfies that $\mathrm{supp}\,\widehat{\psi^{(i)}}$ is compact and bounded away from the origin. Then, the inverse φ-transform $T_\psi : \ddot{h}_w^p(\vec{A}) \to \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is well-defined and continuous. The same result holds if $\ddot{h}_w^p(\vec{A})$ is replaced by $\ell_{p,w}(\vec{A})$, $p \in (0, 1]$.*

Proof. For any $s \in \ddot{h}^p_w(\vec{A})$, by the definition of $\ddot{h}^p_w(\vec{A})$, we have that for all $Q \in \mathcal{R}$,

$$|s_Q| \le \|s\|_{\ddot{h}^p_w(\vec{A})} |Q|^{\frac{1}{2}} |[w(Q)]^{-1/p}.$$

Applying Lemma 3.1 for any $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $P \equiv [0, 1)^n \times [0, 1)^m$ yields that for all $Q \in \mathcal{R}$,

$$|\langle \psi_Q, \phi\rangle| \lesssim \|\phi\|_{N_1, N_2} \prod_{i=1}^2 \left[1 + \frac{\rho_i(x_{Q_i})}{1 \vee |Q_i|}\right]^{-L_i} (|Q_i| \wedge |Q_i|^{-1})^{L_i}.$$

Combining the above estimates with Lemma 3.3 yields that for sufficiently large L_1 and L_2,

$$\begin{aligned}
&\sum_{Q\in\mathcal{R}} |s_Q||\langle \psi_Q, \phi\rangle| \\
&\quad\lesssim \|\phi\|_{N_1, N_2} \|s\|_{\ddot{h}^p_w(\vec{A})} \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{\mathrm{scale}\,(Q)=(j_1, j_2)} \prod_{i=1}^2 b_i^{j_i/2 - |j_i| L_i} \frac{[w(Q)]^{-1/p}}{[1 + \frac{\rho_i(x_{Q_i})}{1 \vee |Q_i|}]^{L_i}} \\
&\quad\lesssim \|\phi\|_{N_1, N_2} \|s\|_{\ddot{h}^p_w(\vec{A})} \sum_{j_1, j_2 \in \mathbb{Z}} \prod_{i=1}^2 b_i^{j_i/2 + |j_i|(2q/p+1) - |j_i| L_i} \lesssim \|\phi\|_{N_1, N_2} \|s\|_{\ddot{h}^p_w(\vec{A})},
\end{aligned}$$

where $q \in (q_w, \infty)$. Thus, by the definition of $T_\psi s$, we have that for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\langle T_\psi s, \phi\rangle = \sum_{Q\in\mathcal{R}} s_Q \langle \psi_Q, \phi\rangle.$$

Moreover, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$|\langle T_\psi s, \phi\rangle| \lesssim \|\phi\|_{N_1, N_2} \|s\|_{\ddot{h}^p_w(\vec{A})},$$

which implies that $T_\psi : \ddot{h}^p_w(\vec{A}) \to \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is continuous.

Moreover, for any $s \in \ell_{p,w}(\vec{A})$, by the definition of $\ell_{p,w}(\vec{A})$, we obtain that $|s_Q| \le \|s\|_{\ell_{p,w}(\vec{A})} [w(Q)]^{1/p} |Q|^{-1/2}$ for all $Q \in \mathcal{R}$. Then, repeating the above proof for $\ddot{h}^p_w(\vec{A})$ yields the desired results for T_ψ on $\ell_{p,w}(\vec{A})$. This finishes the proof of Lemma 3.4. □

Motivated by [2, Definition 3.9], we introduce the notion of majorant sequences.

Definition 3.2. Given a complex-valued sequence $s \equiv \{s_R\}_{R\in\mathcal{R}}$ and $r, \lambda > 0$, define its majorant sequence $s^*_{r,\lambda} \equiv \{(s^*_{r,\lambda})_R\}_{R\in\mathcal{R}}$ by

$$(s^*_{r,\lambda})_R \equiv \left\{ \sum_{P\in\mathcal{R},\ \mathrm{scale}\,(P)=\mathrm{scale}\,(R)} \frac{|s_P|^r}{\prod_{i=1}^2 [1 + |R_i|^{-1} \rho_i(x_{R_i} - x_{P_i})]^\lambda} \right\}^{1/r}.$$

The spaces $\mathcal{L}_{p,\,w}(\vec{A})$, $\ell_{p,\,w}(\vec{A})$, and the sequences $\{(s^*_{r,\,\lambda})_Q\}_{Q\in\mathcal{R}}$ can also be defined "equivalently" via generalized dyadic rectangles associated to $\vec{A}$ in some sense. Precisely, let us begin with recalling the dyadic cubes associated to A introduced in [8, Lemma 2.3], which is a slight variant of [14, Theorem 11].

Lemma 3.5. *Let A be a dilation. Then there exists a set $\dot{\mathcal{Q}}_1\equiv\{\dot{Q}^k_\alpha\subset\mathbb{R}^n:\ k\in\mathbb{Z},\ \alpha\in\mathrm{I}_k\}$ of open subsets, where I_k is some index set, such that*

(i) $|\mathbb{R}^n\setminus(\cup_\alpha\dot{Q}^k_\alpha)|=0$ for each fixed k and $\dot{Q}^k_\alpha\cap\dot{Q}^k_\beta=\emptyset$ if $\alpha\neq\beta$;

(ii) for any $\alpha,\ \beta,\ k,\ \ell$ with $\ell\geq k$, either $\dot{Q}^k_\alpha\cap\dot{Q}^\ell_\beta=\emptyset$ or $\dot{Q}^\ell_\alpha\subset\dot{Q}^k_\beta$;

(iii) for each $(\ell,\ \beta)$ and each $k<\ell$, there exists a unique α such that $\dot{Q}^\ell_\beta\subset\dot{Q}^k_\alpha$;

(iv) there exist certain negative integer v and positive integer u such that for all $\dot{Q}^k_\alpha$ with $k\in\mathbb{Z}$ and $\alpha\in\mathrm{I}_k$, there exists $c_{\dot{Q}^k_\alpha}\in\dot{Q}^k_\alpha$ satisfying that for all $x\in\dot{Q}^k_\alpha$, $B_\rho(c_{\dot{Q}^k_\alpha},\ b^{vk-u})\subset\dot{Q}^k_\alpha\subset B_\rho(x,\ b^{vk+u})$.

In what follows, for convenience, we call $\{\dot{Q}^k_\alpha\}_{k\in\mathbb{Z},\,\alpha\in\mathrm{I}_k}$ in Lemma 3.5 *dyadic cubes.* Also for any dyadic cube $\dot{Q}^k_\alpha$ with $k\in\mathbb{Z}$ and $\alpha\in\mathrm{I}_k$, we always define $\ell(\dot{Q}^k_\alpha)\equiv k$ be its *level.*

Let A_i be a dilation on $\mathbb{R}^{n_i}$, and $\dot{\mathcal{Q}}_i$, $\ell(\dot{Q}_i)$, v_i, u_i *the same as in Lemma 3.5 corresponding to A_i for $i=1,\,2$.* Let $\dot{\mathcal{R}}\equiv\dot{\mathcal{Q}}_1\times\dot{\mathcal{Q}}_2$. For $\dot{R}\in\dot{\mathcal{R}}$, we always write $\dot{R}\equiv\dot{R}_1\times\dot{R}_2$ with $\dot{R}_i\in\dot{\mathcal{Q}}_i$ and call $\dot{R}$ a *dyadic rectangle.* Moreover, we set $\ell(\dot{R})\equiv(\ell(\dot{R}_1),\ \ell(\dot{R}_2))$, and $\ell(\dot{R})\leq\ell(\dot{P})$ always means that $\ell(\dot{R}_i)\leq\ell(\dot{P}_i)$, $i=1,\,2$.

Definition 3.3. For two sets $E_1\subset\mathbb{R}^n$ and $E_2\subset\mathbb{R}^m$, set $\prod_{i=1}^2E_i\equiv E_1\times E_2$. For any locally integrable function f on $\mathbb{R}^n\times\mathbb{R}^m$, the *strong maximal function* $\mathcal{M}_s(f)$ of f is defined by setting, for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$\mathcal{M}_s(f)(x)\equiv\sup_{y\in\mathbb{R}^n\times\mathbb{R}^m,\,r_1,\,r_2>0}\ \sup_{x\in\prod_{i=1}^2B_{\rho_i(y_i,\,r_i)}}\frac{1}{|\prod_{i=1}^2B_{\rho_i}(y_i,\,r_i)|}\int_{\prod_{i=1}^2B_{\rho_i}(y_i,\,r_i)}|f(z)|\,dz.$$

The following lemma comes from [2, Lemma 2.9(a)].

Lemma 3.6. *There exists a positive integer $\tau_i\equiv\tau_i(A_i,\ n_i)$ for all $Q_i\equiv A^{j_i}((0,\,1]^{n_i}+k_i)$ with $j_i\in\mathbb{Z}$ and $k_i\in\mathbb{Z}^{n_i}$, $i=1,\,2$, such that $B_{\rho_i}(c_{Q_i},\ b_i^{j_i-\tau_i})\subset Q_i\subset B_{\rho_i}(c_{Q_i},\ b_i^{j_i+\tau_i})$.*

In what follows, for any $\alpha\in\mathbb{R}$, we denote by $\lfloor\alpha\rfloor$ *the maximal integer not more than* α. Recall that $\sharp E$ denotes the cardinality of the set E.

Lemma 3.7. *(i) For any $\dot{R}\in\dot{\mathcal{R}}$, set $U_{\dot{R}}\equiv U_{\dot{R}_1}\times U_{\dot{R}_2}$ with*

$$U_{\dot{R}_i}\equiv\{R_i\in\mathcal{Q}_i:\ R_i\cap\dot{R}_i\neq\emptyset,\quad\ell(\dot{R}_i)=\lfloor(\mathrm{scale}\,(R_i)-u_i)/v_i\rfloor\},\ i=1,\,2.$$

For any $R\in\mathcal{R}$, set $U_R\equiv U_{R_1}\times U_{R_2}$ with

$$U_{R_i}\equiv\{\dot{R}_i\in\dot{\mathcal{Q}}_i:\ R_i\cap\dot{R}_i\neq\emptyset,\quad\ell(\dot{R}_i)=\lfloor(\mathrm{scale}\,(R_i)-u_i)/v_i\rfloor\},\ i=1,\,2.$$

Then, there exists a positive integer $\widetilde{N}$ such that for all $R\in\mathcal{R}$ and $\dot{R}\in\dot{\mathcal{R}}$, $\sharp U_{\dot{R}}+\sharp U_R\leq\widetilde{N}$. Moreover, for all $w\in\mathcal{A}_\infty(\vec{A})$, $R\in U_{\dot{R}}$ and $\dot{R}\in U_R$, $w(\dot{R})\sim w(R)$.

(ii) There exists a positive constant $\eta_0 \equiv \eta_0(\vec{A}, n, m) \in (0, 1)$ *such that for any open set* $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$,

$$\bigcup_{R \in \mathcal{R},\, R \subset \Omega} \bigcup_{\dot{R} \in U_R} \dot{R} \subset \Omega^{(0)} \equiv \{x \in \mathbb{R}^n \times \mathbb{R}^m : \mathcal{M}_s(\chi_\Omega)(x) > \eta_0\}.$$

Proof. (i) For any $R \in U_{\dot{R}}$, let $\widetilde{x}_i \in R_i \cap \dot{R}_i$, $i = 1, 2$. Then, for any $x_i \in R_i$, by Lemma 3.6, Lemma 3.5(iv) and $\operatorname{scale}(R_i) \le v_i \ell(\dot{R}_i) + u_i$, we obtain

$$\rho_i(x_i - c_{\dot{R}_i}) \le H_i^2[\rho_i(x_i - c_{R_i}) + \rho_i(c_{R_i} - \widetilde{x}_i) + \rho_i(\widetilde{x}_i - c_{\dot{R}_i})] \le 3H_i^2 b_i^{v_i \ell(\dot{R}_i) + u_i + \tau_i},$$

which implies that

$$\bigcup_{R_i \in U_{\dot{R}_i}} R_i \subset B_{\rho_i}(c_{\dot{R}_i}, 3H_i^2 b_i^{v_i \ell(\dot{R}_i) + u_i + \tau_i}),\ i = 1, 2. \tag{3.16}$$

From this, $v_i[\ell(\dot{R}_i) + 1] + u_i < \operatorname{scale}(R_i) \le v_i \ell(\dot{R}_i) + u_i$ and Lemma 3.6, it follows that

$$\sharp U_{\dot{R}} \le v_1 v_2 \frac{|\prod_{i=1}^2 B_{\rho_i}(c_{\dot{R}_i}, 3H_i^2 b_i^{v_i \ell(\dot{R}_i) + u_i + \tau_i})|}{\prod_{i=1}^2 b_i^{v_i[\ell(\dot{R}_i)+1] + u_i - \tau_i}} \lesssim 1.$$

Moreover, by (3.16), Lemma 3.2 and Lemma 3.5(iv), we have

$$w(R) \le w(\textstyle\prod_{i=1}^2 B_{\rho_i}(c_{\dot{R}_i}, 3H_i^2 b_i^{v_i \ell(\dot{R}_i) + u_i + \tau_i})) \lesssim w(\prod_{i=1}^2 B_{\rho_i}(c_{\dot{R}_i}, b_i^{v_i \ell(\dot{R}_i) - u_i})) \lesssim w(\dot{R}).$$

The converse inequality also holds via an argument similar to the above and hence $w(\dot{R}) \sim w(R)$.

Similarly, for any $R \in \mathcal{R}$, we also have that $\sharp U_R \lesssim 1$, and for all $\dot{R} \in U_R$, $w(R) \sim w(\dot{R})$.

(ii) For any $R \in \mathcal{R}$ and $R \subset \Omega$ with $\operatorname{scale}(R) = (j_1, j_2)$, $\dot{R} \in U_R$ and $x \in \dot{R}$, by an estimate similar to that of (3.16), we have $\dot{R}_i \subset B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i})$. From this and Lemma 3.6, it follows that

$$\begin{aligned}
\mathcal{M}_s(\chi_\Omega)(x) &\ge \frac{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i}) \cap \Omega|}{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i})|} \\
&\ge \frac{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, b_i^{j_i - \tau_i})|}{|\prod_{i=1}^2 B_{\rho_i}(c_{R_i}, 3H_i^2 b_i^{j_i - v_i + \tau_i})|} > \frac{1}{10} \prod_{i=1}^2 (H_i^2)^{-1} b_i^{v_i - 2\tau_i} \equiv \eta_0,
\end{aligned}$$

which completes the proof of Lemma 3.7. □

Definition 3.4. For any $R \in \mathcal{R}$ and U_R as in Lemma 3.7(i), let $\chi_{U_R}(\dot{R})$ equals one if $\dot{R} \in U_R$ or else zero. For any $\dot{R} \in \dot{\mathcal{R}}$ and $U_{\dot{R}}$ as in Lemma 3.7(i), let $\chi_{U_{\dot{R}}}(R)$ be similarly defined.

Lemma 3.8. *(i) For any complex-valued sequence* $s \equiv \{s_R\}_{R\in\mathcal{R}}$*, its induced sequence* $\dot{s}$ *is defined by setting* $\dot{s} \equiv \{\dot{s}_{\dot{R}}\}_{\dot{R}\in\dot{\mathcal{R}}}$*, where* $\dot{s}_{\dot{R}} \equiv \sum_{R\in U_{\dot{R}}} |s_R|$ *with* $U_{\dot{R}}$ *as in Lemma 3.7(i). Then, for any* $w \in \mathcal{A}_\infty(\vec{A})$ *and* $p \in (0, 1]$*, there exists a positive constant* C *such that* $\|\dot{s}\|_{\dot{\ell}_{p,\,w}(\vec{A})} \le C\|s\|_{\ell_{p,\,w}(\vec{A})}$*, where the definition of the norm* $\|\cdot\|_{\dot{\ell}_{p,\,w}(\vec{A})}$ *is the same as* $\|\cdot\|_{\ell_{p,\,w}(\vec{A})}$ *but* R *and* $\mathcal{R}$ *are, respectively, replaced by* $\dot{R}$ *and* $\dot{\mathcal{R}}$*.*

(ii) For any $\lambda \in (0, \infty)$*, the majorant sequence* $\{(\dot{s}^*_{2,\,\lambda})_{\dot{R}}\}_{\dot{R}\in\dot{\mathcal{R}}}$ *of* $\dot{s}$ *is defined to be the same as in Definition 3.2 but* R*,* $\mathcal{R}$*,* x_{R_i}*,* x_{P_i}*,* $\mathrm{scale}\,(P)$ *and* $\mathrm{scale}\,(R)$ *are, respectively, replaced by* $\dot{R}$*,* $\dot{\mathcal{R}}$*,* $c_{\dot{R}_i}$*,* $c_{\dot{P}_i}$*,* $\ell(\dot{P})$ *and* $\ell(\dot{R})$*,* $i = 1, 2$*. Then, there exists a positive constant* C *such that for all* $R \in \mathcal{R}$ *and* $\dot{R} \in U_R$*,* $(s^*_{2,\,\lambda})_R \le C(\dot{s}^*_{2,\,\lambda})_{\dot{R}}$*.*

Proof. (i) Let $w \in \mathcal{A}_\infty(\vec{A})$. For any $\dot{R} \in \dot{\mathcal{R}}$, let $U_{\dot{R}}$ be as in Lemma 3.7(i). Then, for any $R \in U_{\dot{R}}$, by Lemma 3.7(i), we have

$$w(R) \sim w(\dot{R}) \text{ and } |\dot{R}| \sim |R|. \tag{3.17}$$

Moreover, for any open set Ω of $\mathbb{R}^n \times \mathbb{R}^m$, similarly to the proof of Lemma 3.7(ii), there exists a positive constant $\widetilde{\eta}_0 \equiv \widetilde{\eta}_0(n,\, m,\, \vec{A}) \in (0, 1)$ such that

$$\bigcup_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\Omega}\ \bigcup_{R\in U_{\dot{R}}} \subset \widetilde{\Omega}^{(0)}, \tag{3.18}$$

where $\widetilde{\Omega}^{(0)} \equiv \{x \in \mathbb{R}^n \times \mathbb{R}^m : \mathcal{M}_s\chi_\Omega(x) > \widetilde{\eta}_0\}$. Furthermore, for any $q \in (q_w, \infty)$, by the $L^q_w(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ (see [8, Proposition 2.10(ii)]), we also have $w(\widetilde{\Omega}^{(0)}) \lesssim w(\Omega)$. Therefore, for any open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$, by (3.17), (3.18), $w(\widetilde{\Omega}^{(0)}) \lesssim w(\Omega)$, $p \in (0, 1]$ and Lemma 3.7, we have

$$\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\Omega} |\dot{s}_{\dot{R}}|^2 |\frac{|\dot{R}|}{w(\dot{R})}\\
&\quad\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\Omega}\ \sum_{R\in\mathcal{R}} |s_R|^2 \frac{|R|}{w(R)}\chi_{U_{\dot{R}}}(R)\\
&\quad\lesssim \frac{1}{[w(\widetilde{\Omega}^{(0)})]^{\frac{2}{p}-1}} \sum_{R\in\mathcal{R},\,R\subset\widetilde{\Omega}^{(0)}} |s_R|^2 \frac{|R|}{w(R)} \sum_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\Omega} \chi_{U_R}(\dot{R}) \lesssim \|s\|^2_{\ell_{p,\,w}(\vec{A})}.
\end{aligned}$$

From this and the arbitrariness of the open set Ω, it further follows that $\|\dot{s}\|_{\dot{\ell}_{p,\,w}(\vec{A})} \lesssim \|s\|_{\ell_{p,\,w}(\vec{A})}$.

(ii) For any $\lambda \in (0, \infty)$, $R \in \mathcal{R}$ with $\mathrm{scale}\,(R) = (j_1, j_2)$ and any $\dot{R} \in U_R$, let us prove that $(s^*_{2,\,\lambda})_R \lesssim (\dot{s}^*_{2,\,\lambda})_{\dot{R}}$. To this end, we choose any other $P \in \mathcal{R}$ with $\mathrm{scale}\,(P) = \mathrm{scale}\,(R)$ and $\dot{P} \in U_P$ with $\ell(\dot{P}) = \ell(\dot{R})$. Let $x_{R_i} \equiv A_i^{j_i} k_i$ and $x_{P_i} \equiv A_i^{j_i}\widetilde{k}_i$ with $k_i, \widetilde{k}_i \in \mathbb{Z}^{n_i}$ and $k_i \ne \widetilde{k}_i$, $i = 1, 2$. Then, by (3.1) and (3.2), there exists a constant c_i such that $\rho_i(k_i - \widetilde{k}_i) \ge b_i^{c_i}$ and hence $\rho_i(x_{R_i} - x_{P_i}) = b_i^{j_i}\rho(k_i - \widetilde{k}_i) \ge b_i^{j_i + c_i}$. Let $\breve{x}_i \in R_i \cap \dot{R}_i$ and $\widetilde{x}_i \in P_i \cap \dot{P}_i$. Therefore, for $i = 1, 2$, by this estimate, Lemma 3.6, $v_i\ell(\dot{P}_i) + u_i < j_i - v_i$, and Lemma 3.5(iv), we have

$$\rho_i(c_{\dot{R}_i} - c_{\dot{P}_i}) \le H_i^6[\rho_i(c_{\dot{R}_i} - \breve{x}_i) + \rho_i(\breve{x}_i - c_{R_i}) + \rho_i(c_{R_i} - x_{R_i}) + \rho_i(x_{R_i} - x_{P_i})$$

$$+\rho_i(x_{P_i}-c_{\dot{P}_i})+\rho_i(c_{\dot{P}_i}-\widetilde{x}_i)+\rho_i(\widetilde{x}_i-c_{\dot{P}_i})]$$
$$\le H_i^6(3b_i^{-v_i-c_i}+3b_i^{\tau_i-c_i}+1)\rho_i(x_{R_i}-x_{P_i}).$$

Thus, for all $R\in\mathcal{R}$ and $\dot{R}\in U_R$, using the above estimate, Lemma 3.7(i) and (3.17), we obtain

$$(s^*_{2,\lambda})^2_R=\sum_{\mathrm{scale}\,(P)=\mathrm{scale}\,(R)}\frac{|s_P|^2}{\prod_{i=1}^2[1+|R_i|^{-1}\rho_i(x_{R_i}-x_{P_i})]^\lambda}$$
$$\lesssim\sum_{\mathrm{scale}\,(P)=\mathrm{scale}\,(R)}\sum_{\ell(\dot{P})=\ell(\dot{R})}\frac{|\dot{s}_{\dot{P}}|^2\chi_{U_P}(\dot{P})}{\prod_{i=1}^2[1+|\dot{R}_i|^{-1}\rho_i(c_{\dot{R}_i}-c_{\dot{P}_i})]^\lambda}\lesssim(\dot{s}^*_{2,\lambda})^2_{\dot{R}},$$

which completes the proof of Lemma 3.8. □

The following lemma is a generalization of [2, Lemma 3.10] and [40, Theorem 1.2 and Lemma 3.1].

Lemma 3.9. *Let $w\in\mathcal{A}_\infty(\vec{A})$ with q_w as in* (3.14).

(i) If $p\in(0,\infty)$, $r\in(0,\infty)$ and $\lambda>q_w\max(1,r/2,r/p)$, then there exists a positive constant C such that for all $s\equiv\{s_R\}_{R\in\mathcal{R}}$, $\|s\|_{\ddot{h}^p_w(\vec{A})}\le\|s^*_{r,\lambda}\|_{\ddot{h}^p_w(\vec{A})}\le C\|s\|_{\ddot{h}^p_w(\vec{A})}$.

(ii) If $p\in(0,1]$ and $\lambda>2q_w/p$, then there exists a positive constant C such that for all $s\equiv\{s_R\}_{R\in\mathcal{R}}\in\ell_{p,w}(\vec{A})$, $\|s\|_{\ell_{p,w}(\vec{A})}\le\|s^*_{2,\lambda}\|_{\ell_{p,w}(\vec{A})}\le C\|s\|_{\ell_{p,w}(\vec{A})}$.

Proof. (i) can be proved by an argument similar to that used in the proof of [2, Lemma 3.10]; see also the proof of [21, Lemma 2.3]. We omit the details.

To show (ii), let $w\in\mathcal{A}_\infty(\vec{A})$, $p\in(0,1]$ and $\lambda\in\mathbb{R}$ satisfying $\lambda>2q_w/p$. Observe that the definition of $\ell_{p,w}(\vec{A})$ is defined via the mean value on open sets, while the corresponding one parameter space $\dot{f}^{0,2}_\infty(A;w)$ is defined via the mean value on dilated cubes. This makes the proof of (ii) quite different from the proof of [2, Lemma 3.10]. We need to use Journé's covering lemma under the setting of expansive dilations (see [8, Lemma 4.9]).

Obviously, for all $R\in\mathcal{R}$, $|s_R|\le(s^*_{2,\lambda})_R$, which implies that $\|s\|_{\ell_{p,w}(\vec{A})}\le\|s^*_{2,\lambda}\|_{\ell_{p,w}(\vec{A})}$. To prove (ii), we still need to show $\|s^*_{2,\lambda}\|_{\ell_{p,w}(\vec{A})}\lesssim\|s\|_{\ell_{p,w}(\vec{A})}$. Let $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$ be any fixed open set satisfying $w(\Omega)<\infty$ and $\Omega^{(0)}$ as in Lemma 3.7. For $i=0,1$, we define inductively the sets $\Omega^{(i+1)}\equiv\{x\in\mathbb{R}^n\times\mathbb{R}^m:\mathcal{M}_s(\chi_{\Omega^{(i)}})(x)>\eta_1\}$, where $\eta_1\equiv\eta_1(\vec{A},n,m)\in(0,1)$ is a constant to be fixed later.

For any $\{s_R\}_{R\in\mathcal{R}}$, define $\{r_R\}_{R\in\mathcal{R}}$ and $\{t_R\}_{R\in\mathcal{R}}$ by letting $r_R\equiv s_R$ if $R\subset\Omega^{(2)}$ or else $r_R\equiv0$ and $t_R\equiv s_R-r_R$ for any $R\in\mathcal{R}$. Moreover, picking any $q\in(q_w,\infty)$, by the $L^q_w(\mathbb{R}^n\times\mathbb{R}^m)$-boundedness of $\mathcal{M}_s$, we have $w(\Omega^{(2)})\sim w(\Omega)$. Thus, we obtain that

$$\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}\\R\subset\Omega}}|(s^*_{2,\lambda})_R|^2\frac{|R|}{w(R)}\lesssim\mathrm{I}+\mathrm{J},$$

where

$$\mathrm{I}\equiv\frac{1}{[w(\Omega^{(2)})]^{\frac{2}{p}-1}}\sum_{\substack{R\in\mathcal{R}\\R\subset\Omega}}|(r^*_{2,\lambda})_R|^2\frac{|R|}{w(R)}$$

and

$$\mathrm{J} \equiv \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} |(t^*_{2,\lambda})_R|^2 \frac{|R|}{w(R)}.$$

Estimate I. For any $l_1,\, l_2 \in \mathbb{Z}_+$ and $R \in \mathcal{R}$, let

$$M_{R,l_1,l_2} \equiv \{P \in \mathcal{R}:\ \mathrm{scale}\,(P) = \mathrm{scale}\,(R),\, b_i^{l_i} \le |R_i|^{-1}\rho_i(x_{P_i} - x_{R_i}) < b_i^{l_i+1},\, i = 1,\, 2\}.$$

In the case that l_i for $i \in \{1,\, 2\}$ is 0, the above condition is replaced by $|R_i|^{-1}\rho_i(x_{P_i} - x_{R_i}) < b_i$. Then we have

$$\mathrm{I} \le \frac{1}{[w(\Omega)^{(2)}]^{\frac{2}{p}-1}} \sum_{j_1,\, j_2\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}\\ \mathrm{scale}\,(R)=(j_1,\, j_2)}} \sum_{l_1,l_2\in\mathbb{Z}_+} \sum_{P\in M_{R,l_1,l_2}} \frac{|r_P|^2|P|[w(R)]^{-1}}{\prod_{i=1}^2[1+|R_i|^{-1}\rho_i(x_{R_i}-x_{P_i})]^{\lambda}}.$$

Since $\lambda > 2q_w/p$ with $p \in (0,\, 1]$, we choose $q \in (q_w,\, \infty)$ sufficiently close to q_w such that $\lambda > 1+q$. For any $P \in M_{R,l_1,l_2}$, by (3.15), we have

$$w(R) \lesssim b_1^{ql_1} b_2^{ql_2} w(P).$$

Moreover, by an elementary lattice counting lemma (see [6, Lemma 2.8]), we obtain that $\sharp M_{R,l_1,l_2} \lesssim b_1^{l_1} b_2^{l_2}$. Thus,

$$\begin{aligned}
\mathrm{I} &\lesssim \frac{1}{[w(\Omega^{(2)})]^{\frac{2}{p}-1}} \sum_{j_1,\, j_2\in\mathbb{Z}} \sum_{\substack{P\in\mathcal{R}\\ \mathrm{scale}\,(P)=(j_1,\, j_2)}} |r_P|^2 \frac{|P|}{w(P)} \sum_{l_1,l_2\in\mathbb{Z}_+} b_1^{-l_1(\lambda-q-1)} b_2^{-l_2(\lambda-q-1)}\\
&\lesssim \frac{1}{[w(\Omega^{(2)})]^{\frac{2}{p}-1}} \sum_{P\subset\Omega^{(2)}} |s_P|^2 \frac{|P|}{w(P)} \lesssim \|s\|^2_{\ell_{p,\, w}(\vec{A})},
\end{aligned}$$

which is a desired estimate for I.

Estimate J. We need to show $\mathrm{J} \lesssim \|s\|_{\ell_{p,\, w}(\vec{A})}$. Notice that for any $\dot{R} \in U_R$, by Lemma 3.7(i), $|\dot{R}| \sim |R|$ and $w(\dot{R}) \sim w(R)$. Moreover, by the $L^q_w(\mathbb{R}^n \times \mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ with $q \in (q_w,\, \infty)$ (see [8, Proposition 2.10(ii)]), we have $w(\Omega^{(0)}) \lesssim w(\Omega)$. Also, observe that for any $R \in \mathcal{R}$ and $\dot{R} \in \dot{\mathcal{R}}$, $R \in U_{\dot{R}}$ if and only if $\dot{R} \in U_R$, which together with Lemma 3.7(i) further implies that $\sum_{R\in U_{\dot{R}}} \chi_{U_R}(\dot{R}) = \sum_{R\in U_{\dot{R}}} \chi_{U_{\dot{R}}}(R) \sim 1$. Using these facts together with the trivial fact that for any $R \in \mathcal{R}$, $\sum_{\dot{R}\in U_R} \chi_{U_R}(\dot{R}) \ge 1$, Lemmas 3.8(ii) and 3.7(ii), we obtain that

$$\begin{aligned}
\mathrm{J} &\le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \sum_{\dot{R}\in U_R} \chi_{U_R}(\dot{R}) |(t^*_{2,\lambda})_R|^2 \frac{|R|}{w(R)}\\
&\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}} \sum_{\dot{R}\in U_R} \chi_{U_{\dot{R}}}(R) |(\dot{t}^*_{2,\lambda})_{\dot{R}}|^2 \chi_{U_R}(\dot{R})| \frac{|\dot{R}|}{w(\dot{R})}\\
&\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{R}\subset\Omega^{(0)},\, \dot{R}\in\dot{\mathcal{R}}} \left[\sum_{R\in U_{\dot{R}}} \chi_{U_R}(\dot{R})\right] |(\dot{t}^*_{2,\lambda})_{\dot{R}}|^2 \frac{|\dot{R}|}{w(\dot{R})}
\end{aligned} \tag{3.19}$$

$$\sim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{R}\subset\Omega^{(0)},\, \dot{R}\in\dot{\mathcal{R}}} |(\dot{t}^*_{2,\lambda})_{\dot{R}}|^2 \frac{|\dot{R}|}{w(\dot{R})}.$$

Denote by $m_i(\Omega^{(0)})$ the *family of all dyadic rectangles* $\dot{R}\subset\Omega^{(0)}$ *which is maximal in the* $\mathbb{R}^{n_i}$ *"direction"*, where $i=1,\,2$. Let $m(\Omega^{(0)})\equiv m_1(\Omega^{(0)})\cap m_2(\Omega^{(0)})$. Notice that for any $\dot{R}\subset\Omega^{(0)}$, there exists at least one dyadic rectangle $\dot{P}\in m(\Omega^{(0)})$ such that $\dot{R}\subset\dot{P}$. Then, by (3.19), we have

$$\text{(3.20)}\quad \mathrm{J}\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P}\in m(\Omega^{(0)})} \sum_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}} \sum_{\substack{\dot{Q}\in\dot{\mathcal{R}}\\ \ell(\dot{Q})=\ell(\dot{R})}} \frac{|\dot{t}_{\dot{Q}}|^2|\dot{Q}|[w(\dot{R})]^{-1}}{\prod_{i=1}^2[1+|\dot{Q}_i|^{-1}\rho_i(c_{\dot{Q}_i}-c_{\dot{R}_i})]^\lambda}.$$

We now need to obtain some subtle decompositions on $\dot{t}_{\dot{Q}}$. For any $\dot{P}\equiv\dot{P}_1\times\dot{P}_2\in m(\Omega^{(0)})$, let $\dot{P}_{1,*}\supset\dot{P}_1$ be the maximal dyadic cube such that

$$\text{(3.21)}\qquad |(\dot{P}_{1,*}\times\dot{P}_2)\cap\Omega^{(0)}|>5H_1^4\eta_1 b_1^{2u_1}b_2^{2u_2}|\dot{P}_{1,*}\times\dot{P}_2|,$$

where we choose $\eta_1\in(0,1)$ small enough such that $5H_1^4\eta_1 b_1^{2u_1}b_2^{2u_2}<1$. For $B_{\dot{P}_{1,*}}\equiv B_{\rho_1}(c_{\dot{P}_{1,*}},\,3H_1^2b_1^{v_1\ell(\dot{P}_{1,*})+u_1})$ and $U_{\dot{P}_{1,*}}\equiv\{\dot{S}_1\in\dot{\mathcal{Q}}_1:\,\ell(\dot{S}_1)=\ell(\dot{P}_{1,*}),\ \dot{S}_1\cap B_{\dot{P}_{1,*}}\neq\emptyset\}$, using Lemma 3.5(iv), we see that

$$B_{U_1}\equiv B_{\rho_1}(c_{\dot{P}_{1,*}},\,5H_1^4b_1^{v_1\ell(\dot{P}_{1,*})+u_1})\supset\bigcup_{\dot{S}_1\in U_{\dot{P}_{1,*}}}\dot{S}_1.$$

Then, for any $\dot{S}_1\in U_{\dot{P}_{1,*}}$ and $x\in\dot{S}_1\times\dot{P}_2$, by Lemma 3.5(iv) and (3.21), we have

$$\mathcal{M}_s(\chi_{\Omega^{(0)}})(x)\ge\frac{|(B_{U_1}\times B_{\rho_2}(c_{\dot{P}_2},\,b_2^{v_2\ell(\dot{P}_2)+u_2}))\cap\Omega^{(0)}|}{|B_{U_1}\times B_{\rho_2}(c_{\dot{P}_2},\,b_2^{v_2\ell(\dot{P}_2)+u_2})|}\ge\frac{|(\dot{P}_{1,*}\times\dot{P}_2)\cap\Omega^{(0)}|}{5H_1^4b_1^{2u_1}b_2^{2u_2}|\dot{P}_{1,*}\times\dot{P}_2|}>\eta_1,$$

which implies that $(\cup_{\dot{S}_1\in U_{\dot{P}_{1,*}}}\dot{S}_1)\times\dot{P}_2\subset\Omega^{(1)}$.

On the other hand, for any $\dot{P}_{1,*}\times\dot{P}_2$, there exists a dyadic rectangle $\dot{P}_{1,\cdot}\supset\dot{P}_{1,*}$ such that $\dot{P}_{1,\cdot}\times\dot{P}_2\in m_1(\Omega^{(1)})$. Then, we further choose the maximal dyadic cube $\dot{P}_{2,*}\supset\dot{P}_2$ such that

$$\text{(3.22)}\qquad |(\dot{P}_{1,\cdot}\times\dot{P}_{2,*})\cap\Omega^{(1)}|>35H_1^6H_2^4b_1^{2u_1}b_2^{2u_2}\eta_1|\dot{P}_{1,\cdot}\times\dot{P}_2|,$$

where we choose $\eta_1\in(0,1)$ small enough such that $35H_1^6H_2^4b_1^{2u_1}b_2^{2u_2}\eta_1<1$.

For $B_{\dot{P}_{2,*}}\equiv B_{\rho_2}(c_{\dot{P}_{2,*}},\,3H_2^2b_2^{v_2\ell(\dot{P}_{2,*})+u_2})$ and

$$U_{\dot{P}_{2,*}}\equiv\{\dot{S}_2\in\dot{\mathcal{Q}}_2:\,\ell(\dot{S}_2)=\ell(\dot{P}_{2,*}),\ \dot{S}_2\cap B_{\dot{P}_{2,*}}\neq\emptyset\},$$

using Lemma 3.5(iv), we obtain that

$$B_{U_2}\equiv B_{\rho_2}(c_{\dot{P}_{2,*}},\,5H_2^4b_2^{v_2\ell(\dot{P}_{2,*})+u_2})\supset\bigcup_{\dot{S}_2\in U_{\dot{P}_{2,*}}}\dot{S}_2.$$

Then, for any $x \equiv (x_1, x_2)$ with $x_1 \in B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1})$ and $x_2 \in \dot{S}_2 \in U_{\dot{P}_{2,*}}$, by Lemma 3.5(iv) and (3.22), we have

$$\begin{aligned}\mathcal{M}_s(\chi_{\Omega^{(1)}})(x) &\geq \frac{|(B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1}) \times B_{U_2}) \cap \Omega^{(1)}|}{|B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1}) \times B_{U_2}|}\\ &\geq \frac{|(\dot{P}_{1,\cdot} \times \dot{P}_{2,*}) \cap \Omega^{(1)}|}{35H_1^6 H_2^4 b_1^{2u_1} b_2^{2u_2} |\dot{P}_{1,\cdot} \times \dot{P}_{2,*}|} > \eta_1,\end{aligned}$$

which together with $\cup_{\dot{S}_1 \in U_{\dot{P}_{1,*}}} \dot{S}_1 \subset B_{U_1} \subset B_{\rho_1}(c_{\dot{P}_{1,\cdot}}, 7H_1^6 b_1^{v_1\ell(\dot{P}_{1,\cdot})+u_1})$ implies that

$$\bigcup_{\dot{S} \in U_{\dot{P}_{1,*}} \times U_{\dot{P}_{2,*}}} \dot{S} \subset \Omega^{(2)}.$$

Therefore, for any $\dot{Q} \in \dot{\mathcal{R}}$ and $\dot{P} \equiv \dot{P}_1 \times \dot{P}_2 \in m(\Omega^{(0)})$ with $\dot{Q} \equiv \dot{Q}_1 \times \dot{Q}_2 \not\subset \Omega^{(2)}$ and $\ell(\dot{Q}_i) \geq \ell(\dot{P}_i)$, $i = 1, 2$, by Lemma 3.5(ii), we have either

$$\dot{Q}_1 \cap \dot{S}_1 = \emptyset \qquad \text{for all } \dot{S}_1 \in U_{\dot{P}_{1,*}}, \tag{3.23}$$

or $\dot{Q}_1 \subset \dot{S}_1$ for some $\dot{S}_1 \in U_{\dot{P}_{1,*}}$. Likewise, we either have

$$\dot{Q}_2 \cap \dot{S}_2 = \emptyset \qquad \text{for all } \dot{S}_2 \in U_{\dot{P}_{2,*}}, \tag{3.24}$$

or $\dot{Q}_2 \subset \dot{S}_2$ for some $\dot{S}_2 \in U_{\dot{P}_{2,*}}$. Observe that either (3.23) or (3.24) must hold. Otherwise, there would exist $\dot{S} \equiv \dot{S}_1 \times \dot{S}_2 \in U_{\dot{P}_{1,*}} \times U_{\dot{P}_{2,*}}$ such that $\dot{Q} \subset \dot{S} \subset \Omega^{(2)}$. This is a contradiction with $\dot{Q} \not\subset \Omega^{(2)}$. Define two *sequences* $\{\widetilde{t}_{\dot{Q}}\}_{\dot{Q} \in \dot{\mathcal{R}}}$ and $\{\check{t}_{\dot{Q}}\}_{\dot{Q} \in \dot{\mathcal{R}}}$, respectively, by setting $\widetilde{t}_{\dot{Q}} \equiv s_{\dot{Q}}$ if $\dot{Q}_1 \cap (\cup_{\dot{S}_1 \in U_{\dot{P}_{1,*}}} \dot{S}_1) = \emptyset$ with $\ell(\dot{Q}) \geq \ell(\dot{P})$ or else $\widetilde{t}_{\dot{Q}} \equiv 0$, and $\check{t}_{\dot{Q}} \equiv s_{\dot{Q}}$ if $\dot{Q}_2 \cap (\cup_{\dot{S}_2 \in U_{\dot{P}_{2,*}}} \dot{S}_2) = \emptyset$ with $\ell(\dot{Q}) \geq \ell(\dot{P})$ or else $\check{t}_{\dot{Q}} \equiv 0$. Notice that by (3.20), if $\dot{Q} \in \dot{\mathcal{R}}$ appears in the sum of J, then $\dot{Q} \not\subset \Omega^{(2)}$ and $\ell(\dot{Q}) \geq \ell(\dot{P})$ for some $\dot{P} \in m(\Omega^{(0)})$. This observation together with (3.20) again yields that

$$\begin{aligned}\mathrm{J} &\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P} \in m(\Omega^{(0)})} \sum_{\dot{R} \in \dot{\mathcal{R}},\, \dot{R} \subset \dot{P}} \sum_{\substack{\dot{Q} \in \dot{\mathcal{R}} \\ \ell(\dot{Q}) = \ell(\dot{R})}} \frac{|\widetilde{t}_{\dot{Q}}|^2 |\dot{Q}| [w(\dot{R})]^{-1}}{\prod_{i=1}^2 [1 + |\dot{Q}_i|^{-1} \rho_i(c_{\dot{Q}_i} - c_{\dot{R}_i})]^\lambda}\\ &\quad + \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P} \in m(\Omega^{(0)})} \sum_{\dot{R} \in \dot{\mathcal{R}},\, \dot{R} \subset \dot{P}} \sum_{\substack{\dot{Q} \in \dot{\mathcal{R}} \\ \ell(\dot{Q}) = \ell(\dot{R})}} \frac{|\check{t}_{\dot{Q}}|^2 |\dot{Q}| [w(\dot{R})]^{-1}}{\prod_{i=1}^2 [1 + |\dot{Q}_i|^{-1} \rho_i(c_{\dot{Q}_i} - c_{\dot{R}_i})]^\lambda}\\ &\equiv \mathrm{J}_1 + \mathrm{J}_2.\end{aligned}$$

Estimate J_1**.** Let us first classify those cubes "$\dot{Q}$" in J_1 by the definition of $\widetilde{t}_{\dot{Q}}$. In what follows, let $\widetilde{U}_{\dot{P}_{1,*}}$ denote the *union of all the dyadic cubes in* $U_{\dot{P}_{1,*}}$. For any

$\dot{P} \equiv \dot{P}_1 \times \dot{P}_2 \in m(\Omega^{(0)})$, we claim that

$$\left\{\dot{Q} \equiv \dot{Q}_1 \times \dot{Q}_2 \in \dot{\mathcal{R}} :\ \dot{Q}_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \ell(\dot{Q}) \geq \ell(\dot{P})\right\} \subset \bigcup_{\{\dot{P}' \equiv \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}}:\, \ell(\dot{P}')=\ell(\dot{P}),\, \dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}}=\emptyset\}} \{\dot{Q} \in \dot{\mathcal{R}} :\ \dot{Q} \subset \dot{P}'\}. \tag{3.25}$$

Indeed, for a fixed $\dot{P} \in m(\Omega^{(0)})$ and any $\dot{Q}_1 \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$ with $\ell(\dot{Q}_1) \geq \ell(\dot{P}_1)$, by (i) and (iii) of Lemma 3.5, there exists a unique $\dot{P}_{1,*}' \in \dot{\mathcal{Q}}_1$ such that $\ell(\dot{P}_{1,*}') = \ell(\dot{P}_{1,*})$, $\dot{Q}_1 \subset \dot{P}_{1,*}'$ and $\dot{P}_{1,*}' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$. Furthermore, by Lemma 3.5(iii) again, there exists a unique $\dot{P}_1' \in \dot{\mathcal{Q}}_1$ such that $\ell(\dot{P}_1') = \ell(\dot{P}_1)$ and $\dot{Q}_1 \subset \dot{P}_1'$. Then, we have $\dot{Q}_1 \subset \dot{P}_{1,*}' \cap \dot{P}_1'$. From this, $\ell(\dot{P}_{1,*}') \leq \ell(\dot{P}_1')$ and Lemma 3.5(iii), it follows that $\dot{P}_1' \subset \dot{P}_{1,*}'$ and hence $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$. By Lemma 3.5(iii), there also exists a unique $\dot{P}_2' \in \dot{\mathcal{Q}}_2$ such that $\dot{Q}_2 \subset \dot{P}_2'$ and $\ell(\dot{P}_2') = \ell(\dot{P}_2)$. Then we have $\dot{Q} \subset \dot{P}' \equiv \dot{P}_1' \times \dot{P}_2'$ satisfying $\ell(\dot{P}') = \ell(\dot{P})$ and $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$, which shows the above claim.

For any $\dot{P}_2 \in \dot{\mathcal{Q}}_2$, let $B_{\dot{P}_2} \equiv B_{\rho_2}(c_{\dot{P}_2}, 3H_2^2 b_2^{v_2\ell(\dot{P}_2)+u_2})$ and

$$U_{\dot{P}_2} \equiv \{\dot{S}_2 \in \dot{\mathcal{Q}}_2 :\ \ell(\dot{S}_2) = \ell(\dot{P}_2),\ \dot{S}_2 \cap B_{\dot{P}_2} \neq \emptyset\}.$$

Denote by $\widetilde{U}_{\dot{P}_2}$ the *union of all cubes in* $U_{\dot{P}_2}$. Define two sets of dyadic cubes

$$W_{\dot{P},1} \equiv \left\{\dot{P}' \equiv \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}} :\ \ell(\dot{P}') = \ell(\dot{P}),\ \dot{P}_1' \bigcap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \dot{P}_2' \bigcap \widetilde{U}_{\dot{P}_2'} = \emptyset\right\},$$
$$W_{\dot{P},2} \equiv \left\{\dot{P}' \equiv \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}} :\ \ell(\dot{P}') = \ell(\dot{P}),\ \dot{P}_1' \bigcap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \dot{P}_2' \subset \widetilde{U}_{\dot{P}_2'}\right\}.$$

Then, for any $\dot{P} \in m(\Omega^{(0)})$, by (i) and (ii) of Lemma 3.5, we rewrite (3.25) as

$$\left\{\dot{Q} \equiv \dot{Q}_1 \times \dot{Q}_2 \in \dot{\mathcal{R}} :\ \dot{Q}_1 \bigcap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset,\ \ell(\dot{Q}) \geq \ell(\dot{P})\right\} \subset \left(\bigcup_{\dot{P}' \in W_{\dot{P},1}} \{\dot{Q} \in \dot{\mathcal{R}} :\ \dot{Q} \subset \dot{P}'\}\right) \bigcup \left(\bigcup_{\dot{P}' \in W_{\dot{P},2}} \{\dot{Q} \in \dot{\mathcal{R}} :\ \dot{Q} \subset \dot{P}'\}\right). \tag{3.26}$$

Notice that for any $\dot{P} \in m(\Omega^{(0)})$ and $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$ with $\ell(\dot{P}_1') = \ell(\dot{P}_1)$, by $\dot{P}_1 \subset \dot{P}_{1,*} \subset B_{\dot{P}_{1,*}} \equiv B_{\rho_1}(c_{\dot{P}_{1,*}}, 3H_1^2 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}) \subset \widetilde{U}_{\dot{P}_{1,*}}$ and $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset$, we obtain that

$$\begin{aligned}\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) &\geq \frac{\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_{1,*}})}{H_1} - \rho_1(c_{\dot{P}_{1,*}} - c_{\dot{P}_1})\\ &\geq \frac{3H_1^2 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}}{H_1} - b_1^{v_1\ell(\dot{P}_{1,*})u_1} \geq 2H_1 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}.\end{aligned}$$

Similarly, for any $\dot{P} \in m(\Omega^{(0)})$ and $\dot{P}_2' \cap \widetilde{U}_{\dot{P}_2} = \emptyset$ with $\ell(\dot{P}_2') = \ell(\dot{P}_2)$, we also have

$$\rho_2(c_{\dot{P}_2'} - c_{\dot{P}_2}) \geq 2H_2 b_2^{v_2\ell(\dot{P}_2)+u_2}.$$

Thus, we have

$$W_{\dot{P},1} \subset \{\dot{P}' \equiv \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}} :\, \ell(\dot{P}') = \ell(\dot{P}),\, \rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) \geq 2H_1 b_1^{v_1\ell(\dot{P}_{1,*})+u_1}, \rho_2(c_{\dot{P}_2'} - c_{\dot{P}_2}) \geq 2H_2 b_2^{v_2\ell(\dot{P}_2)+u_2}\} \tag{3.27}$$

and

$$W_{\dot{P},2} \subset \{\dot{P}' \equiv \dot{P}_1' \times \dot{P}_2' \in \dot{\mathcal{R}} :\, \ell(\dot{P}') = \ell(\dot{P}), \rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) \geq 2H_1 b_1^{v_1\ell(\dot{P}_{1,*})+u_1},\, \dot{P}_2' \subset \widetilde{U}_{\dot{P}_2}\}. \tag{3.28}$$

Let $\gamma_1(\dot{P}) \equiv \ell(\dot{P}_{1,*}) - \ell(\dot{P}_1)$. For any $k_1,\, k_2 \in \mathbb{Z}_+$, set

$$U_{\dot{P},\,k_1,\,k_2} \equiv \left\{\dot{P}' \in \dot{\mathcal{R}} :\, \ell(\dot{P}') = \ell(\dot{P}),\, \rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) \sim b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+k_1}, \rho_2(c_{\dot{P}_2} - c_{\dot{P}_2'}) \sim b_2^{v_2\ell(\dot{P}_2)+k_2}\right\},$$

and

$$U_{\dot{P},\,k_1} \equiv \left\{\dot{P}' \in \dot{\mathcal{R}} :\, \ell(\dot{P}') = \ell(\dot{P}),\, \rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) \sim b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+k_1},\, \dot{P}_2' \subset \widetilde{U}_{\dot{P}_2}\right\},$$

where $\rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) \sim b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+k_1}$ and $\rho_2(c_{\dot{P}_2} - c_{\dot{P}_2'}) \sim b_2^{v_2\ell(\dot{P}_2)+k_1}$ mean, respectively, that

$$2H_1 b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+u_1+k_1} \leq \rho_1(c_{\dot{P}_1} - c_{\dot{P}_1'}) < 2H_1 b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+u_1+k_1+1}$$

and

$$2H_2 b_2^{v_2\ell(\dot{P}_2)+u_2+k_2} \leq \rho_2(c_{\dot{P}_2} - c_{\dot{P}_2'}) < 2H_2 b_2^{v_2\ell(\dot{P}_2)+u_2+k_2+1}.$$

Thus, by this, (3.27) and (3.28), we have

$$W_{\dot{P},1} \subset \bigcup_{k_1,\,k_2\in\mathbb{Z}_+} U_{\dot{P},\,k_1,\,k_2}, \qquad W_{\dot{P},2} \subset \bigcup_{k_1\in\mathbb{Z}_+} U_{\dot{P},\,k_1}.$$

Hence, for any $j_1,\, j_2 \in \mathbb{Z}_+$ and $\dot{P} \in m(\Omega^{(0)})$, using above two decompositions and (3.26), we obtain that

$$\begin{aligned} &\left\{\dot{Q} \in \dot{\mathcal{R}} :\, \ell(\dot{Q}) = \ell(\dot{P}) + (j_1,\, j_2),\, \dot{Q}_1 \bigcap \widetilde{U}_{\dot{P}_{1,*}} = \emptyset\right\} \\ &\quad\subset \left(\bigcup_{k_1,\,k_2\in\mathbb{Z}_+} \bigcup_{\dot{P}'\in U_{\dot{P},\,k_1,\,k_2}} \{\dot{Q} \in \dot{\mathcal{R}} :\, \ell(\dot{Q}) = \ell(\dot{P}) + (j_1,\, j_2),\, \dot{Q} \subset \dot{P}'\}\right) \\ &\quad\bigcup \left(\bigcup_{k_1\in\mathbb{Z}_+} \bigcup_{\dot{P}'\in U_{\dot{P},\,k_1}} \{\dot{Q} \in \dot{\mathcal{R}} :\, \ell(\dot{Q}_1) = \ell(\dot{P}_1) + j_1,\, \dot{Q} \subset \dot{P}'\}\right) \end{aligned} \tag{3.29}$$

$$\equiv \mathrm{V}_{\dot{P},\,j_1,\,j_2}\bigcup \mathrm{V}_{\dot{P},\,j_1}.$$

From this and

$$\sum_{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}} = \sum_{j_1,\,j_2\in\mathbb{Z}_+}\ \sum_{\substack{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}\\ \ell(\dot{R})=\ell(\dot{P})+(j_1,\,j_2)}} = \sum_{j_1\in\mathbb{Z}_+}\ \sum_{\substack{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}\\ \ell(\dot{R}_1)=j_1+\ell(\dot{P}_1)}},$$

it follows that

$$\begin{aligned}
&\mathrm{J}_1 \lesssim \mathrm{J}_1^{(1)}+\mathrm{J}_1^{(2)} \equiv\\
&\quad\frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}}\sum_{\dot{P}\in m(\Omega^{(0)})}\ \sum_{j_1,\,j_2\in\mathbb{Z}_+}\ \sum_{\substack{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}\\ \ell(\dot{R})=\ell(\dot{P})+(j_1,\,j_2)}}\ \sum_{\dot{Q}\in\mathrm{V}_{\dot{P},\,j_1,\,j_2}}\frac{|\widetilde{t}_{\dot{Q}}|^2|\dot{Q}|[w(\dot{R})]^{-1}}{\prod_{i=1}^2[1+|\dot{Q}_i|^{-1}\rho_i(c_{\dot{Q}_i}-c_{\dot{R}_i})]^\lambda}\\
&\quad+\frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}}\sum_{\dot{P}\in m(\Omega^{(0)})}\ \sum_{j_1\in\mathbb{Z}_+}\ \sum_{\substack{\dot{R}\in\dot{\mathcal{R}},\,\dot{R}\subset\dot{P}\\ \ell(\dot{R}_1)=j_1+\ell(\dot{P}_1)}}\ \sum_{\dot{Q}\in\mathrm{V}_{\dot{P},\,j_1}}\frac{|\widetilde{t}_{\dot{Q}}|^2|\dot{Q}|[w(\dot{R})]^{-1}}{\prod_{i=1}^2[1+|\dot{Q}_i|^{-1}\rho_i(c_{\dot{Q}_i}-c_{\dot{R}_i})]^\lambda}.
\end{aligned}$$

Then, let us now estimate $\mathrm{J}_1^{(1)}$ and $\mathrm{J}_1^{(2)}$, respectively.

Estimate $\mathrm{J}_1^{(1)}$. Since $\lambda>2q_w/p+1$, we choose $q\in(q_w,\,\infty)$ to be close enough to q_w such that $\lambda>2q/p+1$. For any $\dot{P}\in m(\Omega^{(0)})$, $\dot{R}\subset\dot{P}$ with $\ell(\dot{R})=\ell(\dot{P})+(j_1,\,j_2)$ and $\dot{Q}\in\mathrm{V}_{\dot{P},\,j_1,\,j_2}$, there exists a unique $\dot{P}'\in U_{\dot{P},\,k_1,\,k_2}$ for some $k_1,\,k_2\in\mathbb{Z}_+$ such that $\dot{P}'\supset\dot{Q}$ and $\ell(\dot{Q})=\ell(\dot{P}')+(j_1,\,j_2)$. Then, by Lemma 3.5(iv) and the definition of $U_{\dot{P},k_1,\,k_2}$, we obtain that $\dot{P},\,\dot{P}'\subset B_{\rho_1}(c_{\dot{P}_1},\,3H_1^2b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+u_1+k_1+1})\times B_{\rho_2}(c_{\dot{P}_2},\,3H_2^2b_2^{v_2\ell(\dot{P}_2)+u_2+k_2+1})$. From this, $\dot{R}\subset\dot{P}$, $\dot{Q}\subset\dot{P}'$, $\ell(\dot{Q})=\ell(\dot{R})=\ell(\dot{P})+(j_1,\,j_2)=\ell(\dot{P}')+(j_1,\,j_2)$, Lemma 3.5(iv) and Lemma 3.2 with $w\in\mathcal{A}_q(\vec{A})$, it follows that

$$\begin{aligned}
w(\dot{R}) &\gtrsim w\left(\prod_{i=1}^2 B_{\rho_i}(c_{\dot{P}_i},\,b_i^{v_i[\ell(\dot{P}_i+j_i]-u_i})\right)\\
&\gtrsim b_1^{qv_1j_1}b_2^{qv_2j_2}w\left(\prod_{i=1}^2 B_{\rho_i}(c_{\dot{P}_i},\,b_i^{v_i\ell(\dot{P}_i)-u_i})\right)\\
&\gtrsim b_1^{qv_1j_1}b_2^{qv_2j_2}w(\dot{P})\\
&\gtrsim b_1^{q\{-v_1[\gamma_1(\dot{P})-j_1]-k_1\}}b_2^{-q(k_2-v_2j_2)}\\
&\quad\times w\left(B_{\rho_1}(c_{\dot{P}_1},\,3H_1^2b_1^{v_1[\ell(\dot{P}_1)+\gamma_1(\dot{P})]+u_1+k_1+1})B_{\rho_2}(c_{\dot{P}_1},\,3H_2^2b_2^{v_2\ell(\dot{P}_2)+u_2+k_2+1})\right)\\
&\gtrsim b_1^{q\{-v_1[\gamma_1(\dot{P})-j_1]-k_1\}}b_2^{-q(k_2-v_2j_2)}w(\dot{P}')\\
&\gtrsim b_1^{q\{-v_1[\gamma_1(\dot{P})-j_1]-k_1\}}b_2^{-q(k_2-v_2j_2)}w(\dot{Q})
\end{aligned}\tag{3.30}$$

and, similarly,

$$w(\dot{P}')\lesssim b_1^{q[v_1\gamma_1(\dot{P})+k_1]}b_2^{qk_2}w(\dot{P}).\tag{3.31}$$

Moreover, for any $j_1,\, j_2,\, k_1,\, k_2 \in \mathbb{Z}_+$, by Lemma 3.5, we obtain that

$$\sharp\{\dot{R} \in \dot{\mathcal{R}} :\ \dot{R} \subset \dot{P},\ \ell(\dot{R}) = \ell(\dot{P}) + (j_1,\, j_2)\} \lesssim b_1^{-v_1 j_1} b_2^{-v_2 j_2} \quad \text{and} \tag{3.32}$$
$$\sharp U_{\dot{P},\, k_1,\, k_2} \lesssim b_1^{v_1\gamma_1(\dot{P})+k_1} b_2^{k_2}.$$

Furthermore, for any $\dot{P} \in m(\Omega^{(0)}), \dot{R} \subset \dot{P},\ \dot{P}' \in U_{\dot{P},\, k_1,\, k_2}$ with any $k_1,\, k_2 \in \mathbb{Z}_+$ and $\dot{Q} \subset \dot{P}'$, by $\dot{P}_1 \subset B_{\dot{P}_{1,\,*}}$ and $\dot{P}_1' \cap B_{\dot{P}_{1,\,*}} = \emptyset$, and (ii) and (iv) of Lemma 3.5, we obtain $\dot{P}_1' \cap \widetilde{U}_{\dot{P}_{1,\,*}} = \emptyset$, $\dot{Q} \subset \dot{P}'$ and

$$\begin{aligned}\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) &\le H_1^2[\rho_1(c_{\dot{P}_1'} - c_{\dot{Q}_1}) + \rho_1(c_{\dot{Q}_1} - c_{\dot{R}_1}) + \rho_1(c_{\dot{R}_1} - c_{\dot{P}_1})]\\ &\le H_1^2[2b_1^{v_1\ell(\dot{P}_{1,\,*})+u_1} + \rho_1(c_{\dot{Q}_1} - c_{\dot{R}_1})],\end{aligned}$$

which together with

$$\rho_1(c_{\dot{Q}_1} - c_{\dot{R}_1}) \ge \frac{\rho_1(c_{\dot{Q}_1} - c_{\dot{P}_{1,\,*}})}{H_1} - \rho_1(c_{\dot{P}_{1,\,*}} - c_{\dot{R}_1}) \ge 2H_1 b_1^{v_1\ell(\dot{P}_{1,\,*})+u_1}$$

implies that

$$\rho_1(c_{\dot{P}_1'} - c_{\dot{P}_1}) \lesssim \rho_1(c_{\dot{R}_1} - c_{\dot{Q}_1}). \tag{3.33}$$

Similarly, for any $\dot{P} \in m(\Omega^{(0)}), \dot{R} \subset \dot{P},\ \dot{P}' \in U_{\dot{P},\, k_1,\, k_2}$ and $\dot{Q} \subset \dot{P}'$, we also have

$$\rho_2(c_{\dot{P}_2'} - c_{\dot{P}_2}) \lesssim \rho_2(c_{\dot{R}_2} - c_{\dot{Q}_2}). \tag{3.34}$$

Then, by (3.29), (3.30), (3.31), (3.32), (3.33), (3.34), $p \in (0,\, 1]$, $q \in (q_w,\, \infty)$ and $\lambda > 2q/p + 1$, we obtain that

$$\begin{aligned}\mathrm{J}_1^{(1)} &\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P}\in m(\Omega^{(0)})} \sum_{j_1,\, j_2\in\mathbb{Z}_+} b_1^{-v_1 j_1} b_2^{-v_2 j_2} \sum_{k_1,\, k_2\in\mathbb{Z}_+} \sum_{\dot{P}'\in U_{\dot{P},\, k_1,\, k_2}}\\ &\quad \times b_1^{q[v_1\gamma_1(\dot{P})+k_1](\frac{2}{p}-1)} b_2^{qk_2(\frac{2}{p}-1)} \left[\frac{w(\dot{P})}{w(\dot{P}')}\right]^{\frac{2}{p}-1}\\ &\quad \times \sum_{\dot{Q}\subset\dot{P}'} \frac{|\widetilde{t}_{\dot{Q}}|^2 |\dot{Q}| b_1^{q\{v_1[\gamma_1(\dot{P})-j_1]+k_1\}} b_2^{q(k_2-v_2 j_2)} [w(\dot{Q})]^{-1}}{[b_1^{v_1[\gamma_1(\dot{P})-j_1]+k_1} b_2^{k_2-v_2 j_2}]^\lambda}\\ &\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \left[\sum_{\dot{P}\in m(\Omega^{(0)})} w(\dot{P}) b_1^{v_1\gamma_1(\dot{P})\frac{2q-p\lambda+p}{2-p}}\right]^{\frac{2}{p}-1} \|\dot{s}\|_{\dot{\ell}_{p,\, w}(\vec{A})}^2\\ &\quad \times \prod_{i=1}^{2} \sum_{j_i\in\mathbb{Z}_+} b_i^{j_i v_i(\lambda-q-1)} \sum_{k_i\in\mathbb{Z}_+} b_i^{-k_i(\lambda-\frac{2q}{p}-1)} \lesssim \|\dot{s}\|_{\dot{\ell}_{p,\, w}(\vec{A})}^2,\end{aligned} \tag{3.35}$$

where in the last inequality, we used that $\sum_{\dot{P}\in m(\Omega^{(0)})} w(\dot{P}) b_1^{v_1\gamma_1(\dot{P})\frac{2q-p\lambda+p}{2-p}} \lesssim w(\Omega^{(0)})$, which holds by Journé's covering lemma (see [8, Lemma 4.9]).

Estimate $\mathrm{J}_1^{(2)}$. For any $\dot{Q}_2$ and $k_2 \in \mathbb{Z}_+$, let

$$G_{\dot{Q}_2, k_2} \equiv \{\dot{R}_2 \in \dot{\mathcal{Q}}_2 : \ell(\dot{R}_2) = \ell(\dot{Q}_2),\ \rho_2(c_{\dot{Q}_2} - c_{\dot{R}_2}) \sim b_2^{v_2\ell(\dot{Q}_2)+k_2}\},$$

where $\rho_2(c_{\dot{Q}_2} - c_{\dot{R}_2}) \sim b_2^{v_2\ell(\dot{Q}_2)+k_2}$ always means that $\rho_2(c_{\dot{Q}_2} - c_{\dot{R}_2}) < b_2^{v_2\ell(\dot{Q}_2)+u_2+k_2}$ when $k_2 = 0$ and $b_2^{v_2\ell(\dot{Q}_2)+u_2+k_2-1} \le \rho_2(c_{\dot{Q}_2} - c_{\dot{R}_2}) < b_2^{v_2\ell(\dot{Q}_2)+u_2+k_2}$ when $k_2 \ge 1$. Moreover, for any $\dot{P} \in m(\Omega^{(0)})$, by Lemmas 3.5 and 3.2, we have

$$\sharp\{\dot{R}_1 : \ell(\dot{R}_1) = \ell(\dot{P}_1) + j_1,\ \dot{R}_1 \subset \dot{P}_1\} \lesssim b_1^{-v_1 j_1},\ \sharp G_{\dot{Q}_2, k_2} \lesssim b_2^{k_2}$$
$$\text{and } \sharp U_{\dot{P}, k_1} \lesssim b_1^{v_1\gamma_1(\dot{P})+k_1}. \tag{3.36}$$

For any $\dot{P} \in m(\Omega^{(0)})$, $\dot{P}' \in U_{\dot{P}, k_1}$, $\dot{Q} \subset \dot{P}'$ with $\ell(\dot{Q}_1) = \ell(\dot{P}_1) + j_1$, $\dot{R} \in \dot{\mathcal{R}}$ with $\dot{R} \subset \dot{P}$, $\ell(\dot{R}) = \ell(\dot{P}) + j_1$ and $\dot{R}_2 \in G_{\dot{Q}_2, k_2}$, by $\dot{P}_1' \cap B_{\dot{P}_{1,*}} = \emptyset$, and Lemmas 3.5 and 3.2, we obtain that

$$w(\dot{R}) \gtrsim b_1^{qv_1j_1} w(\dot{P}_1 \times \dot{R}_2) \gtrsim b_1^{-q\{v_1[\gamma_1(\dot{P})-j_1]+k_1\}} w(\dot{P}_1 \times \dot{R}_2)$$
$$\gtrsim b_1^{-q\{v_1[\gamma_1(\dot{P})-j_1]+k_1\}} b_2^{-qk_2} w(\dot{Q}) \quad \text{and} \quad w(\dot{P}') \lesssim b_1^{q[v_1\gamma_1(\dot{P})+k_1]} w(\dot{P}). \tag{3.37}$$

Therefore, by (3.29), (3.33), (3.36), (3.37), $p \in (0, 1]$, $q \in (q_w, \infty)$ and $\lambda > 2q/p + 1$, we have

$$\begin{aligned}
\mathrm{J}_1^{(2)} &\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \sum_{\dot{P}\in m(\Omega^{(0)})} \sum_{j_1\in\mathbb{Z}_+} b_1^{-v_1j_1} \\
&\quad \times \sum_{k_1\in\mathbb{Z}_+} \sum_{\dot{P}'\in U_{\dot{P}, k_1}} b_1^{(\frac{2}{p}-1)q[v_1\gamma_1(\dot{P})+k_1]} \left[\frac{w(\dot{P})}{w(\dot{P}')}\right]^{\frac{2}{p}-1} \\
&\quad \times \sum_{\dot{Q}\subset\dot{P}'} \sum_{k_2\in\mathbb{Z}_+} \sum_{\dot{R}_2\in G_{\dot{Q}_2, k_2}} b_1^{q\{v_1[\gamma_1(\dot{P})-j_1]+k_1\}} b_2^{qk_2} \frac{|\widetilde{t}_{\dot{Q}}|^2 |\dot{Q}| [w(\dot{Q})]^{-1}}{[b_1^{v_1[\gamma_1(\dot{P})-j_1]+k_1} b_2^{k_2}]^{\lambda}} \\
&\lesssim \frac{1}{[w(\Omega^{(0)})]^{\frac{2}{p}-1}} \left[\sum_{\dot{P}\in m(\Omega^{(0)})} w(\dot{P}) b_1^{v_1\gamma_1(\dot{P})\frac{2q-p\lambda+p}{2-p}}\right]^{\frac{2}{p}-1} \|\dot{s}\|^2_{\dot{\ell}_{p,\,w}(\vec{A})} \\
&\quad \times \sum_{j_1\in\mathbb{Z}_+} b_1^{j_1v_1(\lambda-q-1)} \sum_{k_1\in\mathbb{Z}_+} b_1^{-k_1(\lambda-\frac{2q}{p}-1)} \sum_{k_2\in\mathbb{Z}_+} b_2^{-k_2(\lambda-q-1)} \lesssim \|\dot{s}\|^2_{\dot{\ell}_{p,\,w}(\vec{A})},
\end{aligned} \tag{3.38}$$

where in the last inequality, we used that $\sum_{\dot{P}\in m(\Omega^{(0)})} w(\dot{P}) b_1^{v_1\gamma_1(\dot{P})\frac{2q-p\lambda+p}{2-p}} \lesssim w(\Omega^{(0)})$, which holds again by Journé's covering lemma (see [8, Lemma 4.9]).

Combining (3.35) and (3.38) yields that $\mathrm{J}_1 \lesssim \mathrm{J}_1^{(1)} + \mathrm{J}_1^{(2)} \lesssim \|\dot{s}\|^2_{\dot{\ell}_{p,\,w}(\vec{A})}$.

Symmetrically, we also have $\mathrm{J}_2 \lesssim \|\dot{s}\|^2_{\dot{\ell}_{p,\,w}(\vec{A})}$. Combining the estimates of J_1 and J_2 and Lemma 3.8(i) yields that $\mathrm{J} \lesssim \|s\|^2_{\ell_{p,\,w}(\vec{A})}$, which completes the proof of Lemma 3.9. □

We also need to generalize Peetre's mean value inequality to our setting, which is an extension of [2, Lemma 8.3] and [21, Lemma A.4] from one parameter setting to two parameter setting. Since the proof of Lemma 3.10 is similar to the proof of [2, Lemma 8.3], we omit the details.

Lemma 3.10. *Let K be a compact subset of $\mathbb{R}^n \times \mathbb{R}^m$ and $\lambda \in (0, \infty)$. Suppose that $g \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ with $\operatorname{supp} \widehat{g} \subset K$. For any $\gamma \in \mathbb{N}$, define two sequences $\{a_Q\}_{Q\in\mathcal{R}}$ and $\{b_Q\}_{Q\in\mathcal{R}}$, respectively, by setting, for all $Q \in \mathcal{R}$,*

$$a_Q \equiv \sup_{y\in Q} |g(y)| \quad \text{and} \tag{3.39}$$

$$b_Q \equiv \sup\left\{ \inf_{y\in P} |g(y)| : \operatorname{scale}(P) = \operatorname{scale}(Q) - (\gamma, \gamma),\ P \cap Q \neq \emptyset \right\}.$$

Then, for any sufficiently large positive integer γ and $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (0, 0)$, $(a^*_{2,\lambda})_Q \sim (b^*_{2,\lambda})_Q$ with equivalent constants independent of g and Q.

Lemma 3.11. *Let $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14). Suppose $\varphi \equiv \varphi^{(1)} \otimes \varphi^{(2)}$ with $\varphi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ satisfying that $\operatorname{supp} \varphi^{(i)}$ is compact and bounded away from the origin, where $i = 1, 2$. For any $f \in \mathcal{S}'_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $\gamma \in \mathbb{Z}_+$, define the sequences $\sup(f) = \{\sup_Q(f)\}_{Q\in\mathcal{R}}$ and $\inf(f) = \{\inf_Q(f)\}_{Q\in\mathcal{R}}$ by setting, for all $Q \in \mathcal{R}$ with $\operatorname{scale}(Q) = (-j_1, -j_2)$,*

$$\sup_Q(f) \equiv |Q|^{\frac12} \sup_{y\in Q} |\widetilde{\varphi}_{j_1, j_2} * f(y)| \quad \text{and}$$

$$\inf_Q(f) \equiv |Q|^{\frac12} \sup\left\{ \inf_{y\in P} |\widetilde{\varphi}_{j_1, j_2} * f(y)| : \operatorname{scale}(P) = (-j_1 - \gamma, -j_2 - \gamma),\ P \cap Q \neq \emptyset \right\},$$

where $\widetilde{\varphi}(\cdot) = \overline{\varphi(-\cdot)}$.

(i) If $p \in (0, \infty)$, then for any sufficiently large $\gamma \in \mathbb{Z}_+$,

$$\|f\|_{\ddot{H}^p_w(\vec{A})} \sim \|\sup(f)\|_{\ddot{h}^p_w(\vec{A})} \sim \|\inf(f)\|_{\ddot{h}^p_w(\vec{A})} \tag{3.40}$$

with equivalent constants independent of f.

(ii) If $p \in (0, 1]$, then (3.40) also holds with $\ddot{H}^p_w(\vec{A})$ and $\ddot{h}^p_w(\vec{A})$ replaced, respectively, by $\mathcal{L}_{p,w}(\vec{A})$ and $\ell_{p,w}(\vec{A})$.

Proof. We shall only prove Lemma 3.11 for $\mathcal{L}_{p,w}(\vec{A})$ and $\ell_{p,w}(\vec{A})$; the proofs for the spaces $\ddot{H}^p_w(\vec{A})$ and $\ddot{h}^p_w(\vec{A})$ are similar to the proof of [2, Lemma 3.11] and we omit the details.

Let us first prove that for all $f \in \mathcal{L}_{p,w}(\vec{A})$ with $p \in (0, 1]$, $\|\inf(f)\|_{\ell_{p,w}(\vec{A})} \lesssim \|f\|_{\mathcal{L}_{p,w}(\vec{A})}$. For any fixed $\gamma \in \mathbb{Z}_+$, define the *sequence* $s \equiv \{s_P\}_{P\in\mathcal{R}}$ by setting

$$s_P \equiv |P|^{1/2} \inf_{y\in P} |\widetilde{\varphi}_{\ell_1,\ell_2} * f(y)|$$

for any $P \in \mathcal{R}$ with $\operatorname{scale}(P) \equiv (-\ell_1, -\ell_2)$. Clearly, we have

$$|Q|^{-\frac12} \inf_Q(f) = \sup\{|P|^{-\frac12} |s_P| : P \cap Q \neq \emptyset,\ \operatorname{scale}(P) = \operatorname{scale}(Q) - (\gamma, \gamma)\}.$$

Fix $j_1,\, j_2 \in \mathbb{Z}$ and $Q \in \mathcal{R}$ with $\mathrm{scale}\,(Q) = (-j_1,\, -j_2)$. Suppose that $P,\, R \in \mathcal{R}$ satisfy

$$\mathrm{scale}\,(P) = \mathrm{scale}\,(R) = (-j_1 - \gamma,\, -j_2 - \gamma),\ y \in P \cap Q \neq \emptyset,\ z \in R \cap Q \neq \emptyset. \tag{3.41}$$

Then, we have

$$\rho_i(x_{P_i} - x_{R_i}) \le H_i^2\left[\rho_i(x_{P_i} - y_i) + \rho_i(y_i - z_i) + \rho_i(z_i - x_{R_i})\right] \lesssim |Q_i|,$$

where $i = 1,\, 2$. Thus, for any $\lambda > 1$, we obtain

$$s_P \le (s^*_{2,\,\lambda})_R \prod_{i=1}^{2} [1 + |P_i|^{-1}\rho_i(x_{P_i} - x_{R_i})]^{\lambda/2} \lesssim b_1^{\gamma\lambda/2} b_2^{\gamma\lambda/2} (s^*_{2,\,\lambda})_R. \tag{3.42}$$

Moreover, for any $Q \in \mathcal{R}$ with $\mathrm{scale}\,(Q) = (-j_1,\, -j_2)$, there exists a positive constant $C_1 > 1$ such that

$$\mathcal{U}_Q \equiv \{R \in \mathcal{R} :\ \mathrm{scale}\,(R) \le \mathrm{scale}\,(Q),\, R \cap Q \neq \emptyset\} \subset B_{\rho_1}(c_{Q_1},\, C_1 b_1^{-j_1}) \times B_{\rho_2}(c_{Q_2},\, C_1 b_2^{-j_2})$$

and $B_{\rho_1}(c_{Q_1},\, C_1^{-1} b_1^{-j_1}) \times B_{\rho_2}(c_{Q_2},\, C_1^{-1} b_2^{-j_2}) \subset Q$. Then, for any fixed open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, $Q \in \mathcal{R}$ with $Q \subset \Omega$, $\widetilde{\Omega} \equiv \{x \in \mathbb{R}^n \times \mathbb{R}^m :\ \mathcal{M}_s(\chi_\Omega)(x) > C_1^{-4}\}$ and $x \in R \in \mathcal{U}_Q$, we have

$$\mathcal{M}_s(\chi_\Omega)(x) \ge \frac{|(B_{\rho_1}(x_{Q_1},\, C_1 b_1^{j_1}) \times B_{\rho_2}(x_{Q_2},\, C_1 b_2^{j_2})) \cap \Omega|}{|B_{\rho_1}(x_{Q_1},\, C_1 b_1^{j_1}) \times B_{\rho_2}(x_{Q_2},\, C_1 b_2^{j_2})|} > C_1^{-4},$$

which implies that

$$\bigcup_{R \in \mathcal{U}_Q} R \subset \widetilde{\Omega}. \tag{3.43}$$

Then, by (3.41), (3.42) and (3.43), we obtain

$$\begin{aligned}
&\sum_{\substack{Q \subset \Omega \\ \mathrm{scale}\,(Q) = (-j_1,\, -j_2)}} \left[\inf\nolimits_Q(f)|Q|^{-\frac{1}{2}}\right]^2 \chi_Q \\
&\lesssim (b_1 b_2)^{\lambda\gamma} \sum_{\substack{Q \subset \Omega \\ \mathrm{scale}\,(Q) = (-j_1,\, -j_2)}} \sum_{\mathrm{scale}\,(P) = (-j_1 - \gamma,\, -j_2 - \gamma)} \left[(s^*_{2,\,\lambda})_P |P|^{-\frac{1}{2}}\right]^2 \chi_P \chi_{\mathcal{U}_Q}(P) \chi_Q \\
&\lesssim (b_1 b_2)^{\lambda\gamma} \sum_{\substack{P \subset \widetilde{\Omega} \\ \mathrm{scale}\,(P) = (-j_1 - \gamma,\, -j_2 - \gamma)}} \left[(s^*_{2,\,\lambda})_P |P|^{-\frac{1}{2}}\right]^2 \chi_P.
\end{aligned}$$

Thus, for any open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ and $p \in (0,\, 1]$, choosing $\lambda > 2q_w/p + 1$, by the above estimate, $w(\Omega) \sim w(\widetilde{\Omega})$ (by [8, Proposition 2.10(ii)]), $w(Q) \sim w(P)$ (by Lemma 3.2), $|P| \sim |Q|$ and Lemma 3.9, we obtain that

$$\frac{1}{[w(\Omega)]^{\frac{2}{p} - 1}} \int_\Omega \sum_{j_1,\, j_2 \in \mathbb{Z}} \sum_{\substack{Q \subset \Omega \\ \mathrm{scale}\,(Q) = (-j_1,\, -j_2)}} \left[\inf\nolimits_Q(f)|Q|^{-\frac{1}{2}} \chi_Q(x)\right]^2 \frac{|Q|^2}{[w(Q)]^2} w(x)\, dx$$

$$\begin{aligned}
&\lesssim \frac{(b_1b_2)^{\lambda\gamma}}{[w(\widetilde{\Omega})]^{\frac{2}{p}-1}}\int_{\widetilde{\Omega}}\sum_{P\subset\widetilde{\Omega}}\left[(s^*_{2,\lambda})_P|P|^{-\frac12}\chi_P(x)\right]^2\frac{|P|^2}{[w(P)]^2}w(x)\,dx\\
&\lesssim \|s^*_{2,\lambda}\|^2_{\ell_{p,w}(\vec{A})}\lesssim\|s\|^2_{\ell_{p,w}(\vec{A})}\\
&\lesssim \sup_{w(\Omega)<\infty}\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\int_{\Omega}\sum_{j_1,j_2\in\mathbb{Z}}\sum_{\substack{Q\subset\Omega\\ \mathrm{scale}\,(Q)=(-j_1,-j_2)}}\\
&\quad\times\sum_{\substack{Q\cap P\neq\emptyset\\ \mathrm{scale}\,(P)=(-j_1-\gamma,-j_2-\gamma)}}|\widetilde{\varphi}_{j_1+\gamma,j_2+\gamma}*f(x)|^2\chi_P(x)\frac{|Q|^2}{[w(Q)]^2}\chi_Q(x)w(x)\,dx\\
&\lesssim \sup_{w(\Omega)<\infty}\frac{1}{[w(\ddot{\Omega})]^{\frac{2}{p}-1}}\int_{\ddot{\Omega}}\sum_{j_1,j_2\in\mathbb{Z}}\sum_{\substack{P\subset\ddot{\Omega}\\ \mathrm{scale}\,(P)=(-j_1-\gamma,-j_2-\gamma)}}\\
&\quad\times|\widetilde{\varphi}_{j_1+\gamma,j_2+\gamma}*f(x)|^2\chi_P(x)\frac{|P|^2}{[w(P)]^2}\sum_{\substack{Q\subset\Omega\\ \mathrm{scale}\,(Q)=(-j_1,-j_2)}}\chi_Q(x)w(x)\,dx\lesssim\|f\|^2_{\mathcal{L}_{p,w}(\vec{A})},
\end{aligned}$$

where $\cup_{\{P\in\mathcal{R}:\,Q\subset\Omega,\,P\cap Q\neq\emptyset,\,\mathrm{scale}\,(P)=\mathrm{scale}\,(Q)-(\gamma,\gamma)\}}P\subset\ddot{\Omega}\equiv\{x\in\mathbb{R}^n\times\mathbb{R}^m:\,\mathcal{M}_s(\chi_\Omega)(x)>C_0\}$ which is obtained by a proof similar to that of (3.43). Here $C_0\in(0,1)$ is some positive constant independent of Ω. From the above estimate and the arbitrariness of Ω, we deduce that $\|\inf(f)\|_{\ell_{p,w}(\vec{A})}\lesssim\|f\|_{\mathcal{L}_{p,w}(\vec{A})}$.

Obviously, for any $f\in\mathcal{L}_{p,w}(\mathbb{R}^n\times\mathbb{R}^m;\,\widetilde{\varphi})$, $\|f\|_{\mathcal{L}_{p,w}(\vec{A})}\le\|\sup(f)\|_{\ell_{p,w}(\vec{A})}$. To finish the proof of Lemma 3.11, it still needs to prove $\|\sup(f)\|_{\ell_{p,w}(\vec{A})}\lesssim\|\inf(f)\|_{\ell_{p,w}(\vec{A})}$. Fix any $Q\in\mathcal{R}$ with $\mathrm{scale}\,(Q)=(-j_1,-j_2)$, $j_1,\,j_2\in\mathbb{Z}$. Let $g(x)\equiv(\widetilde{\varphi}_{j_1,j_2}*f)(A_1^{-j_1}x_1,\,A_2^{-j_2}x_2)$. Then we have $\mathrm{supp}\,\widehat{g}\subset K\equiv(\mathrm{supp}\,\widehat{\varphi^{(1)}}\times\mathrm{supp}\,\widehat{\varphi^{(2)}})$. Let $\{a_Q\}_{Q\in\mathcal{R}}$ and $\{b_Q\}_{Q\in\mathcal{R}}$ be as in (3.39). A direct calculation shows that for any fixed $Q\in\mathcal{R}$ with $\mathrm{scale}\,(Q)=(-j_1,-j_2)$,

$$a_{A_1^{j_1}Q_1\times A_2^{j_2}Q_2}\equiv|Q|^{-\frac12}\sup\nolimits_Q(f),\quad b_{A_1^{j_1}Q_1\times A_2^{j_2}Q_2}\equiv|Q|^{-\frac12}\inf\nolimits_Q(f),\quad Q\in\mathcal{R}.$$

Hence, applying Lemma 3.10 to the dilated rectangle $\widetilde{Q}\equiv A_1^{j_1}Q_1\times A_2^{j_2}Q_2$, we have

$$(\sup(f)^*_{2,\lambda})_Q=|Q|^{\frac12}(a^*_{2,\lambda})_{\widetilde{Q}}\lesssim|Q|^{\frac12}(b^*_{2,\lambda})_{\widetilde{Q}}\sim(\inf(f)^*_{2,\lambda})_Q.\tag{3.44}$$

Since $Q\in\mathcal{R}$ is arbitrary, let $p\in(0,1]$ and $\lambda\in(2q_w/p+1,\infty)$ be as in Lemma 3.9, by Lemma 3.9(ii) and (3.44), we obtain

$$\|\sup(f)\|_{\ell_{p,w}(\vec{A})}\lesssim\|\inf(f)\|_{\ell_{p,w}(\vec{A})},$$

which completes the proof of Lemma 3.11. □

Proof of Theorem 2.1. Let $w\in\mathcal{A}_\infty(\vec{A})$. By Lemmas 3.9 and 3.11 together with an argument similar to the proofs of [5, Theorem 3.5] and [40, Theorem 1.4], we obtain the desired results for Theorem 2.1 on the spaces $\ddot{h}^p_w(\vec{A})$ and $\ddot{H}^p_w(\vec{A})$. We omit the details by similarity.

In what follows, let us prove Theorem 2.1 on the spaces $\ell_{p,w}(\vec{A})$ and $\mathcal{L}_{p,w}(\vec{A})$ with $p\in(0,1]$. We first prove that T_ψ is bounded from $\ell_{p,w}(\vec{A})$ to $\mathcal{L}_{p,w}(\vec{A})$. Let $f\equiv T_\psi s=\sum_{Q\in\mathcal{R}} s_Q\psi_Q$. Then by Lemma 3.4, we obtain that f is a well-defined element of $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, which implies that for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$f*\varphi_{j_1,j_2}(x)=\sum_{\ell_1,\ell_2\in\mathbb{Z}}\ \sum_{\mathrm{scale}\,(Q)=(-\ell_1,-\ell_2)} s_Q\psi_Q*\varphi_{j_1,j_2}(x).$$

Since $\mathrm{supp}\,\widehat{\varphi^{(i)}}$ and $\mathrm{supp}\,\widehat{\psi^{(i)}}$ are compact and bounded away from the origin, $i=1,2$, then for any $Q_i\in\mathcal{Q}_i$ with $\mathrm{scale}\,(Q_i)=-\ell_i$, $i=1,2$, there exists a sufficiently large integer M such that when $|j_i-\ell_i|>M$ and $\mathrm{scale}\,(Q_i)=-\ell_i$, $i=1,2$, we have $\mathrm{supp}\,\widehat{\psi^{(i)}_{Q_i}}\cap\mathrm{supp}\,\widehat{\varphi^{(i)}_{j_i}}=\emptyset$, and hence for any $\xi\in\mathbb{R}^n\times\mathbb{R}^m$, $|\ell_1-j_1|>M$ or $|\ell_2-j_2|>M$, we further have $(\psi_Q*\varphi_{j_1,j_2})\widehat{\ }(\xi)\equiv0$. From this, it follows that for any $x\in\mathbb{R}^n\times\mathbb{R}^m$, $\psi_Q*\varphi_{j_1,j_2}(x)\equiv0$ when $|\ell_1-j_1|>M$ or $|\ell_2-j_2|>M$. Therefore, we obtain that for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$f*\varphi_{j_1,j_2}(x)=\sum_{\substack{|\ell_i-j_i|\le M\\ i=1,2}}\ \sum_{\mathrm{scale}\,(Q)=(-\ell_1,-\ell_2)} s_Q\varphi_{j_1,j_2}*\psi_Q(x).$$

For any $p\in(0,1]$, we take $\lambda\in(2q_w/p+1,\infty)$. Since $\varphi_{j_1,j_2}*\psi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$, for all $x\in\mathbb{R}^n\times\mathbb{R}^m$, we have

$$\begin{aligned}\varphi_{j_1,j_2}*\psi_Q(x)&=(\varphi_{j_1-\ell_1,j_2-\ell_2}*\psi)_Q(x)\\&=|Q|^{-\frac12}(\varphi_{j_1-\ell_1,j_2-\ell_2}*\psi)(A_1^{\ell_1}(x_1-x_{Q_1}),\,A_2^{\ell_2}(x_2-x_{Q_2}))\\&\lesssim\frac{|Q|^{-\frac12}}{\prod_{i=1}^2[1+\rho_i(A_i^{\ell_i}x_i-x_{Q_i})]^{\lambda/2}}.\end{aligned}$$

Moreover, for any $x\in\mathbb{R}^n\times\mathbb{R}^m$ and $\ell_1,\ell_2\in\mathbb{Z}$, there exists a unique $Q^x\in\mathcal{R}$ such that $x\in Q^x$ and $\mathrm{scale}\,(Q^x)=(-\ell_1,-\ell_2)$. Then for any $Q\in\mathcal{R}$ with $\mathrm{scale}\,(Q)\equiv(-\ell_1,-\ell_2)$, it is easy to show $1+\rho_i(A_i^{\ell_i}(x_{Q_i^x}-x_{Q_i}))\lesssim1+\rho_i(A_i^{\ell_i}(x_i-x_{Q_i}))$, $i=1,2$. From this, the above estimate, $p\in(0,1]$, $\lambda\in(2q_w/p+1,\infty)$ and Hölder's inequality, it follows that for all $x\in\mathbb{R}^n\times\mathbb{R}^m$,

$$\begin{aligned}|f*\varphi_{j_1,j_2}(x)|&\lesssim\sum_{\substack{|\ell_i-j_i|\le M\\ i=1,2}}\chi_{Q^x}(x)|Q^x|^{-\frac12}\sum_{\mathrm{scale}\,(Q)=(-\ell_1,-\ell_2)}\frac{|s_Q|}{\prod_{i=1}^2[1+\rho_i(A_i^{\ell_i}(x_{Q_i^x}-x_{Q_i}))]^{\lambda/2}}\\&\lesssim\sum_{\substack{|\ell_i-j_i|\le M\\ i=1,2}}\chi_{Q^x}(x)|Q^x|^{-\frac12}(s^*_{2,\lambda})_{Q^x}\\&\lesssim\sum_{\substack{|\ell_i-j_i|\le M\\ i=1,2}}\ \sum_{\mathrm{scale}\,(Q)=(-\ell_1,-\ell_2)}(s^*_{2,\lambda})_Q\chi_Q(x)|Q|^{-\frac12}.\end{aligned}$$

Moreover, for any open set $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$, using an argument similar to the estimate for (3.43), there exists a positive constant $\eta_2\in(0,1)$ such that

$$\bigcup_{\substack{|\ell_i-j_i|\le M\\ i=1,2}}\ \bigcup_{\substack{\mathrm{scale}\,(Q)=(-\ell_1,-\ell_2)\\ Q\subset\Omega}} Q\subset\check{\Omega}\equiv\{x\in\mathbb{R}^n\times\mathbb{R}^m:\,\mathcal{M}_s(\chi_\Omega)(x)>\eta_2\}.$$

Consequently, for any $p \in (0, 1]$, $\lambda \in (2q_w/p + 1, \infty)$, $q \in (q_w, \infty)$ and open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with $w(\Omega) < \infty$, by the last two estimates, (3.43), $w(\check{\Omega}) \lesssim w(\Omega)$ (see [8, Proposition 2.10(ii)]) and Lemma 3.11, we have

$$\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{\substack{R \subset \Omega \\ \mathrm{scale}\,(R) = (-j_1, -j_2)}} |f * \varphi_{j_1, j_2}(x)|^2 \chi_R(x) \frac{|R|^2}{[w(R)]^2} w(x)\,dx \\
&\quad \lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \int_\Omega \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{\substack{R \subset \Omega,\, R \in \mathcal{R} \\ \mathrm{scale}\,(R) = (-j_1, -j_2)}} \\
&\qquad \times \left[\sum_{\substack{|\ell_i - j_i| \le M \\ i = 1, 2}} \sum_{\mathrm{scale}\,(Q) = (-\ell_1, -\ell_2)} (s^*_{2,\lambda})_Q \chi_Q(x) |Q|^{-\frac{1}{2}} \right]^2 \chi_R(x) \frac{|R|^2}{[w(R)]^2} w(x)\,dx \\
&\quad \lesssim \frac{1}{[w(\check{\Omega})]^{\frac{2}{p}-1}} \int_{\check{\Omega}} \sum_{\ell_1, \ell_2 \in \mathbb{Z}} \sum_{\substack{Q \subset \check{\Omega} \\ \mathrm{scale}\,(Q) = (-\ell_1, -\ell_2)}} (s^*_{2,\lambda})_Q^2 \chi_Q(x) \frac{|Q|}{[w(Q)]^2} \\
&\qquad \times \sum_{\substack{|\ell_i - j_i| \le M \\ i = 1, 2}} \sum_{\mathrm{scale}\,(R) = (-j_1, -j_2)} \chi_R(x) w(x)\,dx \\
&\quad \lesssim \|s^*_{2,\lambda}\|^p_{\ell_{p,w}(\vec{A})} \lesssim \|s\|^2_{\ell_{p,w}(\vec{A})},
\end{aligned}$$

which implies that $\|T_\psi(s)\|_{\mathcal{L}_{p,w}(\vec{A})} \lesssim \|s\|_{\ell_{p,w}(\vec{A})}$.

Suppose that $f \in \mathcal{L}_{p,w}(\vec{A})$ and $Q \equiv A_1^{-j_1}([0, 1)^n + k_1) \times A_2^{-j_2}([0, 1)^m + k_2)$, $j_1, j_2 \in \mathbb{Z}$, $k_1 \in \mathbb{Z}^n$, $k_2 \in \mathbb{Z}^m$. Then,

$$|(S_\varphi f)_Q| = |\langle f, \varphi_Q \rangle| = |Q|^{\frac{1}{2}} |(\widetilde{\varphi}_{j_1, j_2} * f)(x_Q)| \le \sup\nolimits_Q(f).$$

Therefore, by Lemma 3.11, we obtain the boundedness of S_φ from $\ell_{p,w}(\vec{A})$ to $\mathcal{L}_{p,w}(\vec{A}; \widetilde{\varphi})$.

Finally, if (ψ, φ) is an admissible pair of frame wavelets as in Definition 2.2, then by Lemma 2.1(ii), we have that $\mathcal{L}_{p,w}(\vec{A}; \widetilde{\varphi}) \hookrightarrow \mathcal{L}_{p,w}(\vec{A}; \varphi)$ is a bounded inclusion. Hence, by reversing the roles of φ and $\widetilde{\varphi}$, we have

$$\mathcal{L}_{p,w}(\vec{A}; \widetilde{\varphi}) = \mathcal{L}_{p,w}(\vec{A}; \varphi),$$

which completes the proof of Theorem 2.1. □

To prove Theorem 2.2, we need two technical lemmas first. By [8, Lemma 5.5] and a basic fact that $\psi_j * \varphi_k \equiv (\psi_{j-k} * \varphi)_k$, $j, k \in \mathbb{Z}$, we obtain the following lemma.

Lemma 3.12. *For $i = 1, 2$, let $M_i \in (0, \infty)$, $N_i \in \mathbb{Z}_+$, $\varphi^{(i)}, \psi^{(i)} \in \mathcal{S}_{N_i}(\mathbb{R}^{n_i})$, $\varphi \equiv \varphi^{(1)} \otimes \varphi^{(2)}$ and $\psi \equiv \psi^{(1)} \otimes \psi^{(2)}$. Then there exists a positive constant C, depending on M_i and N_i with $i \in \{1, 2\}$, such that for all $j_i, k_i \in \mathbb{Z}$ with $j_i \ge k_i$ for $i \in \{1, 2\}$ and for all $x \in \mathbb{R}^n \times \mathbb{R}^m$,*

$$|\varphi_{j_1, j_2} * \psi_{k_1, k_2}(x)| \le C \prod_{i=1}^2 b_i^{k_i + (k_i - j_i)(N_i + 1)\zeta_{i,-}} [1 + \rho_i(A_i^{k_i} x_i)]^{-M_i}.$$

We skip the proof of the following Lemma 3.13 since it is similar to the proofs of [5, Theorem 4.1] and [40, Theorem 2.1].

Lemma 3.13. *Let $p \in (0, 1]$ and $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14). An operator $\mathcal{A}$ is called almost diagonal, if its associated matrix $\{a_{R,Q}\}_{R,Q\in\mathcal{R}}$, where $a_{R,Q} \equiv (\mathcal{A}e^Q)_R$, satisfies that there exists some positive constant ϵ such that*

$$\sup_{R,\,Q\in\mathcal{R}} |a_{R,Q}|/\kappa_{R,Q}(\epsilon) < \infty,$$

where

$$\kappa_{R,Q}(\epsilon) \equiv \prod_{i=1}^{2}\left[1+\frac{\rho_i(x_{R_i}-x_{Q_i})}{|R_i|\vee|Q_i|}\right]^{-\frac{q_w}{p}-\epsilon}\left[\left(\frac{|R_i|}{|Q_i|}\right)^{\frac{1+\epsilon}{2}}\wedge\left(\frac{|Q_i|}{|R_i|}\right)^{\frac{q_w}{p}+\frac{\epsilon-1}{2}}\right]. \tag{3.45}$$

Then, the almost diagonal operator $\mathcal{A}$ is bounded on $\ddot{h}^p_w(\vec{A})$.

Proof of Theorem 2.2. Let $p \in (0, 1]$ and $w \in \mathcal{A}_\infty(\vec{A})$. We first show that for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, $\|f\|_{\widetilde{H}^p_w(\vec{A})} \lesssim \|f\|_{\ddot{H}^p_w(\vec{A})}$. For any $k_1, k_2 \in \mathbb{Z}$, $x \in \mathbb{R}^n\times\mathbb{R}^m$, and $\Phi \in \mathscr{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, we have

$$\begin{aligned} & b_1^{k_1}b_2^{k_2}\int_{\prod_{i=1}^2 B_{\rho_i}(x_i,\, b_i^{-k_i})} |f*\Phi_{k_1,k_2}(y)|^2\,dy \\ &\quad = b_1^{k_1}b_2^{k_2}\int_{\prod_{i=1}^2 B_{\rho_i}(x_i,\, b_i^{-k_i})} \sum_{\substack{R\in\mathcal{R}\\ \mathrm{scale}\,(R)=(-k_1,\,-k_2)}} |f*\Phi_{k_1,k_2}(y)|^2\chi_R(y)\,dy. \end{aligned} \tag{3.46}$$

Let (φ,ψ) be the admissible pair of frame wavelets as in Definition 2.2. For any $f \in \ddot{H}^p_w(\vec{A})$, by Lemma 2.1(ii), we obtain that $f = \sum_{Q\in\mathcal{R}}\langle f,\,\varphi_Q\rangle\psi_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, which together with $\Phi \in \mathcal{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$ implies that

$$f*\Phi_{k_1,k_2} = \sum_{Q\in\mathcal{R}}\langle f,\,\varphi_Q\rangle\psi_Q*\Phi_{k_1,k_2} \tag{3.47}$$

holds pointwise. Moreover, since $\psi \equiv \psi^{(1)}\otimes\psi^{(2)}$ and $\Phi \equiv \Phi^{(1)}\otimes\Phi^{(2)}$ with $\psi^{(i)}, \Phi^{(i)} \in \mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i = 1, 2$, then for any $y \in R$ with $\mathrm{scale}\,(R) = (-k_1, -k_2)$, $Q \in \mathcal{R}$ with $\mathrm{scale}\,(Q) = (-j_1, -j_2)$, $M_i \in (0, \infty)$ and $N_i \in \mathbb{Z}_+$ to be fixed later, by Lemma 3.12, we obtain that

$$\begin{aligned} |\psi_Q*\Phi_{k_1,k_2}(y)| &\lesssim \prod_{i=1}^{2} b_i^{-j_i/2-|j_i-k_i|(N_i+1)\zeta_{i,-}+j_i\wedge k_i}[1+b_i^{j_i\wedge k_i}\rho_i(y_i-x_{Q_i})]^{-M_i} \\ &\lesssim |R|^{-\frac12}\prod_{i=1}^{2} b_i^{-|j_i-k_i|[(N_i+1)\zeta_{i,-}+\frac12]}[1+b_i^{j_i\wedge k_i}\rho_i(x_{R_i}-x_{Q_i})]^{-M_i}. \end{aligned} \tag{3.48}$$

Since $p \in (0, 1]$ and q_w is as in (3.14), if we let

$$a_{R,Q} \equiv \prod_{i=1}^{2} b_i^{-|j_i-k_i|[(N_i+1)\zeta_{i,-}+\frac12]}[1+b_i^{j_i\wedge k_i}\rho_i(x_{R_i}-x_{Q_i})]^{-M_i},$$

$N_i > (q_w/p - 1)\zeta_{i,-}^{-1} - 1$, $M_i > q_w/p$, $N_i \in \mathbb{Z}_+$ and $\epsilon \in \mathbb{R}_+$ such that

$$\epsilon/2 < \min_{i=1,2}\{(N_i + 1)\zeta_{i,-} + 1 - q_w/p,\, M_i - q_w/p,\, (N_i + 1)\zeta_{i,-}\},$$

then it is easy to show that for any $R, Q \in \mathcal{R}$, $a_{R,Q} \lesssim \kappa_{R,Q}(\epsilon)$ uniformly, where $\kappa_{R,Q}(\epsilon)$ satisfies (3.45), which implies that $\{a_{R,Q}\}_{R,Q\in\mathcal{R}}$ induces an almost diagonal operator. Therefore, for any $f \in \ddot{H}^p_w(\vec{A})$ and $s \equiv \{s_Q\}_{Q\in\mathcal{R}}$ with $s_Q \equiv \langle f, \varphi_Q\rangle$, by (3.46), (3.47), (3.48), Lemma 3.13 and Theorem 2.1 with the inverse φ-transform $S_\varphi(f)$, we have

$$\begin{aligned}\|f\|_{\widetilde{H}^p_w(\vec{A})} &= \|\vec{S}_\Phi(f)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \left\|\left\{\sum_{Q\in\mathcal{R}} a_{R,Q}s_Q\right\}_{R\in\mathcal{R}}\right\|_{\ddot{h}^p_w(\vec{A})}\\ &\lesssim \|S_\varphi(f)\|_{\ddot{h}^p_w(\vec{A})} \lesssim \|f\|_{\ddot{H}^p_w(\vec{A})},\end{aligned}$$

which is desired.

Finally, we show that for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, $\|f\|_{\ddot{H}^p_w(\vec{A})} \lesssim \|f\|_{\widetilde{H}^p_w(\vec{A})}$. To this end, letting $\gamma \in \mathbb{N}$ be as in Lemma 3.11, for any $Q \in \mathcal{R}$ with $\mathrm{scale}\,(Q) = (-k_1, -k_2)$, $k_1, k_2 \in \mathbb{Z}$, and any $x \in Q$, by [2, Lemma 2.9(b)], there exists some constant $k_0 \in \mathbb{N}$, independent of the choice of Q, such that

$$\bigcup_{\{P\in\mathcal{R}:\, P\cap Q\neq\emptyset,\ \mathrm{scale}\,(P_i)=(-k_1-\gamma,\,-k_2-\gamma)\}} P \subset B_{\rho_1}(x_1, b_1^{-k_1+k_0}) \times B_{\rho_2}(x_2, b_2^{-k_2+k_0}).$$

Thus, for any $x \in Q$ and $\Phi \in \mathscr{S}_\infty(\mathbb{R}^n\times\mathbb{R}^m)$, we have

$$\begin{aligned}(3.49)\qquad &\sum_{\substack{P\cap Q\neq\emptyset\\ \mathrm{scale}\,(P)=(-k_1-\gamma,\,-k_2-\gamma)}} \inf_{y\in P} |f*\Phi_{k_1-k_0,\,k_2-k_0}(y)|^2\chi_Q(x)\\ &\quad\le b_1^{k_1-k_0}b_2^{k_2-k_0}\int_{B_{\rho_1}(x_1,b_1^{-k_1+k_0})\times B_{\rho_2}(x_2,b_2^{-k_2+k_0})} |f*\Phi_{k_1-k_0,\,k_2-k_0}(y)|^2\,dy.\end{aligned}$$

Let

$$\begin{aligned}&\inf\nolimits_Q(f)\\ &\quad\equiv |Q|^{\frac12}\sup\left\{\inf_{y\in P}|f*\widetilde{\Phi}_{k_1-k_0,\,k_2-k_0}(y)| :\ \mathrm{scale}\,(P) = (-k_1-\gamma,\,-k_2-\gamma),\ P\cap Q\neq\emptyset\right\}.\end{aligned}$$

Then, for any $f \in \widetilde{H}^p_w(\vec{A})$ with $p \in (0, 1]$ and $w \in \mathcal{A}_\infty(\vec{A})$, by Corollary 2.1, Lemma 3.11 and (3.49), we obtain that

$$\begin{aligned}\|f\|_{\ddot{H}^p_w(\vec{A})} &\sim \left\|\left\{\sum_{k_1,k_2\in\mathbb{Z}} |f*\widetilde{\Phi}_{k_1-k_0,\,k_2-k_0}|^2\right\}^{\frac12}\right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\lesssim \left\|\left\{\sum_{k_1,k_2\in\mathbb{Z}}\ \sum_{\substack{Q\in\mathcal{R}\\ \mathrm{scale}\,(Q)=(-k_1,\,-k_2)}} [\inf\nolimits_Q(f)]^2\,|Q|^{-1}\chi_Q\right\}^{\frac12}\right\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)}\end{aligned}$$

$$\lesssim \|\vec{S}_\Phi(f)\|_{L^p_w(\mathbb{R}^n\times\mathbb{R}^m)} \sim \|f\|_{\widetilde{H}^p_w(\mathbb{R}^n\times\mathbb{R}^m)},$$

which completes the proof of Theorem 2.2. □

Proof of Proposition 2.2. Let $p \in (0, 1]$ and $w \in \mathcal{A}_\infty(\vec{A})$ with q_w as in (3.14). We first prove that for any $t \equiv \{t_R\}_{R\in\mathcal{R}} \in \ell_{p,\,w}(\vec{A})$, its induced map L_t, defined by $L_t(s) \equiv \sum_{R\in\mathcal{R}} s_R\bar{t}_R$ for any $s \in \ddot{h}^p_w(\vec{A})$, belongs to $(\ddot{h}^p_w(\vec{A}))^*$. We show this by using some ideas from the proof of [40, Theorem 3.5]. For any $x \in \mathbb{R}^n\times\mathbb{R}^m$, $k \in \mathbb{Z}$ and $R \in \mathcal{R}$, set $G(x) \equiv \{\sum_{R\in\mathcal{R}} |s_R|^2|R|^{-1}\chi_R(x)\}^{\frac12}$, $\Omega_k \equiv \{x \in \mathbb{R}^n\times\mathbb{R}^m : G(x) > 2^k\}$,

$$\mathcal{R}_k \equiv \{R \in \mathcal{R} : |R\cap\Omega_k| > |R|/2, |R\cap\Omega_{k+1}| \le |R|/2\}$$

and $E_{R,\,k} \equiv R\cap(\Omega_{k+1})^\complement$. Then, for any $k \in \mathbb{Z}$ and $R \in \mathcal{R}_k$, by Lemma 3.2 with $w \in \mathcal{A}_q(\vec{A})$ and $q \in (q_w, \infty)$, we obtain

$$\frac{1}{2^q} \le \frac{|E_{R,\,k}|^q}{|R|^q} \lesssim \frac{w(E_{R,\,k})}{w(R)}. \tag{3.50}$$

We choose a positive integer $c_0 > 2$ such that $b_1^{-c_0u_1}b_2^{-c_0u_2} < b_1^{-2u_1}b_2^{-2u_2}/2$ and set $\widetilde{\Omega}_k \equiv \{x \in \mathbb{R}^n\times\mathbb{R}^m : \mathcal{M}_s(\chi_{\Omega_k})(x) > b_1^{-c_0u_1}b_2^{-c_0u_2}\}$ for all $k \in \mathbb{Z}$. Then, for all $R \in \mathcal{R}_k$ and all $x \in R$, by Lemma 3.5(iv), we obtain

$$\begin{aligned}\mathcal{M}_s(\chi_{\Omega_k})(x) &\ge \frac{1}{b_1^{v_1\ell(\dot{R}_1)+u_1}b_2^{v_2\ell(\dot{R}_2)+u_2}}\int_{x_R+B^{(1)}_{v_1\ell(\dot{R}_1)+u_1}\times B^{(2)}_{v_2\ell(\dot{R}_2)R+u_2}}\chi_{\Omega_k}(y)\,dy\\ &\ge b_1^{-2u_1}b_2^{-2u_2}\frac{|\Omega_k\cap R|}{|R|} > b_1^{-c_0u_1}b_2^{-c_0u_2},\end{aligned}$$

which implies that

$$\bigcup_{R\in\mathcal{R}_k} R \subset \widetilde{\Omega}_k. \tag{3.51}$$

Moreover, for any $w \in \mathcal{A}_q(\vec{A})$ with $q \in (q_w, \infty)$, by the $L^q_w(\mathbb{R}^n\times\mathbb{R}^m)$-boundedness of $\mathcal{M}_s$ (see [8, Proposition 2.10(ii)]), we obtain $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$ for all $k \in \mathbb{Z}$.

Therefore, for all $s \in \ddot{h}^p_w(\vec{A})$, by (3.50), (3.51), Hölder's inequality and $w(\widetilde{\Omega}_k) \lesssim w(\Omega_k)$, we have

$$\begin{aligned}|L_t(s)| &\le \sum_{k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_k}|t_R||s_R|\\ &\lesssim \sum_{k\in\mathbb{Z}}\int_{\widetilde{\Omega}_k}\sum_{R\in\mathcal{R}_k}|t_R|\frac{|R|^{\frac12}}{w(R)}\chi_R(x)|s_R||R|^{-\frac12}\chi_{E_{R,\,k}}(x)w(x)\,dx\\ &\lesssim \sum_{k\in\mathbb{Z}}\left\{\int_{\widetilde{\Omega}_k}\sum_{R\in\mathcal{R}_k}|t_R|^2\frac{|R|}{[w(R)]^2}\chi_R(x)w(x)\,dx\right\}^{\frac12}\end{aligned} \tag{3.52}$$

$$\times\left\{\int_{\widetilde{\Omega}_k}\sum_{R\in\mathcal{R}_k}|s_R|^2|R|^{-1}\chi_{E_{R,\,k}}(x)w(x)\,dx\right\}^{\frac12}$$
$$\lesssim \|t\|_{\ell_{p,\,w}(\vec{A})}\sum_{k\in\mathbb{Z}}[w(\widetilde{\Omega}_k)]^{(\frac{2}{p}-1)\frac12}\left\{\int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}}[G(x)]^2w(x)\,dx\right\}^{\frac12}$$
$$\lesssim \|t\|_{\ell_{p,\,w}(\vec{A})}\sum_{k\in\mathbb{Z}}2^k[w(\Omega_k)]^{\frac1p}\lesssim \|t\|_{\ell_{p,\,w}(\vec{A})}\|s\|_{\ddot{h}^p_w(\vec{A})},$$

which implies that the induced fractional $L_t\in(\ddot{h}^p_w(\vec{A}))^*$ and $|L_t|\lesssim\|t\|_{\ell_{p,\,w}(\vec{A})}$.

Now let us prove the converse by borrowing some ideas from the proof of [21, Theorem 5.9]. For any $N\in\mathbb{N}$, $w\in\mathcal{A}_\infty(\vec{A})$ and $R\in\mathcal{R}$, let $B_N\equiv B_{\rho_1}(0,\,b_1^N)\times B_{\rho_2}(0,\,b_2^N)$,

$$I_N\equiv\{R\in\mathcal{R}:\ R\subset B_N,\ |\operatorname{scale}(R_i)|\le N,\ i=1,\,2\},$$

and $\ell^2(B_N)$ be the *set of all* $s^{(N)}\equiv\{s_R^{(N)}\}_{R\in I_N}$ *satisfying that*

$$\|s^{(N)}\|_{\ell^2(B_N)}\equiv\left\{\sum_{R\in I_N}\left|s_R^{(N)}\right|^2\right\}^{1/2}<\infty.$$

For any $t\in\ddot{h}^p_w(\vec{A})$ and $N\in\mathbb{N}$, let $t^{(N)}\equiv\{t_R^{(N)}\}_{R\in I_N}$ with $t_R^{(N)}\equiv t_R$ if $R\in I_N$. We denote by $\ddot{h}^p_w(\vec{A};\,B_N)$ the *set of all such* $t^{(N)}$. Obviously, $\ddot{h}^p_w(\vec{A};\,B_N)$ endowed with the norm $\|\cdot\|_{\ddot{h}^p_w(\vec{A})}$ is a subspace of $\ddot{h}^p_w(\vec{A})$.

Notice that for any $s^{(N)}\in\ell^2(B_N)$, $p\in(0,\,1]$ and $w\in\mathcal{A}_\infty(\vec{A})$, by Hölder's inequality, we have that $\|s^{(N)}\|_{\ddot{h}^p_w(\vec{A})}\le\|s^{(N)}\|_{\ell^2(B_N)}[w(B_N)]^{1-\frac{p}{2}}$, which implies that $\ell^2(B_N)\subset\ddot{h}^p_w(\vec{A};\,B_N)$ and hence $(\ddot{h}^p_w(\vec{A};\,B_N))^*\subset(\ell^2(B_N))^*=\ell^2(B_N)$. Then, for any $L\in(\ddot{h}^p_w(\vec{A}))^*$, by the above estimate and $(\ddot{h}^p_w(\vec{A}))^*\subset(\ddot{h}^p_w(\vec{A};\,B_N))^*$, there exists some $t^{(N)}\in\ell^2(B_N)$ such that for all $s^{(N)}\in\ddot{h}^p_w(\vec{A};\,B_N)$,

$$L(s^{(N)})=\sum_{R\in I_N}s_R^{(N)}\overline{t_R^{(N)}}. \tag{3.53}$$

For $N+1$, repeating the above process, there exists some $t^{(N+1)}\in\ell^2(B_N)$ such that for all $s^{(N+1)}\in\ddot{h}^p_w(\vec{A};\,B_{N+1})$,

$$L(s^{(N+1)})=\sum_{R\in I_{N+1}}s_R^{(N+1)}\overline{t_R^{(N+1)}},$$

and $t^{(N+1)}|_{\ddot{h}^p_w(\vec{A};\,B_N)}=t^{(N)}$. By this extension, we obtain a sequence $t^*\equiv\{t^*_R\}_{R\in\mathcal{R}}$, where $t^*_R\equiv t_R^{(N)}$ if $R\in I_N$ for all $N\in\mathbb{N}$.

We now show that $t^*\in\ell_{p,\,w}(\vec{A})$. To this end, let Ω be any open set with $w(\Omega)<\infty$ and ϑ the measure on $\mathcal{R}$ such that for any $R\in\mathcal{R}$, $\vartheta(R)\equiv[w(\Omega)]^{1-2/p}|R|[w(R)]^{-1}$ if

$R \subset \Omega$, or else $\vartheta(R) \equiv 0$. Define $\ell^2(\Omega;\,\vartheta)$ to be the *set of all complex-valued sequences* $s \equiv \{s_R\}_{R\in\mathcal{R},\,R\subset\Omega}$ *such that*

$$\|s\|_{\ell^2(\Omega;\,\vartheta)} \equiv \left\{\sum_{R\subset\Omega} |s_R|^2[w(\Omega)]^{1-\frac{2}{p}}|R|[w(R)]^{-1}\right\}^{\frac{1}{2}} < \infty.$$

Then, by (3.53) and $(\ell^2(\Omega;\,\vartheta))^* = \ell^2(\Omega;\,\vartheta)$, we have

$$\begin{aligned}
&\left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\chi_{I_N}(R)|t^*_R|^2|R|[w(R)]^{-1}\right\}^{\frac{1}{2}}\\
&\quad= \|\chi_{I_N}t^*\|_{\ell^2(\Omega,\,\vartheta)} = \sup_{\|s\|_{\ell^2(\Omega;\,\vartheta)}\le 1}\left|\sum_{R\in I_N,R\subset\Omega} t_R^{(N)}\overline{s_R}[w(\Omega)]^{1-\frac{2}{p}}|R|[w(R)]^{-1}\right|\\
&\quad\le \|L\|_{(\ddot{h}^p_w(\vec{A}))^*}\sup_{\|s\|_{\ell^2(\Omega;\,\vartheta)}\le 1}\|\{s_R[w(\Omega)]^{1-\frac{2}{p}}|R|[w(R)]^{-1}\}_{R\subset\Omega}\|_{\ddot{h}^p_w(\vec{A})} \le \|L\|_{(\ddot{h}^p_w(\vec{A}))^*},
\end{aligned}$$

where in the last step we used the following inequality

$$\begin{aligned}
&\|\{s_R w(\Omega)^{1-\frac{2}{p}}|R|[w(R)]^{-1}\}_{R\subset\Omega}\|_{\ddot{h}^p_w(\vec{A})}\\
&\quad= \left\{\int_\Omega\left[\sum_{R\subset\Omega}|s_R[w(\Omega)]^{1-\frac{2}{p}}|^2|R|[w(R)]^{-2}\chi_R(x)\right]^{p/2}w(x)\,dx\right\}^{1/p}\\
&\quad\le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\left\{\int_\Omega\sum_{R\subset\Omega}|s_R|^2|R|[w(R)]^{-2}\chi_R(x)w(x)\,dx\right\}^{\frac{1}{2}}[w(\Omega)]^{\frac{1}{p}-\frac{1}{2}} = \|s\|_{\ell^2(\Omega,\,\vartheta)} \le 1.
\end{aligned}$$

From this and the Fatou lemma, it follows that

$$\left\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{R\subset\Omega}|t^*_R|^2\frac{|R|}{w(R)}\right\}^{\frac{1}{2}} \le \|L\|_{(\ddot{h}^p_w(\vec{A}))^*},$$

which together with the arbitrariness of Ω implies that $t^* \in \ell_{p,\,w}(\vec{A})$ and $\|t^*\|_{\ell_{p,\,w}(\vec{A})} \le \|L\|_{(\ddot{h}^p_w(\vec{A}))^*}$.

By (3.52), for all $s \in \ddot{h}^p_w(\vec{A})$, we have

$$\sum_{R\in\mathcal{R}}|t^*_R s_R| \lesssim \|t^*\|_{\ell_{p,\,w}(\vec{A})}\|s\|_{\ddot{h}^p_w(\vec{A})} \lesssim \|L\|_{(\ddot{h}^p_w(\vec{A}))^*}\|s\|_{\ddot{h}^p_w(\vec{A})},$$

which together with the Lebesgue dominated convergence theorem on series and (3.53) yields that for all $s \in \ddot{h}^p_w(\vec{A})$,

$$L(s) = \lim_{N\to\infty} L\left(s^{(N)}\right) = \lim_{N\to\infty}\sum_{R\in I_N} s_R^{(N)}\overline{t^*_R} = \sum_{R\in\mathcal{R}} s_R\overline{t^*_R}.$$

This finishes the proof of Proposition 2.2. □

From Theorem 2.1 and Proposition 2.2, the proof of Theorem 2.3 follows by a straightforward adaption of methods by Frazier and Jawerth [21, Theorem 5.13].

Proof of Theorem 2.3. Let $p \in (0, 1]$ and $w \in \mathcal{A}_\infty(\vec{A})$. Let (φ, ψ) be an admissible pair of frame wavelets as in Definition 2.2 such that $\varphi = \psi$. In other words, φ is an admissible Parseval wavelet. Using Theorem 2.1, we obtain that $T_\varphi \circ S_\varphi$ is also an identity on $\ddot{H}^p_w(\vec{A})$.

For $s \equiv \{s_R\}_{R\in\mathcal{R}}$ and $t \equiv \{t_R\}_{R\in\mathcal{R}}$, let $\langle s, t\rangle \equiv \sum_{R\in\mathcal{R}} s_R\overline{t_R}$. Then for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, the φ-transform S_φ and the inverse φ-transform T_φ, we have

$$\langle S_\varphi(f), t\rangle = \sum_{R\in\mathcal{R}} \langle f, \varphi_R\rangle \overline{t_R} = \langle f, T_\varphi(t)\rangle. \tag{3.54}$$

For any $g \in \mathcal{L}_{p,w}(\vec{A})$, define a linear functional $\widetilde{L}_g$ by $\widetilde{L}_g(f) \equiv \langle g, f\rangle$ for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

Then, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$, by Theorem 2.1, (3.54) and Proposition 2.2, we have

$$\begin{aligned}|\widetilde{L}_g(f)| &= |\langle g, f\rangle| = |\langle T_\varphi(S_\varphi(g)), f\rangle| = |\langle S_\varphi(g), S_\varphi(f)\rangle| \\ &\lesssim \|S_\varphi(g)\|_{\ell_{p,w}(\vec{A})} \|S_\varphi(f)\|_{\ddot{h}^p_w(\vec{A})} \lesssim \|g\|_{\mathcal{L}_{p,w}(\vec{A})} \|f\|_{\ddot{H}^p_w(\vec{A})},\end{aligned}$$

which implies that $\|\widetilde{L}_g\|_{(\ddot{H}^p_w(\vec{A}))^*} \lesssim \|g\|_{\mathcal{L}_{p,w}(\vec{A})}$, and hence $\widetilde{L}_g$ defines a continuous linear functional on $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Moreover, since $\mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ is a dense subspace of $\ddot{H}^p_w(\vec{A})$, using Theorem 2.1, we obtain that $\widetilde{L}_g$ is uniquely extended to a continuous linear functional L_g on $\ddot{H}^p_w(\vec{A})$.

Conversely, for any $L \in (\ddot{H}^p_w(\vec{A}))^*$ and the inverse φ-transform T_φ, by Theorem 2.1, we have $\ell_1 \equiv L \circ T_\varphi \in (\ddot{h}^p_w(\vec{A}))^*$. Then, by Proposition 2.2 and Theorem 2.1, there exists $t = \{t_R\}_{R\in\mathcal{R}} \in \ell_{p,w}(\vec{A})$ such that $\ell_1(s) = \sum_{R\in\mathcal{R}} s_R\overline{t_R}$ for any $s \equiv \{s_R\}_{R\in\mathcal{R}} \in \ddot{h}^p_w(\vec{A})$ and $\|t\|_{\ell_{p,w}(\vec{A})} \sim \|\ell_1\|_{(\ddot{h}^p_w(\vec{A}))^*} \lesssim \|L\|_{(\ddot{H}^p_w(\vec{A}))^*}$. Hence, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $g \equiv T_\varphi(t) = \sum_{R\in\mathcal{R}} t_R\varphi_R$, by $\ell_1 \circ S_\varphi = L \circ T_\varphi \circ S_\varphi = L$, $T_\varphi \circ S_\varphi = Id$ on $\ddot{H}^p_w(\vec{A})$ and (3.54), we have

$$L(f) = \ell_1 \circ (S_\varphi(f)) = \langle S_\varphi(f), t\rangle = \langle f, g\rangle = L_g(f),$$

which implies that $L = L_g$. Moreover, by Theorem 2.1, we obtain that $\|g\|_{\mathcal{L}_{p,w}(\vec{A})} \lesssim \|t\|_{\ell_{p,w}(\vec{A})} \lesssim \|L_g\|_{(\ddot{H}^p_w(\vec{A}))^*}$, which completes the proof of Theorem 2.3. □

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Baode Li

Department of Mathematics, Xinjiang University, Urumqi 830046, People's Republic of China

E-mail address: `baodeli@mail.bnu.edu.cn`

Marcin Bownik

Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA

E-mail address: `mbownik@uoregon.edu`

Dachun Yang (Corresponding author)

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

E-mail address: dcyang@bnu.edu.cn