A COMBINATORIAL CHARACTERIZATION OF TIGHT FUSION FRAMES

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Abstract. In this paper we give a combinatorial characterization of tight fusion frame (TFF) sequences using Littlewood-Richardson skew tableaux. The equal rank case has been solved recently by Casazza et al. [8]. Our characterization does not have this limitation. We also develop some methods for generating TFF sequences. The basic technique is a majorization principle for TFF sequences combined with spatial and Naimark dualities. We use these methods and our characterization to give necessary and sufficient conditions which are satisfied by the first three highest ranks. We also give a combinatorial interpretation of spatial and Naimark dualities in terms of Littlewood-Richardson coefficients. We exhibit four classes of TFF sequences which have unique maximal elements with respect to majorization partial order. Finally, we give several examples illustrating our techniques including an example of tight fusion frame which can not be constructed by the existing spectral tetris techniques [5, 6, 8]. We end the paper by giving a complete list of maximal TFF sequences in dimensions ≤ 9.

1. Introduction

Fusion frames were introduced by Casazza, Kutyniok in [9] (under the name frames of subspaces) and [10]. A fusion frame for $\mathbb{R}^N$ is a finite collection of subspaces $\{W_i\}_{i=1}^K$ in $\mathbb{R}^N$ such that there exists constants $0 < \alpha \leq \alpha' < \infty$ satisfying

$$\alpha||x||^2 \leq \sum_{i=1}^{K} ||P_i x||^2 \leq \alpha'||x||^2 \qquad \text{for all } x \in \mathbb{R}^N,$$

where P_i is the orthogonal projection onto W_i. Equivalently, $\{W_i\}_{i=1}^K$ is a fusion frame if and only if

$$\alpha\mathbf{I} \leq \sum_{i=1}^{K} P_i \leq \alpha'\mathbf{I},$$

where $\mathbf{I}$ is the identity on $\mathbb{R}^N$. The constants α and α' are called fusion frame bounds. An important class of fusion frames are *tight fusion frames* (TFF), for which $\alpha = \alpha'$ and hence $\sum_{i=1}^K P_i = \alpha\mathbf{I}$. We note that the definition of fusion frames given in [9, 10] applies to closed subspaces in any Hilbert space together with a collection of weights

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associated to each subspace W_i. Since the scope of this paper is limited to non-weighted finite dimensional TFF, the definition of a fusion frame is only presented for this case.

Fusion frames have been a very active area of research in the frame theory. A lot of effort was devoted into developing the basic properties and constructing fusion frames with desired properties. In particular, the construction and existence of sparse tight fusion frames was studied in [5]. Fusion frame potentials have been studied in [7] and [27]. Applications of fusion frames include sensor networks [10], coding theory [3, 26], compressed sensing [4], and filter banks [11]. In this paper we consider a problem of classifying TFF sequences.

Problem 1.1. *Given $N \in \mathbb{N}$, characterize sequences $(L_1, \dots, L_K)$ for which there exists a tight fusion frame $\{W_i\}_{i=1}^K$ with $\dim W_i = L_i$ in N dimensional space. Equivalently, given $\alpha > 1$ such that $\alpha N \in \mathbb{N}$, characterize sequences $(L_1, \dots, L_K)$ such that $\alpha \mathbf{I}$ can be decomposed as a sum of projections $P_1 + \ldots + P_K$ with $\operatorname{rank} P_i = L_i$, $i = 1, \dots, K$.*

Casazza, Fickus, Mixon, Wang, and Zhou [8] have recently achieved significant progress in this direction by solving the equal rank case. That is, the authors have classified all triples (K, L, N) such that there exists a tight fusion frame consisting of K subspaces $\{W_i\}_{i=1}^K$ with the same dimension $\dim W_i = L$ in $\mathbb{R}^N$. The answer given in [8] is highly non-trivial in the most interesting case when L does not divide N and $2L < N$. The authors show that a necessary condition for such sequences (K, L, N) is that $K \geq \lceil N/L \rceil + 1$, whereas a sufficient condition is $K \geq \lceil N/L \rceil + 2$. In a gray area, where $K = \lceil N/L \rceil + 1$, the authors have devised a reduction procedure which replaces the original sequence by another one with the equivalent TFF property (existence or non-existence). Then, it is shown that after a finite number of steps the original sequence (K, L, N) is reduced to one for which either the necessary condition fails or the sufficient condition holds. However, the results of [8] do not say much about a more general problem of classifying TFF sequences with non-equal ranks. In this paper we answer Problem 1.1 by giving a combinatorial characterization of TFF sequences using Littlewood-Richardson skew tableaux.

While the concept of fusion frames is relatively new, the problem of representing an operator as a sum of orthogonal projections has been studied for a long time in the operator theory. The first fundamental result of this kind belongs to Fillmore [12] who characterized finite rank operators which are finite sums of projections, see Theorem 3.1. Fong and Murphy [13] characterized operators which are positive combinations of projections. Analogous results were recently investigated for C-$*$ algebras and von Neumann algebras, see [16, 18]. However, the most relevant results for us are due to Kruglyak, Rabanovich, and Samoĭlenko [25, 24] who characterized the set of all (α, N) such that $\alpha \mathbf{I}$ is the sum of K orthogonal projections. In other words, their main result [24, Theorem 7] gives a minimal length K of a TFF sequence in $\mathbb{R}^N$ with the frame bound α. However, [24] does not say anything about the ranks of projections which is a focus of this paper.

In the finite dimensional setting the existence of TFF sequences is intimately related to Horn's problem [17] which has been solved by Klyachko [20], and Knutson and Tao [21, 23], for a survey see [15, 22]. Problem 1.1 can be thought of as a very special kind of Horn's problem where hermitian matrices have only two eigenvalues: 0 and 1, and their sum has only one eigenvalue α. Using Klyachko's result [20] we show that the existence of TFF sequence $(L_1, \ldots, L_K)$ is equivalent to the non-vanishing of a certain Littlewood-Richardson coefficient, see Theorem 4.3. In turn, the latter condition is equivalent to the existence of a matrix satisfying some computationally explicit properties such as: constant row and column sums, and row and column sum dominance, see Corollary 4.4. Our combinatorial characterization enables us to deduce several properties that TFF sequences must satisfy. In addition, it enables us to give an explicit construction procedure of a tight fusion frame corresponding to a given TFF sequence, see Example 7.2.

A fundamental technique of our paper is a majorization principle involving the majorization partial order $\preccurlyeq$ as in the Schur-Horn theorem [2, 19], which is also known as the dominance order in algebraic combinatorics [14]. In Section 2 we show that a sequence majorized by a TFF sequence is also a TFF sequence. We also establish the spatial and Naimark dualities for general TFF sequences extending the equal rank results in [8]. In Section 3 we find necessary and sufficient conditions on the first three largest ranks of projections using Filmore's theorem [12] and a description of possible spectra of a sum of two projections, see Lemma 3.2. The latter result might be of independent interest since its proof uses honeycomb models developed by Knutson and Tao [21, 22]. In the same section we also exhibit classes of TFF sequences which have only one maximal element. These include not only the expected case of integer α, but also half-integer scenario, and the corresponding conjugate α's via the Naimark duality. In Section 4 we prove our main characterization result of TFF sequences using Littlewood-Richardson skew tableaux. In addition to illustrating it on specific examples, in Section 5 we give a complete proof of Theorem 3.3 using the combinatorics of the Schur functions. This leads to a partial characterization of TFF sequences which are of the hook type, i.e., sequences ending in repeated 1's. In Section 6 we show that the spatial and Naimark dualities manifest themselves as identities for the corresponding Littlewood-Richardson coefficients. In the final Section 7 we give several examples of existence of tight fusion frames using skew Littlewood-Richardson tableaux. In particular, we give an explicit construction of TFF corresponding to the sequence $(4, 2, 2, 2, 1)$ in dimension $N = 6$. This example is remarkable for two reasons. It is the first TFF sequence which is missed by brute force generation involving recursive spatial and Naimark dualities. Furthermore, this example can not be constructed by the existing spectral tetris construction [5, 6], which is an algorithmic method of constructing sparse fusion frames utilized in the equal rank characterization [8]. We end the paper by giving a complete list of maximal TFF sequences for $\alpha \leq 2$ in dimensions $N \leq 9$.

2. Basic majorization and duality results

Definition 2.1. Fix a positive integer N. Let $L_1 \geq L_2 \geq \ldots \geq L_K > 0$ be a weakly decreasing sequence of positive integers. Such sequence is also known as a *partition* in number theory [1] and algebraic combinatorics [14]. We say that $(L_1, L_2, \ldots, L_K)$ is a tight fusion frame (TFF) sequence if there exists orthogonal projections $P_1, \dots, P_K$ such that

$$\alpha \mathbf{I} = \sum_{i=1}^{K} P_i, \qquad \text{and } \operatorname{rank} P_i = L_i, \tag{2.1}$$

where $\alpha \in \mathbb{R}$ and $\mathbf{I}$ is the identity on $\mathbb{R}^N$. A trace argument shows that $\alpha = \sum_{i=1}^{K} L_i/N \geq 1$. Given $\alpha \geq 1$ such that $\alpha N \in \mathbb{N}$, we define $\mathrm{TFF}(\alpha, N)$ to be the set of all TFF sequences in $\mathbb{R}^N$ with the frame bound α.

2.1. **Majorization.** The following definition comes from the majorization theory of the Schur-Horn theorem, see [19]. In algebraic combinatorics the majorization partial order on partitions is known as the *dominance order*, see [14].

Definition 2.2. Suppose that $\mathbf{L} = (L_1, L_2, \dots, L_K)$ and $\mathbf{L}' = (L'_1, L'_2, \dots, L'_{K'})$ be two weakly decreasing sequences of non-negative integers. We say that $\mathbf{L}'$ majorizes $\mathbf{L}$, and write $\mathbf{L} \preccurlyeq \mathbf{L}'$ if

$$\sum_{i=1}^{K} L_i = \sum_{i=1}^{K'} L'_i \qquad \text{and} \qquad \sum_{i=1}^{k} L_i \leq \sum_{i=1}^{k} L'_i,$$

for all $k \leq \min(K, K')$.

Observe that appending zeros at the tails of sequences $\mathbf{L}, \mathbf{L}'$ does not affect majorization relation. Moreover, for sequences with only positive terms, the majorization $\mathbf{L} \preccurlyeq \mathbf{L}'$ forces that $K \geq K'$.

The majorization principle for TFF sequences takes the following form.

Theorem 2.3. *Let $\mathbf{L}$ and $\mathbf{L}'$ be two weakly decreasing sequences of positive integers such that $\mathbf{L} \preccurlyeq \mathbf{L}'$. Then, $\mathbf{L}' \in \mathrm{TFF}(\alpha, N)$ implies that $\mathbf{L} \in \mathrm{TFF}(\alpha, N)$.*

In the proof of Theorem 2.3 we use the following elementary result on a sum of two projections.

Lemma 2.4. *Fix positive integers $p > q \geq 0$. Let P and Q be two orthogonal projection of ranks p and q, resp. Then, there exists orthogonal projections P' and Q' of ranks $p-1$ and $q+1$, resp., such that $P + Q = P' + Q'$.*

Proof. Assume we have two projections P and Q with ranks $p > q$ that act on an N dimensional vector space V. Then, we can decompose V into the eigenspaces of P and Q such that

$$V = V_P \oplus V_P^{\perp}, \qquad V = V_Q \oplus V_Q^{\perp},$$

where V_P and $V_P^{\perp}$ denote the 1-eigenspace and 0-eigenspace, resp. Since $p > q$, we have that $p + (N - q) > N$ and hence $\dim(V_P \cap V_Q^{\perp}) > 0$. Choose a nonzero vector in $V_P \cap V_Q^{\perp}$ and let R denote the corresponding rank 1 projection. Then, we can decompose $P = \bar{P} + R$, where $\bar{P}$ is a rank $p-1$ projection. Moreover, $Q + R$ is a projection of rank $q+1$. Thus, $P + Q = \bar{P} + (Q + R)$, which completes the proof of the lemma. $\square$

Proof of Theorem 2.3. Since $\mathbf{L} \preccurlyeq \mathbf{L}'$ we can find a sequence of partitions $\mathbf{L} = \mathbf{L}^0 \preccurlyeq \mathbf{L}^1 \preccurlyeq \ldots \preccurlyeq \mathbf{L}^n = \mathbf{L}'$ such that any two consecutive partitions $\mathbf{L}^{j-1}$ and $\mathbf{L}^j$, $j = 1, \ldots, n$, differ at exactly two positions by ± 1. That is, for each $j = 1, \ldots, n$, there exist two positions $m < m' \in \mathbb{N}$ such that

$$\begin{aligned} \mathbf{L}^{j-1} &= (*, \ldots, *, \tilde{L}_m \qquad, *, \ldots, *, \tilde{L}_{m'} \qquad, *, \ldots, *), \\ \mathbf{L}^{j} &= (*, \ldots, *, \tilde{L}_m + 1, *, \ldots, *, \tilde{L}_{m'} - 1, *, \ldots, *), \end{aligned} \tag{2.2}$$

where the remaining values, denoted by $*$, are the same. Such $\mathbf{L}^j$'s can be easily constructed by the following recursive procedure.

Given the initial partitions $\mathbf{L}$ and $\mathbf{L}'$ we append extra zeros to $\mathbf{L}'$ so that $\mathbf{L}$ and $\mathbf{L}'$ have the same length. Define m to be the first position such that initial subsequences $(L_1, \ldots, L_m)$ and $(L'_1, \ldots, L'_m)$ are not the same. Likewise, m' is the last position such that the ending subsequences $(L_{m'}, \ldots)$ and $(L'_{m'}, \ldots)$ are not the same. Define $\mathbf{L}^1$ from $\mathbf{L}$ by replacing $L_m \to L_m + 1$ and $L_{m'} \to L_{m'} - 1$. It is not difficult to see that $\mathbf{L}^1$ forms a weakly decreasing sequence and $\mathbf{L} = \mathbf{L}_0 \preccurlyeq \mathbf{L}^1 \preccurlyeq \mathbf{L}'$. Repeating this procedure recursively we define a sequence $\mathbf{L}^1 \preccurlyeq \mathbf{L}^2 \preccurlyeq \ldots \preccurlyeq \mathbf{L}'$. After a finite number of steps we must arrive at $\mathbf{L}^n = \mathbf{L}'$.

Observe that the ranks in (2.2) satisfy $\tilde{L}_m \geq \tilde{L}_{m'}$. By Lemma 2.4 applied to two projections with ranks $p = \tilde{L}_m + 1 > q = \tilde{L}_{m'} - 1 \geq 0$, if $\mathbf{L}^j \in \mathrm{TFF}(\alpha, N)$, then $\mathbf{L}^{j-1} \in \mathrm{TFF}(\alpha, N)$. Thus, a repetitive application of Lemma 2.4 proves Theorem 2.3. □

We remark that the above proof does not use the tightness assumption in any way. Consequently, Theorem 2.3 holds for general (not necessarily tight) fusion frames with a prescribed frame operator.

2.2. **Dualities.** In this subsection we shall establish two dualities for TFF sequences. The first duality involves taking orthogonal projections of the same ambient space and is a straightforward generalization of [8, Theorem 6].

Theorem 2.5. *Suppose that* $(L_1, L_2, \ldots, L_K) \in \mathrm{TFF}(\alpha, N)$*. Then,* $(N - L_K, N - L_{K-1}, \ldots, N - L_1) \in \mathrm{TFF}(K - \alpha, N)$*.*

Proof. Let $P_1, \ldots, P_K$ be the orthogonal projections with $\operatorname{rank} P_i = L_i$ such that $\sum_{i=1}^K P_i = \alpha \mathbf{I}$. Clearly, $\sum_{i=1}^K (\mathbf{I} - P_i) = (K - \alpha)\mathbf{I}$ and $\operatorname{rank}(\mathbf{I} - P_i) = N - L_i$. This shows the theorem. □

The second result relies on taking more subtle orthogonal complements based on a dilation theorem for tight frames with bound 1, also known as Parseval frames. It is known that every Parseval frame can be obtained as a projection of an orthogonal basis of some higher dimensional space. The complementary projection gives rise to another Parseval frame, which is often called the *Naimark complement* of the original frame. This leads to the following result

Theorem 2.6. *Suppose that* $(L_1, L_2, \ldots, L_K) \in \mathrm{TFF}(\alpha, N)$*. Then, the same sequence* $(L_1, L_2, \ldots, L_K) \in \mathrm{TFF}(\tilde{\alpha}, \tilde{N})$*, where the dimension* $\tilde{N} = (\sum_{i=1}^K L_i - N)$ *and the frame bound* $\tilde{\alpha} = \alpha/(\alpha - 1) = \alpha N/\tilde{N}$*.*

Proof. For each $k = 0, \dots, K$, define $\sigma_k = \sum_{i=1}^{k} L_i$ with the convention that $\sigma_0 = 0$. Our assumption implies that there exists a tight frame $\{v_j\}_{j=1}^{\sigma_K}$ in $\mathbb{R}^N$ such that for each $k = 1, \dots, K$, the subcollection $\{v_j\}_{j=1+\sigma_{k-1}}^{\sigma_k}$ is an orthonormal sequence which spans the L_k dimensional space W_k from the definition of a TFF. Treating $v_1, \dots, v_{\sigma_K}$ as column vectors we obtain an $N \times \sigma_K$ matrix U with orthogonal rows each of norm $\alpha = \sigma_K/N$. This is due to the fact that $\{v_j\}_{j=1}^{\sigma_K}$ is a tight frame with constant α.

Let $\tilde{U}$ be an extension of U to a $\sigma_K \times \sigma_K$ matrix with all orthogonal rows of norm α. In other words, $\frac{1}{\alpha}\tilde{U}$ is a unitary extension of $\frac{1}{\alpha}U$ which has orthonormal rows. Let $\{w_j\}_{j=1}^{\sigma_K}$ be the column vectors constituting the $(\sigma_K - N) \times \sigma_K$ submatrix of the bottom rows of $\tilde{U}$. Since $\frac{1}{\alpha}\tilde{U}$ is an orthogonal matrix we have

$$\langle v_j, v_{j'} \rangle + \langle w_j, w_{j'} \rangle = \alpha \delta_{j,j'} \qquad \text{for all } j, j' = 1, \dots, \sigma_K.$$

By the block orthogonality of v_j's we have that for each block $k = 1, \dots, K$,

$$\langle w_j, w_{j'} \rangle = (\alpha - 1)\delta_{j,j'} \qquad \text{for all } j, j' = 1 + \sigma_{k-1}, \dots, \sigma_k.$$

This means that the vectors $\{w_j\}_{j=1+\sigma_{k-1}}^{\sigma_k}$ form an orthogonal sequence which span some L_k dimensional space $\tilde{W}_k$. Moreover, $\{w_j\}_{j=1}^{\sigma_K}$ is a tight frame with a constant α for $(\sigma_K - N)$ dimensional space. Consequently, unit norm vectors $\{\frac{1}{\alpha-1} w_j\}_{j=1}^{\sigma_K}$, which are block orthonormal, form a tight frame with a constant $\frac{\alpha}{\alpha-1}$. This leads to the decomposition $\tilde{P}_1 + \dots \tilde{P}_K = \frac{\alpha}{\alpha-1}\mathbf{I}$, where $\tilde{P}_k$ is an orthogonal projection onto $\tilde{W}_k$. This completes the proof of the theorem. □

As an immediate corollary of Theorem 2.6 we can reduce the study of TFF sequences to the case when $1 < \alpha < 2$; the case $\alpha = 2$ does not cause any difficulties as we will see later.

Corollary 2.7. *If $\alpha > 1$ is such that $\alpha N \in \mathbb{N}$, then* $\mathrm{TFF}(\alpha, N) = \mathrm{TFF}(\tilde{\alpha}, \tilde{N})$, *where* $1/\alpha + 1/\tilde{\alpha} = 1$ *and* $\tilde{N} = N(\alpha - 1)$.

Observe that if there exists a TFF sequence with parameters (α, N), then by computing traces we necessarily have that $\alpha N \in \mathbb{N}$. Hence, without loss of generality we shall always make this assumption.

3. Estimates on first 3 ranks

In this section we find necessary and sufficient conditions on the first three largest ranks of TFF projections. Our analysis is based on two fundamental results. Theorem 3.1 is due to Fillmore [12, Theorem 1]. Lemma 3.2 describes the spectral properties of the sum of two projections, and it can be thought of as a generalization of Lemma 2.4.

Theorem 3.1. *A non-negative definite hermitian matrix S is a sum of projections if and only if*

$$\mathrm{trace}(S) \in \mathbb{N}_0 \qquad \text{and} \qquad \mathrm{trace}(S) \geq \mathrm{rank}(S). \tag{3.1}$$

Lemma 3.2. *Let P, Q be two orthogonal projections on an N dimensional vector space V with ranks p, q, resp. For any $\lambda \in \mathbb{R}$, let $m(\lambda)$ be the multiplicity of λ as an eigenvalue of $P+Q$. Then, the following are true:*

(i) $m(\lambda) > 0 \implies \lambda \in [0,2]$,
(ii) $\sum_{\lambda\in[0,2]} m(\lambda) = N$,
(iii) $m(1) \geq |p-q|$,
(iv) $\lambda \in (0,2) \implies m(\lambda) = m(2-\lambda)$,
(v) $m(0) - m(2) = N - p - q$.

Conversely, if $0 \leq p, q \leq N$, and $m : \mathbb{R} \to \mathbb{N}_0$ satisfies (i)–(v), then there exists orthogonal projections P, Q of ranks p, q, such that m is a multiplicity function of $P+Q$.

Proof. Since P, Q are hermitian, we can decompose V as a direct sum of eigenspaces

$$V = V_P \oplus V_P^{\perp} = V_Q \oplus V_Q^{\perp}$$

where V_P denotes the 1-eigenspace and $V_P^{\perp}$ the 0 eigenspace of P. Thus, $p = \dim(V_P)$ and $q = \dim(V_Q)$. Parts (i)–(iii) follow by basic linear algebra.

To prove part (iv) we define $f_\lambda : V \to V$ by

$$f_\lambda(v) := v_P + \left(\frac{\lambda}{\lambda - 2}\right) v_P',$$

where $v = v_P + v_P'$ is induced by the orthogonal decomposition $V = V_P \oplus V_P^{\perp}$ and $\lambda \in (0,2)$. Since f_λ is an invertible and linear map, it suffices to show that if $(P+Q)v = \lambda v$, then $(P+Q)f_\lambda(v) = (2-\lambda)f_\lambda(v)$. Write

$$v_P = x_Q + x_Q' \quad \text{and} \quad v_P' = y_Q + y_Q'$$

according to the decomposition $V = V_Q \oplus V_Q^{\perp}$. Then,

$$(P+Q)v = v_P + x_Q + y_Q = 2x_Q + y_Q + x_Q' = \lambda(x_Q + x_Q' + y_Q + y_Q')$$

and hence

$$(2-\lambda)x_Q + (1-\lambda)y_Q = (\lambda - 1)x_Q' + \lambda y_Q'.$$

This implies that

$$(2-\lambda)x_Q = (\lambda-1)y_Q \quad \text{and} \quad (1-\lambda)x_Q' = \lambda y_Q' \tag{3.2}$$

since $V_Q \cap V_Q^{\perp} = \{0\}$.

By equation (3.2), we have that

$$\begin{aligned}(P+Q)f_\lambda(v) &= 2x_Q + x'_Q + \left(\frac{\lambda}{\lambda-2}\right) y_Q \\ &= (2-\lambda)v_P + \lambda x_Q + (\lambda-1)x'_Q + \left(\frac{\lambda}{\lambda-2}\right) y_Q \\ &= (2-\lambda)v_P + \left(\frac{\lambda(1-\lambda)}{\lambda-2}\right) y_Q - \lambda y'_Q + \left(\frac{\lambda}{\lambda-2}\right) y_Q \\ &= (2-\lambda)v_P - \lambda y_Q - \lambda y'_Q \\ &= (2-\lambda)\left(v_P + \left(\frac{\lambda}{\lambda-2}\right) v'_P\right) = (2-\lambda)f_\lambda(v).\end{aligned}$$

This proves part (iv). To prove part (v), we consider the projection map

$$g : V \to V_P + V_Q$$

where $V_P + V_Q$ denotes the span of vectors in V_P, V_Q. We have that

$$\dim(V_P + V_Q) = \dim(V_P) + \dim(V_Q) - m(2) = p + q - m(2).$$

But

$$\dim(V_P + V_Q) = N - \dim(\ker g) = N - m(0).$$

This shows that the properties (i)–(v) are necessary.

A quick way to see the converse direction is to utilize the honeycomb model of Knutson and Tao [21, 22]. The honeycombs corresponding to triples $(P, Q, -(P+Q))$, where $p > q$ can be represented by one of the following diagrams. In the case $p = q$ the line corresponding the eigenvalue -1 of $-(P+Q)$ might not be present. We leave the details to the reader. This involves finding multiplicities of unlabelled line segments to satisfy the "zero-tension" property.

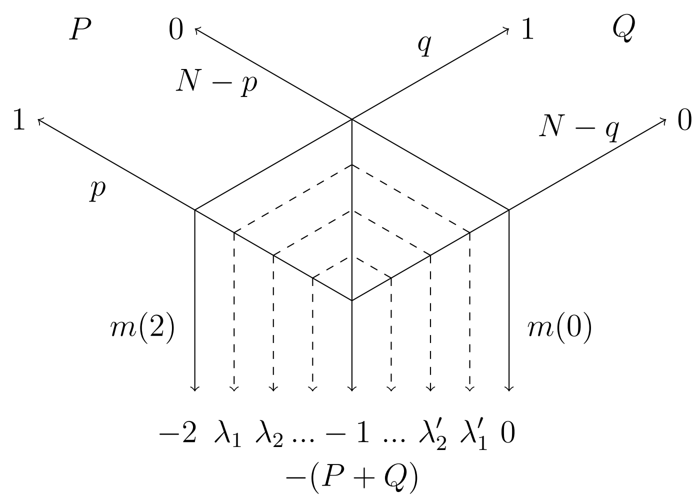


FIGURE 1. Honeycomb with $m(2) > 0$, $m(0) > 0$ and $\lambda'_i := -2 - \lambda_i$.

□

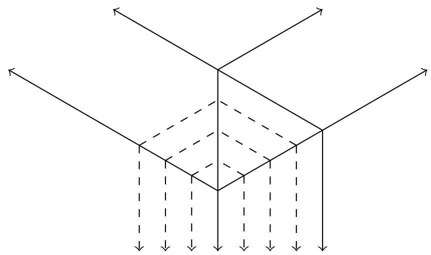
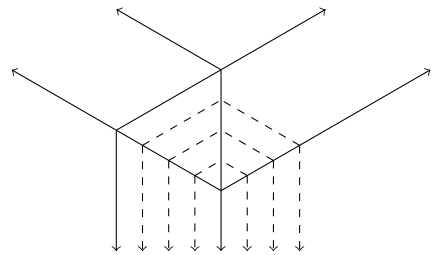



FIGURE 2. Honeycombs with $m(2) = 0$ and $m(0) = 0$, respectively.

Using Theorem 3.1 and Lemma 3.2 our goal is to find necessary and sufficient conditions on the first three largest ranks of projections in a TFF.

Theorem 3.3. *Suppose that* $1 < \alpha < 2$ *and* $(L_1 \geq L_2 \geq \cdots \geq L_K) \in \mathrm{TFF}(\alpha, N)$. *Then, we have the following necessary conditions:*

$$L_1 \leq (\alpha - 1)N, \tag{3.3}$$

$$L_1 + L_2 \leq N, \tag{3.4}$$

$$L_1 + L_2 + L_3 \leq \begin{cases} N & \alpha < 3/2, \\ 2(\alpha - 1)N & \alpha > 3/2. \end{cases} \tag{3.5}$$

Conversely, if $L_1 \geq L_2 \geq L_3$ *satisfy* (3.3), (3.4), *and* (3.5), *then there exists* $\mathbf{L} \in \mathrm{TFF}(\alpha, N)$ *which starts with the sequence* (L_1, L_2, L_3).

Proof. Suppose $\alpha\mathbf{I}$ is written as in (2.1). Then, $S = \alpha\mathbf{I} - P_1$ is an operator with 2 eigenvalues: α with multiplicity $N - L_1$ and $(\alpha - 1)$ with multiplicity L_1. By Theorem 3.1 we must have that

$$\alpha N - L_1 \geq N.$$

Solving this for L_1 yields (3.3).

By Lemma 3.2 the sum $P_1 + P_2$ has eigenvalue 1 with multiplicity at least $L_1 - L_2$. Moreover, all other positive eigenvalues of this sum must come in pairs $(2 - \lambda, \lambda)$, where $1 \leq \lambda \leq \alpha < 2$. Thus, by Lemma 3.2(v), $L_1 + L_2 \leq N$. Let $S = \alpha\mathbf{I} - P_1 - P_2$. By Theorem 3.1, S must satisfy (3.1). Note that the trace of S remains constant regardless of choices of P_1 and P_2,

$$\mathrm{trace}(S) = \alpha N - L_1 - L_2.$$

Thus, the rank of S must be minimized to guarantee that it can be written as a sum of projections. The minimal rank of S occurs if $P_1 + P_2$ has eigenvalue α with multiplicity L_2, and thus eigenvalue $2-\alpha$ with the same multiplicity. Then, the rank of the corresponding S is $N - L_2$. Thus, we have

$$\alpha N - L_1 - L_2 \geq N - L_2.$$

This leads again to (3.3). Thus, Fillmore's theorem does not introduce new constraints in this case. In other words, (3.3) and (3.4) are both necessary and sufficient conditions for the existence of an element of $\mathrm{TFF}(\alpha, N)$ starting with (L_1, L_2).

Suppose next that $1 < \alpha < 3/2$. Repeating the above arguments, by Lemma 3.2, P_1+P_2 must have all of its L_1+L_2 non-zero eigenvalues (counted with multiplicities) in the interval $[2-\alpha,\alpha]$. Thus, if $L_1+L_2+L_3 > N$, then at least one eigenvalue of $P_1+P_2+P_3$ would be at least $(2-\alpha)+1 > 3/2 > \alpha$, which is impossible. Thus, (3.5) is necessary.

To prove the converse, assume that $L_1+L_2+L_3 \le N$. Using honeycomb models as in the proof of Lemma 3.2 one can show that there exist projections P_i such that their sum $P_1+P_2+P_3$ has the eigenvalue α with multiplicity L_2+L_3, and no eigenvalues bigger than α. This is shown in a two step process. First, we construct P_2 and P_3 such that their sum has eigenvalues: α and $2-\alpha$ both with multiplicities L_3 and 1 with multiplicity L_2-L_3. Then, using a honeycomb model we can add on another projection P_1, such that $P_1+P_2+P_3$ has eigenvalue α with multiplicity L_2+L_3. This leads to an operator $S = \alpha\mathbf{I}-(P_1+P_2+P_3)$ with the rank $N-L_2-L_3$. The trace of S remains constant regardless of the choice of such projections,

$$\operatorname{trace}(S) = \alpha N - L_1 - L_2 - L_3.$$

Since $L_1 \le (\alpha-1)N$, Fillmore's Theorem 3.1 can be applied to represent S as a sum of projections. This proves that (3.3)–(3.5) are both necessary and sufficient conditions for the first 3 ranks of a TFF sequence in the case $1 < \alpha < 3/2$. Unfortunately, the case $3/2 < \alpha < 2$ does not seem to be easily approachable with the techniques of this section. Instead, in Section 5 we shall give another combinatorial proof of Theorem 3.3 which works in the entire range $1 < \alpha < 2$. □

We end this section by an explicit characterization of TFF sequences for some special values α.

Theorem 3.4. *The set* $\operatorname{TFF}(\alpha, N)$ *has exactly one maximal element* $\mathbf{L}$ *with respect to majorization relation* $\preccurlyeq$ *in the following four cases indexed by* $n \in \mathbb{N}$*:*

$$\alpha = n, \qquad \mathbf{L} = (\underbrace{N, N, \dots, N}_{n}), \tag{3.6}$$

$$\alpha = 1+\frac{1}{n},\ n|N, \qquad \mathbf{L} = \Bigg(\underbrace{\frac{N}{n}, \frac{N}{n}, \dots, \frac{N}{n}}_{n+1}\Bigg), \tag{3.7}$$

$$\alpha = n+\frac{1}{2},\ 2|N, \qquad \mathbf{L} = \Bigg(\underbrace{N, \dots, N}_{n-1}, \frac{N}{2}, \frac{N}{2}, \frac{N}{2}\Bigg), \tag{3.8}$$

$$\alpha = 1+\frac{2}{2n-1},\ (2n-1)|N,\ \mathbf{L} = \Bigg(\underbrace{\frac{2N}{2n-1}, \dots, \frac{2N}{2n-1}}_{n-1}, \frac{N}{2n-1}, \frac{N}{2n-1}, \frac{N}{2n-1}\Bigg). \tag{3.9}$$

Proof. The case (3.6) is the easiest and it follows immediately from Theorem 2.3. The case (3.7) is obtained by the duality argument. Indeed, note that if $\alpha = 1+1/n$, then n must divide N. Then, by Corollary 2.7, $\operatorname{TFF}(\alpha, N) = \operatorname{TFF}(\tilde\alpha, \tilde N)$, where $\tilde\alpha = \alpha/(\alpha-1) = n+1$ and $\tilde N = (\alpha-1)N = N/n$.

In particular, we have that $\mathrm{TFF}(3/2, N) = \mathrm{TFF}(3, N/2)$ has a unique maximal element $(N/2, N/2, N/2)$. By appending $(n-1)$ N's in the front of this sequence we obtain a maximal element of $\mathrm{TFF}(n+1/2, N)$. It remains to show that this is the only maximal element.

Suppose that we have another element $(L_1, \ldots, L_K) \in \mathrm{TFF}(n+1/2, N)$. Let P_i's be the corresponding projections. Given two hermitian matrices S and T we write $S \le T$ if $\langle Sx, x\rangle \le \langle Tx, x\rangle$ for all $x \in \mathbb{R}^N$. Since $\sum_{i=1}^n P_i \le n\mathbf{I}$, $S = \sum_{i=n+1}^K P_i$ must have full rank N. By Fillmore's Theorem 3.1, this implies that

$$\mathrm{trace}(S) = \sum_{i=n+1}^{K} L_i \ge N.$$

Thus, $L_1 + \ldots + L_n \le (n-1/2)N$.

Suppose on the contrary that $L_1+\ldots+L_{n+1} > nN$. Let W_i's be the corresponding subspaces with $\dim W_i = L_i$. By basic linear algebra the intersection satisfies

$$\dim\left(\bigcap_{i=1}^{n+1} W_i\right) = L_1 + \ldots + L_{n+1} - nN > 0.$$

This implies that $P_1 + \ldots + P_{n+1}$ has eigenvalue $n+1$ exceeding $\alpha = n+1/2$, which is a contradiction. Thus, we have necessarily that $L_1 + \ldots + L_{n+1} \le nN$. Clearly,

$$L_1 + \ldots + L_{n+2} \le L_1 + \ldots + L_K = (n+1/2)N.$$

Consequently, $(L_1, \ldots, L_K) \preccurlyeq \mathbf{L}$ proving (3.8).

Finally, (3.9) is shown by the duality argument. Indeed, note that if $\alpha = 1 + 2/(2n-1)$, then $2n-1$ must divide N. Then, by Corollary 2.7, $\mathrm{TFF}(\alpha, N) = \mathrm{TFF}(\tilde{\alpha}, \tilde{N})$, where $\tilde{\alpha} = \alpha/(\alpha-1) = n+1/2$ and $\tilde{N} = (\alpha-1)N = 2N/(2n-1)$. □

Section 7 provides the list of all maximal elements in $\mathrm{TFF}(\alpha, N)$ for all $\alpha \le 2$ and dimensions $N \le 9$. It is easy to observe that all unique maximal elements in our tables are covered by Theorem 3.4. Hence, it is very tempting to conjecture that for general α and N, if $\mathrm{TFF}(\alpha, N)$ has only one maximal element, then α must necessarily come from one of the four cases of Theorem 3.4.

4. A combinatorial characterization of tight fusion frames

In this section we give a combinatorial characterization of tight fusion frames in the context of Schur functions. The main result of this section, Theorem 4.3, is a direct consequence of Horn's recursion for the hermitian eigenvalue problem (for a survey of this problem see [15]). For completeness, we state the main results of this body of work. For any partition

$$\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_d > 0),$$

let

$$|\lambda| = \sum_{i=1}^{d} \lambda_i$$

denote the size of λ and let d denote the length. We say λ is a rectangular partition if $\lambda = (a^b) := \underbrace{(a, \dots, a)}_{b}$ for some positive integers a, b. For any partition λ, let s_λ denote the corresponding Schur polynomial. The polynomial s_λ is a homogeneous polynomial of degree $|\lambda|$. It is well known that the Schur polynomials form a linear basis of the ring of symmetric polynomials with integer coefficients. Hence for any collection of partitions $\lambda^1, \dots, \lambda^K$ we can define the corresponding *Littlewood-Richardson coefficients* $c(\lambda^1, \dots, \lambda^K; \mu)$ as the product structure constants of

$$\prod_{i=1}^{K} s_{\lambda^i} = \sum_{\mu} c(\lambda^1, \dots, \lambda^K; \mu)\, s_\mu.$$

The Littlewood-Richardson coefficients defined above play an important role in the hermitian eigenvalue problem. To state these results, we first need some notation. There is a standard identification between sets of positive integers of size r and partitions of length at most r. For any set $I = \{i_1 < i_2 < \cdots < i_r\}$, define the partition

$$\lambda(I) := (i_r - r, i_{r-1} - r + 1, \dots, i_1 - 1).$$

Let $(\beta^1, \dots, \beta^{K+1}) \in (\mathbb{R}^N)^{K+1}$ denote a collection of sequences where each $\beta^i := (\beta^i_1 \geq \cdots \geq \beta^i_N)$. The goal of the hermitian eigenvalue problem is to determine for which sequences $(\beta^1, \dots, \beta^{K+1})$ do there exist $N \times N$ hermitian matrices $H_1, \dots, H_{K+1}$ such that the eigenvalues of H_i are given by the sequence β^i and

$$\sum_{i=1}^{K} H_i = H_{K+1}.$$

The following theorem, proved by Klyachko in [20], gives a remarkable characterization in terms of collection of a inequalities parametrized by non-zero Littlewood-Richardson coefficients.

Theorem 4.1. *Let $(\beta^1, \dots, \beta^{K+1}) \in (\mathbb{R}^N)^{K+1}$ be a collection of sequences of non-increasing real numbers such that*

$$\sum_{i=1}^{K} \sum_{j=1}^{N} \beta^i_j = \sum_{j'=1}^{N} \beta^{K+1}_{j'}.$$

Then the following are equivalent:

(1) *There exist $N \times N$ hermitian matrices $H_1, \dots, H_{K+1}$ with spectra $(\beta^1, \dots, \beta^{K+1})$ such that*

$$\sum_{i=1}^{K} H_i = H_{K+1}.$$

(2) *For every $r < N$, the sequence $(\beta^1, \dots, \beta^{K+1})$ satisfies the inequality*

$$\sum_{i=1}^{K} \sum_{j \in I^j} \beta^i_j \geq \sum_{j' \in I^{K+1}} \beta^{K+1}_{j'} \tag{4.1}$$

for every collection of subsets $I^1, \ldots, I^{K+1}$ of size r of the integers $\{1, 2, \ldots, N\}$ where the Littlewood-Richardson coefficient

$$c(\lambda(I^1), \ldots, \lambda(I^K); \lambda(I^{K+1})) \neq 0.$$

The inequalities given in (4.1) are called Horn's inequalities and were initially defined in a very different way by Horn in [17]. While Horn's list of inequalities in [17] are, a priori, different than Klyachko's list (4.1), they were shown to be equivalent as a consequence of the saturation theorem of Knutson and Tao in [21]. What is amazing about this equivalence is that Horn's initial definition of the inequalities (4.1) uses a recursion unrelated to Littlewood-Richardson coefficients. Horn's recursion in light of Theorem 4.1 can be stated as follows:

Theorem 4.2. *Let $I^1, \ldots, I^{K+1}$ be subsets of size r of the integers $\{1, 2, \ldots, N\}$ such that*

$$\sum_{i=1}^{K} \sum_{j=1}^{r} \lambda(I^i)_j = \sum_{j'=1}^{r} \lambda(I^{K+1})_{j'}. \tag{4.2}$$

The following are equivalent:

1. *The Littlewood-Richardson coefficient*

$$c(\lambda(I^1), \ldots, \lambda(I^K); \lambda(I^{K+1})) \neq 0.$$

2. *There exist $r \times r$ hermitian matrices $H_1, \ldots, H_{K+1}$ with spectra $(\lambda(I^1), \ldots, \lambda(I^{K+1}))$ such that*

$$\sum_{i=1}^{K} H_i = H_{K+1}. \tag{4.3}$$

The recursion says that a collection of subsets $I^1, \ldots, I^{K+1}$ corresponds to a Horn inequality if and only if the corresponding collection of partitions are eigenvalues of some $r \times r$ hermitian matrices which satisfy (4.3). Hence Horn's inequalities can be defined recursively by induction on N. We also remark that equation (4.2) is a necessary condition for the corresponding Littlewood-Richardson coefficient to be nonzero.

We now apply Theorem 4.2 to the case of tight fusion frames. Suppose that $(L_1 \geq L_2 \geq \cdots \geq L_K) \in \mathrm{TFF}(\alpha, N)$ and that $M := \sum_{i=1}^{K} L_i$. Then there exist orthogonal projections $P_1, \ldots, P_K$ such that

$$\sum_{i=1}^{K} NP_i = M\mathbf{I}. \tag{4.4}$$

Since P_i is an orthogonal projection, the spectra of the hermitian matrix NP_i is given by

$$(\underbrace{N, \ldots, N}_{L_i}, \underbrace{0, \ldots, 0}_{N-L_i}).$$

Let (N^{L_i}) denote the corresponding rectangular partition to the spectra above. The following is a direct corollary of Theorem 4.2.

Theorem 4.3. *Fix an integer N and let $(L_1 \geq L_2 \cdots \geq L_K)$ be a sequence of nonnegative integers such that $L_1 \leq N$. Let $M := \sum_{i=1}^{K} L_i$ and $\alpha = M/N$. The following are equivalent:*

(1) The sequence $(L_1 \geq L_2 \geq \cdots \geq L_K) \in \mathrm{TFF}(\alpha, N)$.
(2) The Littlewood-Richardson coefficient

$$c((N^{L_1}), \dots, (N^{L_K}); (M^N)) \neq 0.$$

Proof. Assume part (1). Then there exist orthogonal projections $P_1, \dots, P_K$ with ranks $(L_1, \dots, L_K)$ such that

$$\sum_{i=1}^{K} P_i = \alpha \mathbf{I}. \tag{4.5}$$

Multiplying both sides of equation (4.5) by N gives equation (4.4). Applying Theorem 4.2 gives part (2).

Conversely, if we assume part (2) then by Theorem 4.2, there exists a collection of $N \times N$ matrices which satisfy equation (4.4) and have spectra $(N^{L_1}), \dots, (N^{L_K})$. Scaling by $1/N$ yields the desired tight fusion frame. □

The condition that $c((N^{L_1}), \dots, (N^{L_K}); (M^N)) \neq 0$ can be made computationally explicit by the following existence condition. With the notation of Theorem 4.3 we consider the following properties for an $N \times M$ matrix $A = A[i, j]$.

(i) (integral nonnegativity) $A[i, j] \in \mathbb{Z}_{\geq 0}$

(ii) (row sum) $\displaystyle\sum_{j=1}^{M} A[i, j] = M \quad \forall i$

(iii) (column sum) $\displaystyle\sum_{i=1}^{N} A[i, j] = N \quad \forall j$

(iv) (row sum dominance) $\displaystyle\sum_{j=1}^{l} (A[i, j] - A[i+1, j]) \geq A[i+1, l+1] \quad \forall i, l$

(v) (column sum dominance) $\displaystyle\sum_{i=1}^{l} (A[i, j] - A[i, j+1]) \geq A[l+1, j+1] \quad \forall j, l$

Observe that properties (iv) and (v) require dominance with one additional summand in the later row or column. Also note that (ii) and (iii) are the only properties dependant on the size of the matrix A. Let A be an $N \times M$ matrix and consider the sequence $(L_1, \dots L_K)$. We can partition A into a sequence of column block matrices

$$A = [A_1 | A_2 | \cdots | A_K]$$

where each A_i is the corresponding $N \times L_i$ sub-matrix of A. We now have the following addition to Theorem 4.3.

Corollary 4.4. *Conditions (1) and (2) in Theorem 4.3 are equivalent to the following:*

(3) There exists an $N \times M$ matrix A which satisfies properties (i)-(iv) and whose column block sub-matrices $A_1, \dots, A_K$ each satisfy property (v).

Moreover, the coefficient $c((N^{L_1}), \dots, (N^{L_K}); (M^N))$ equals the number of $N \times M$ matrices A which satisfy (3).

Proof. We refer to [14] for definitions and details of Littlewood-Richardson skew tableaux. Consider the Littlewood-Richardson coefficients $c^{\nu}_{\lambda,\mu}$ corresponding to the product of two Schur functions

$$s_\lambda s_\mu = \sum_\nu c^{\nu}_{\lambda,\,\mu}\; s_\nu.$$

It is well known that the number $c^{\mu}_{\lambda,\,\nu}$ is precisely equal to the number of Littlewood-Richardson skew tableaux ν/λ of content μ. Now suppose there exists a $N \times M$ matrix A which satisfies the conditions of Corollary 4.4 with respect to a sequence $\mathbf{L} = (L_1, \cdots, L_K)$. For any $k \leq K$ let

$$A(k) := [A_1|\cdots|A_k]$$

denote the submatrix of A consisting of the matrices $A_1, \dots, A_k$. By properties (i) and (iv), the row sums of $A(k)$ yield a partition

$$\mu^k := \left(\sum_j A(k)[i,j]\right)_{i=1}^{N} \tag{4.6}$$

given in the standard weakly decreasing form. It is easy to see that μ^k/μ^{k-1} is a well defined skew partition. Consider the Young diagram corresponding to μ^k/μ^{k-1}. We can fill the boxes of the j^{th} row of this diagram with $A_k[j,1]$ 1's, $A_k[j,2]$ 2's, $A_k[j,3]$ 3's and so forth in weakly increasing order. Properties (iv) and (v) imply that the resulting skew tableau is a Littlewood-Richardson skew tableau. Property (iii) implies that content of the tableau is that of the rectangular partition (N^{L_k}). Hence the existence of the matrix $A(k)$ implies that the Littlewood-Richardson coefficient

$$c^{\mu^k}_{\mu^{k-1},\,(N^{L_k})} \neq 0.$$

Finally, properties (ii) and (iii) imply that $\mu^K = (M^N)$. By induction on k, multiplying the Schur functions $s_{(N^{L_1})}, \dots s_{(N^{L_K})}$ gives that

$$c((N^{L_1}), \dots, (N^{L_K}); (M^N)) \neq 0.$$

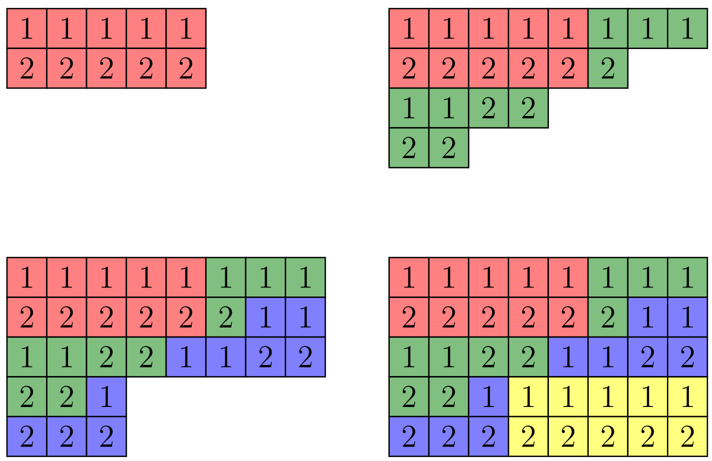
It is easy to see that this argument can be reversed. This bijection together with Littlewood-Richardson rule for counting $c^{\mu}_{\lambda,\,\nu}$ implies that second part of Corollary 4.4. This completes the proof. □

Example 4.5. We consider two examples where tight fusion frames exist for $N = 5$ and $M = 8$.

First, consider the sequence $\mathbf{L} = (2,2,2,2)$. The following matrix

$$A = \left(\begin{array}{cc|cc|cc|cc} 5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 5 \end{array}\right)$$

satisfies the conditions in Corollary 4.4. We write out the corresponding Young tableaux to the partitions μ^1, μ^2, μ^3 and μ^4 with content given by the sub-matrices $A(1)$, $A(2)$, $A(3)$, $A(4)$:

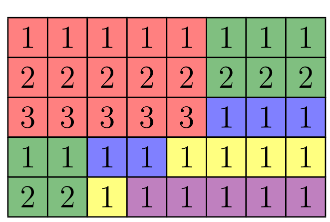


Note that the all the data can be encoded in the final partition μ^4 as a union of skew Littlewood-Richardson tableaux.

For the second example, we consider $\mathbf{L} = (3,2,1,1,1)$ and the matrix

$$A = \left(\begin{array}{ccc|cc|c|c|c} 5 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 5 \end{array}\right)$$

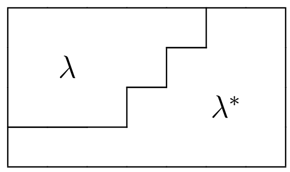
The corresponding union of Littlewood-Richardson tableaux is given by



5. Combinatorial majorization and hook type sequences

In this section we give alternate proofs of Theorem 2.3 on majorization and Theorem 3.3 on estimates using the combinatorics of Schur functions and Theorem 4.3. We begin with some fundamental definitions and lemmas on Schur functions. Let $\lambda \subseteq (M^N)$. We define the dual partition of λ in (M^N) to be the partition

$$\lambda^* := (M - \lambda_N \geq M - \lambda_{N-1} \geq \cdots \geq M - \lambda_1).$$



Lemma 5.1. *Let $\lambda \subseteq (M^N)$ and let $p(\lambda)$ denote the number of parts of λ equal to M. Assume that for some positive integer k we have that*

$$|\lambda| = N(M-k).$$

Then

$$c(\lambda, \underbrace{(N), \ldots, (N)}_{k}; (M^N)) \neq 0$$

if and only if $k \geq N - p(\lambda)$.

Proof. The lemma follows from two elementary facts about Schur functions. Consider the product

$$(s_{(N)})^k = \sum_{\mu} c((N), \ldots, (N); \mu)\ s_\mu$$

By the Pieri rule, we have that $c((N), \ldots, (N); \mu) \neq 0$ if and only if μ has length less than or equal to k and $|\mu| = Nk$. Furthermore, if $\lambda, \mu \subseteq (M^N)$, then $c_{\lambda,\mu}^{(M^N)} \neq 0$ if and only if $\mu = \lambda^*$. It is easy to check that λ^* appears as a summand in the product $(s_{(N)})^k$ precisely when $k \geq N - p(\lambda)$. □

The following theorem on the product of Schur functions corresponding to rectangular partitions is proved by Okada in [28, Theorem 2.4].

Theorem 5.2. *Fix integers a, b, N_1, N_2 with $a \geq b$. The product of Schur functions*

$$s_{(N_1^a)} s_{(N_2^b)} = \sum_{\lambda} s_\lambda, \tag{5.1}$$

where the sum is over all partitions λ with length $\leq a+b$ such that

- $\lambda_{b+1} = \lambda_{b+2} = \cdots = \lambda_a = N_1$.
- $\lambda_b \geq \max\{N_1, N_2\}$.
- $\lambda_i + \lambda_{a+b+1-i} = N_1 + N_2 \qquad \forall i \in \{1, \ldots, b\}$

We now give an alternate proof of Theorem 2.3 using Theorem 5.2 in the case when $N_1 = N_2$.

Lemma 5.3. *Fix a positive integer N and let $0 < a < b$. Then the Littlewood-Richardson coefficients*

$$c^{\lambda}_{(N^b),\,(N^a)} \leq c^{\lambda}_{(N^{b-1}),\,(N^{a+1})}.$$

In particular, Theorem 2.3 on majorization of tight fusion frames follows.

Proof. It is easy to check the λ that appear in the summation (5.1) for the pair $((N^b),(N^a))$ are contained in the λ that appear in the summation (5.1) for the pair $((N^{b-1}),(N^{a+1}))$. This proves the inequality. The application to tight fusion frames follows from Theorem 4.3. □

It is easy to see that by majorization, the following theorem is equivalent to Theorem 3.3 on estimates.

Theorem 5.4. *Assume the conditions in Theorem 4.3. Further assume that $\alpha = M/N < 2$. If $(L_1 \geq L_2 \geq \cdots \geq L_K) \in \mathrm{TFF}(\alpha, N)$, then we have the following necessary conditions.*

(1) $L_1 \leq M - N$.
(2) $L_1 + L_2 \leq N$
(3) If $\alpha > 3/2$, *then* $L_1 + L_2 + L_3 \leq 2(M - N)$.
(4) If $\alpha < 3/2$, *then* $L_1 + L_2 + L_3 \leq N$

Conversely, suppose L_1, L_2, L_3 satisfy the above conditions and $L_4 = \cdots = L_K = 1$. Then $(L_1 \geq L_2 \geq \cdots \geq L_K) \in \mathrm{TFF}(\alpha, N)$.

Proof. Recall that for any partition $\lambda \subseteq (M^N)$, we let $p(\lambda)$ denote the number of parts of λ equal to M. First we prove part (1). By majorization, it suffices to assume that $L_2 = 1$. Part (1) now follows from Lemma 5.1 by setting $\lambda = (N^{L_1})$ and observing that $p((N^{L_1})) = 0$.

We now prove part (2). By majorization, it suffices to assume that $L_3 = 1$. Consider the product

$$s_{(N^{L_1})} s_{(N^{L_2})} = \sum_{\lambda} s_{\lambda}. \tag{5.2}$$

By Theorem 5.2, we have that $\lambda_1 + \lambda_{L_1+L_2} = 2N$ for every λ in the sum (5.2). If $\lambda \subseteq (M^N)$, then $\lambda_1 \leq M$. Hence

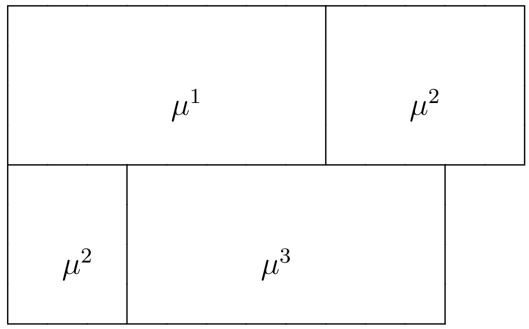
$$\lambda_{L_1+L_2} = 2N - \lambda_1 \geq 2N - M > 0$$

since $\alpha < 2$. This implies that $L_1 + L_2 \leq N$ since (M^N) has only N parts.

For part (3), we assume that $L_4 = 1$. First, if $L_2 + L_3 \leq L_1$, then by part (1), $L_1 + L_2 + L_3 \leq 2(M - N)$. Next, we assume $L_1 \leq L_2 + L_3$. Consider the product

$$s_{(N^{L_1})} s_{(N^{L_2})} s_{(N^{L_3})} = \sum_{\lambda} c((N^{L_1}), (N^{L_2}), (N^{L_3}); \lambda)\ s_{\lambda} \tag{5.3}$$

Since $\alpha > 3/2$, for any $\lambda \subseteq (M^N)$ such that $c((N^{L_1}), (N^{L_2}), (N^{L_3}); \lambda) \neq 0$, we have that $p(\lambda) \leq L_1$. This can be seen by considering L_2 and L_3 as large as possible, hence $L_1 = L_2 = L_3$. One can show using the Littlewood-Richardson rule that since $3N < 2M$, 3 layered bricks of width N cannot span M more than once, see diagram below.



By Lemma 5.1,

$$M - L_1 - L_2 - L_3 \geq N - p(\lambda) \geq N - L_1.$$

Hence $L_2 + L_3 \leq M - N$. This proves part (3).

For part (4), fix any λ in the summand found in equation (5.2) such that $\lambda \subseteq (M^N)$. Since $\alpha < \frac{3}{2}$, we have that

$$\lambda_{L_1+L_2} = 2N - \lambda_1 \geq 2N - M > M - N.$$

Hence the rectangular partition $((M-N+1)^{L_1+L_2}) \subseteq \lambda$. Comparing the two products

$$s_\lambda s_{(N^{L_3})} = \sum_{\mu'} c^{\mu'}_{\lambda,(N^{L_3})} \, s_{\mu'} \tag{5.4}$$

and

$$s_{((M-N+1)^{L_1+L_2})} s_{(N^{L_3})} = \sum_{\mu} s_\mu \tag{5.5}$$

we have that any partition μ' from equation (5.4) such that $c^{\mu'}_{\lambda,(N^{L_3})} \neq 0$ contains some μ from equation (5.5). Therefore it is enough to consider the partitions μ from (5.5). By Theorem 5.2, we get that

$$\mu_1 + \mu_{L_1+L_2+L_3} = M - N + 1 + N = M + 1$$

for every μ in the sum (5.5). Hence if $\mu \subseteq (M^N)$, then $\mu_{L_1+L_2+L_3} > 0$ since $\mu_1 \leq M$. Thus $L_1 + L_2 + L_3 \leq N$. This proves part (4).

To prove sufficiency, we construct λ in the sum (5.3) such that $\lambda \subseteq (M^N)$ and $c((N^{L_1}), (N^{L_2}), (N^{L_3}); \lambda) \neq 0$. One can show using the Littlewood-Richardson rule that parts (1)–(4) imply that such a λ exists. Furthermore, we can construct λ such that $p(\lambda) = L_1$ if $L_1 \leq L_2 + L_3$ or $p(\lambda) = L_2 + L_3$ if $L_2 + L_3 \leq L_1$, see Figures 3 and 4. In either case, Lemma 5.1 implies that we only need to check that $L_1 + L_2 + L_3 \leq 2(M - N)$. However, this is already a necessary condition. This completes the proof of the theorem. □

Remark 5.5. Parts (2) and (4) of Theorem 5.4 can be generalized to the following statement.

Let $2 \leq k \leq K$. If $\alpha < \dfrac{k}{k-1}$, then $L_1 + \cdots + L_k \leq N$.

The proof follows the same argument as the proof of Theorem 5.4 part (4).

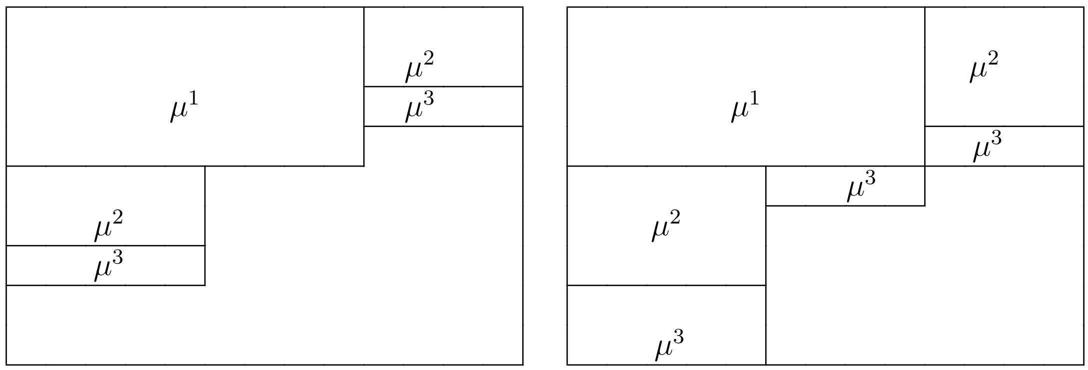


Figure 3. Construction of λ for $\alpha < 3/2$ as a union of Littlewood-Richardson skew tableaux μ^1, μ^2, μ^3 when $L_2 + L_3 \leq L_1$ and $L_1 \leq L_2 + L_3$, resp. This construction is possible since $L_1 + L_2 + L_3 \leq N$.

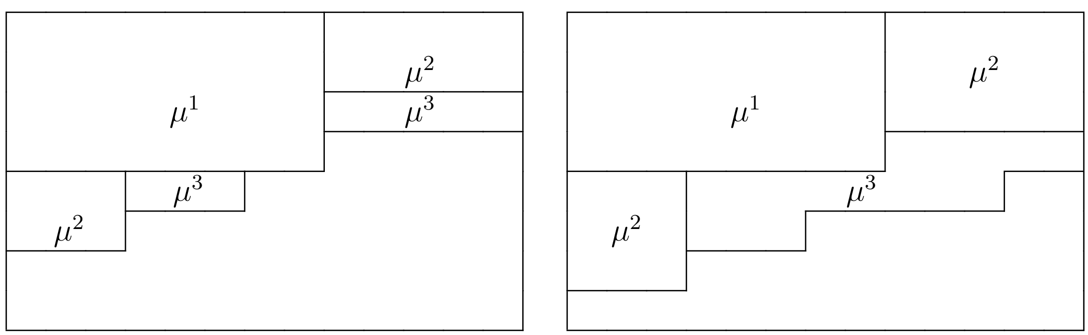


Figure 4. Construction of λ for $\alpha > 3/2$ as a union of Littlewood-Richardson skew tableaux μ^1, μ^2, μ^3 when $L_2 + L_3 \leq L_1$ and $L_1 \leq L_2 + L_3$, resp.

6. Combinatorial spatial and Naimark duality

Theorems 2.5 and 2.6 establish spatial and Naimark dualities for tight fusion frames. By Theorem 4.3, we have the analogous results for Littlewood-Richardson coefficients.

Corollary 6.1. *Assume we have a sequence of integers* $(L_1 \geq \cdots \geq L_K)$ *as in Theorem 4.3. Then*

$$c((N^{L_1}),\dots,(N^{L_K});(M^N)) \neq 0 \Leftrightarrow c((N^{N-L_1}),\dots,(N^{N-L_K});((KN-M)^N)) \neq 0$$

and

$$c((N^{L_1}),\dots,(N^{L_K});(M^N)) \neq 0 \Leftrightarrow c(((M-N)^{L_1}),\dots,((M-N)^{L_K});(M^{(M-N)})) \neq 0.$$

In this section we prove a much stronger version of the corollary above. In particular, we prove that these Littlewood-Richardson coefficients are equal. We will frequently reference properties $(i)-(v)$ for matrices defined in the paragraph preceding Corollary 4.4 using lower case roman numerals. We first consider spatial duality.

Theorem 6.2. *The Littlewood-Richardson coefficients*

$$c((N^{L_1}),\dots,(N^{L_K});(M^N)) = c((N^{N-L_1}),\dots,(N^{N-L_K});((KN-M)^N)). \tag{6.1}$$

The coefficient $c((N^{L_1}),\dots,(N^{L_K});(M^N))$ is precisely the number of $N\times M$ matrices A which satisfy the conditions given in the Corollary 4.4. We will call such a collection of matrices the set of configuration matrices corresponding to $(L_1,\dots,L_K;N)$. We prove Theorem 6.2 by providing a bijection between the configuration matrices corresponding to the coefficients in (6.1).

Suppose that $c((N^{L_1}),\dots,(N^{L_K});(M^N))\neq 0$ and fix a configuration matrix $A=[A_1|A_2|\cdots|A_K]$. For each A_i, we construct a $N\times(N-L_i)$ matrix B_i as follows. Decompose

$$A_i=\sum_{j=1}^{N} C_j$$

as a sum of binary matrices which satisfy the following conditions for all integers y,j

(1) $\displaystyle\sum_{x=1}^{N} C_j[x,y]=1$

(2) $\displaystyle\sum_{x=1}^{N'} (C_j[x,y]-C_j[x,y+1])\geq 0 \quad \forall\ N'<N$

(3) $\displaystyle\sum_{x=1}^{N'} (C_j[x,y]-C_{j+1}[x,y])\geq 0 \quad \forall\ N'<N.$

Consider A_2 from Example 4.5. We have that

$$\begin{pmatrix}3&0\\0&1\\2&2\\0&2\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\\1&0\\0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\\1&0\\0&1\\0&0\end{pmatrix}.$$

It is easy to see that this decomposition of A_i is unique since A_i satisfies properties $(i),(iii)$ and (v). For each C_j, define the $N\times(N-L_i)$ matrix C'_j to be the unique binary matrix which satisfies conditions $(1),(2)$ and that $[C_j|C'_j]$ is invertible. For example, if $N=5$ then

$$C_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \rightsquigarrow \quad C'_j = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define

$$B_i := \sum_{i=1}^{N} C'_j$$

and consider the $N \times (KN - M)$ matrix

$$B := [B_K|B_{K-1}|\cdots|B_1].$$

Note that the binary decomposition of B_i into C'_j also satisfies conditions $(1)-(3)$ if we order the C'_j in reverse. Moreover, if we apply this algorithm to the matrix B, we will recover the matrix A. We now record some important observations on the submatrices A_i and B_i. First, if $x < y$, then

$$A_i[x, y] = B_i[x, y] = 0. \tag{6.2}$$

Second, we have that

$$A_i[x, y] + B_i[x, x - y] = A_i[x + 1, y + 1] + B_i[x + 1, x - y + 1]. \tag{6.3}$$

In the equations above we take $A_i[x, y] = 0$ (resp. $B_i[x, y] = 0$) if x, y lie outside the boundaries of A_i (resp. B_i). In the case when $x = y$, we get

$$A_i[x, x] = A_i[x + 1, x + 1] + B_i[x + 1, 1]. \tag{6.4}$$

Theorem 6.2 follows from the proceeding proposition.

Proposition 6.3. *The matrix $N \times (KN - M)$ matrix B is a configuration matrix for the sequence $(N - L_K, \ldots, N - L_1; KN - M)$.*

Proof. The most challenging part of this proof is to show that the matrix B satisfies property (iv). Hence the majority of this argument is dedicated to the proof this property. We first consider the other properties. Properties $(i)-(iii)$ are immediate by construction of B. Property (v) follows form the fact that each B_i is a sum of binary matrices which satisfy conditions $(1) - (3)$. We now prove that B satisfies property (iv) by contradiction. Suppose there exists integers i, l such that

$$\sum_{j=1}^{l} (B[i, j] - B[i + 1, j]) < B[i + 1, l + 1]. \tag{6.5}$$

We define the integers k, l' as follows. Let k denote largest integer for which the partial sum

$$l' := \sum_{j=1}^{k} (N - L_{K-j+1}) \leq l.$$

Hence l' is the number of columns of the submatrix $[B_K|\cdots|B_{K-k+1}]$ of B.

Observe that each row sum of the matrix $[A_j|B_j]$ is equal to N. Combining this observation with equation (6.5) gives that

$$\begin{aligned}\sum_{j=1}^{l}(B[i,j]-B[i+1,j]) &= \sum_{j=1}^{l'}(B[i,j]-B[i+1,j])+\sum_{j=l'+1}^{l}(B[i,j]-B[i+1,j]) \\ &= \sum_{j=M-(kN-l'-1)}^{M}(A[i+1,j]-A[i,j]) \\ &\qquad +\sum_{j=l'+1}^{l}(B[i,j]-B[i+1,j])<B[i+1,l+1].\end{aligned}$$

Rewriting this inequality yields

$$\begin{aligned}\sum_{j=M-kN+l'+1}^{M}(A[i+1,j]-A[i,j]) &< B[i+1,l+1]-\sum_{j=l'+1}^{l}(B[i,j]-B[i+1,j]) \\ &= B[i+1,l'+1]+\sum_{j=l'+1}^{l}(B[i+1,j+1]-B[i,j]).\end{aligned}$$

The matrix entries of B appearing on the right hand side of the above equation are all contained in the submatrix B_{K-k}. Applying equations (6.3),(6.4), we get that

$$\sum_{j=M-kN+l'+1}^{M}(A[i+1,j]-A[i,j])<\sum_{j=0}^{l-l'}(A_{K-k}[i,i-j]-A_{K-k}[i+1,i-j+1]). \tag{6.6}$$

By equation (6.2), $A_{K-k}[x,y]=0$ if $y>x$. Hence we can extend the right hand side of equation (6.6) to

$$\begin{aligned}\sum_{j=M-kN+l'+1}^{M}(A[i+1,j]-A[i,j]) &< A_{K-k}[i,i-l+l'] \\ &+\sum_{j=0}^{L_{K-k}-(i+1)+(l-l')}(A_{K-k}[i,L_{K-k}-j]-A_{K-k}[i+1,L_{K-k}-j]).\end{aligned}$$

Now the fact that A satisfies properties (ii), contradicts the fact that it also satisfies property (iv). This completes the proof. □

Example 6.4. Let $N=4$ and consider the sequence $\mathbf{L}=(2,2,2,1)$. By Corollary 4.4, the matrix A below implies that $\mathbf{L}\in \mathrm{TFF}(7/4,4)$.

$$A=\left(\begin{array}{cc|cc|cc|c} 4 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 4 \end{array}\right)$$

We get that

$$B = \left(\begin{array}{ccc|cc|cc|cc} 4 & 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 4 \end{array}\right)$$

and hence $(3, 2, 2, 2) \in \mathrm{TFF}(9/4, 4)$.

We now give the analogous theorem on combinatorial Naimark duality.

Theorem 6.5. *The Littlewood-Richardson coefficients*

$$c((N^{L_1}), \ldots, (N^{L_K}); (M^N)) = c(((M-N)^{L_1}), \ldots, ((M-N)^{L_K}); (M^{(M-N)})). \tag{6.7}$$

As with Theorem 6.2, we define a bijection between configuration matrices corresponding to the Littlewood-Richardson coefficients in (6.7). Fix a configuration matrix A corresponding to the sequence $(L_1, \ldots, L_K; N)$ and consider the Littlewood-Richardson skew tableaux μ^k/μ^{k-1} where μ^k is defined in equation (4.6). To each μ^k/μ^{k-1} we define the $L_k \times M$ binary matrix T_k by

$$T_k[x, y] := \begin{cases} 1 & \text{if } x \text{ appears in column } y \text{ of } \mu^k/\mu^{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

The partition shape of μ^k can be recovered from the matrices $T_1, \ldots T_K$ as follows. Define the matrix $T(k)$ by "stacking" the matrices $T_1, \ldots, T_k$ (see Example 6.7 below). In other words,

$$T(k) := \left(\begin{array}{c} T_1 \\ \hline \vdots \\ \hline T_k \end{array}\right).$$

Since A satisfies property (iv), the partition μ^k can be recovered by upward justifying the nonzero entries of $T(k)$. In particular, the entire collection $T_1, \ldots, T_K$ uniquely determines the matrix A.

We now define the "complementary" $M \times L_k$ matrix S_k by

$$S_k[x, y] := 1 - T_k[x, M - y + 1]$$

and $S(k)$ as the corresponding column matrix with block entries $S_1, \ldots, S_k$. It is easy that if the nonzero entries of $S(k)$ are justified upwards, we get the dual partition $(\mu^k)^*$ in rectangle (M^{M_k}) where $M_k := \sum_{i=1}^k L_k$. Hence $S_1, \ldots, S_K$ determines some matrix B in the same way that $T_1, \ldots, T_K$ determines A. Also note that we can recover T_k from S_k by applying the complementary operation to S_k. Theorem 6.5 follows from the proceeding proposition.

Proposition 6.6. *The collection $S_1, \ldots, S_K$ determines a configuration matrix for the sequence $(L_1, \ldots, L_K; M - N)$.*

Proof. Let $B = [B_1 | \cdots | B_K]$ denote the matrix corresponding to the collection S_1, …, S_K. We will show that B is a configuration matrix for the sequence $(L_1, \ldots, L_K; M-$

N). In this case, property (v) is the most challenging to prove. Hence most the argument to dedicated to this part of the proof.

First, note that B trivially satisfies properties (i) and (iv). Next, we observe that A satisfies properties (ii) and (iii) if and only if the matrix $T(K)$ has M columns where each column sum is equal to N. Since $S(K)$ has the same number of columns as $T(K)$ with column sums of $M-N$, we get that B also satisfies properties (ii) and (iii).

We now prove that B satisfies property (v) by contradiction. Suppose there exists B_k and integers j, l such that

$$\sum_{i=1}^{l}(B_k[i,j]-B_k[i,j+1]) < B_k[l+1,j+1].$$

This implies there exists an integer l' such that

$$\sum_{i=l'+1}^{M}(S_k[j,i]-S_k[j+1,i]) < 0 \tag{6.8}$$

with

$$S_k[j,l'+1]=0 \quad \text{and} \quad S_k[j+1,l'+1]=1. \tag{6.9}$$

Conversely, assume there exists an integer l' such that equations (6.8) and (6.9) are true. By equation (6.9), there exists an integer l'' such that

$$\sum_{i=l'+1}^{M} S_k[j,i]=\sum_{i=1}^{l''} B_k[i,j] \quad \text{and} \quad \sum_{i=l'+1}^{M} S_k[j+1,i] \le \sum_{i=1}^{l''+1} B_k[i,j+1].$$

Hence by equation (6.8),

$$-B_k[l''+1,j+1]+\sum_{i=1}^{l''}(B_k[i,j]-B_k[i,j+1]) \le \sum_{i=l'+1}^{M}(S_k[j,i]-S_k[j+1,i]) < 0.$$

Observe that if (6.8) is true for l, then there is always some integer $l' \le l$ for which both (6.8) and (6.9) are true. Thus the failure of property (v) is equivalent to equation (6.8). By definition of S_k and equation (6.8), we have that

$$\sum_{i=1}^{M-l'}(T_k[j+1,i]-T_k[j,i]) < 0.$$

Since the row sums of T_k equal N,

$$\sum_{i=M-l'+1}^{M}(T_k[j,i]-T_k[j+1,i]) < 0.$$

Therefore the matrix A also fails to satisfy property (v) which is a contradiction. This completes the proof. $\square$

Example 6.7. Consider $N = 4$ and $\mathbf{L} = (2, 2, 2, 1)$ as in Example 6.4. Then μ^4, as a union of Littlewood-Richardson skew tableaux, is equal to

1	1	1	1	1	1	1
2	2	2	2	2	1	1
1	2	2	1	1	2	2
2	2	2	1	1	1	1

We have that

$$T(4) = \left(\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \quad \rightsquigarrow \quad S(4) = \left(\begin{array}{ccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Upward justifying the nonzero entries of $S(4)$ gives the union of Littlewood-Richardson skew tableaux

1	1	1	1	1	1	1
2	2	2	2	1	1	2
2	2	2	2	1	1	1

The corresponding configuration matrix is

$$B = \left(\begin{array}{cc|cc|cc|c} 3 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 3 \end{array}\right).$$

and hence $(2, 2, 2, 1) \in \mathrm{TFF}(7/3, 3)$.

7. Examples and tables of TFF sequences

This section is divided into two parts. In the first part we give several examples of existence of tight fusion frames using skew Littlewood-Richardson tableaux as in Example 4.5. In the second part, we give a complete list of tight fusion frame sequences for $N \leq 9$ and $\alpha \leq 2$ by listing all maximal elements in the partial order induced by majorization.

7.1. Examples of skew Littlewood-Richardson tableaux.

The following are some examples of Littlewood-Richardson tableaux in the cases of $N = 3, 5, 7$. Readers who are interested in combinatorial spatial and Naimark duality as discussed in Section 6 are encouraged to apply the bijective constructions to these examples.

$N = 3$ and $\mathbf{L} = (3, 2, 1)$, $\mathbf{L} = (2, 1, 1, 1)$, and $\mathbf{L} = (1, 1, 1, 1)$,

1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	1	1	1

1	1	1	1	1
2	2	2	1	1
1	1	1	1	1

1	1	1	1
1	1	1	1
1	1	1	1

$N = 5$ and $\mathbf{L} = (2, 2, 2, 2)$ and $\mathbf{L} = (3, 3, 3, 3)$

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	1	2	2	2
3	3	3	3	3	3	1	1	1	1	1	1
2	2	3	3	1	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3

$N = 7$ and $\mathbf{L} = (4, 3, 3, 1, 1)$ and $\mathbf{L} = (3, 2, 2, 2, 1)$

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	1	1	1
4	4	4	4	4	4	4	1	1	2	2	2
1	1	3	3	3	1	1	2	2	3	3	3
2	2	2	2	3	3	1	1	1	1	1	1
3	3	3	3	1	1	1	1	1	1	1	1

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	1	1	1
1	1	1	1	1	1	1	2	2	2
2	2	2	2	2	1	1	1	1	1
1	2	2	1	1	2	2	2	2	2
2	2	2	1	1	1	1	1	1	1

7.2. **Tables of maximal tight fusion frames.** At the end of this section we give a complete list of tight fusion frames for $N \leq 9$ and $\alpha \leq 2$ by listing all maximal elements in the partial order induced by majorization. These lists are generated by applying the techniques developed in this paper. In particular, we use the following methods

- Constructing Littlewood-Richardson tableaux as in Corollary 4.4.
- Recursive construction using spatial and Naimark duality.
- Recursive construction using Lemma 7.1.
- Applying inequalities of Theorem 3.3/5.4.

The following lemma follows from Naimark duality.

Lemma 7.1. *Assume that* $L_1 = N(\alpha - 1)$*. Then,* $\mathbf{L} \in \mathrm{TFF}(\alpha, N)$ *if and only if* $\mathbf{L}' \in \mathrm{TFF}(\tilde{\alpha}, N(\alpha - 1))$ *where* $\mathbf{L}' = (L_2 \geq \cdots \geq L_k)$ *and* $1/\alpha + 1/\alpha' = 1$*.*

It is easy to see that maximality under the majorization partial order is preserved under these dualities and the lemma above. Unfortunately, there are several TFF sequences missed by majorization and the recursive generation techniques mentioned above. These sequences were only found by brute force construction of Littlewood-Richardson tableaux. The first maximal tight fusion frame sequence missed by recursion is $(4, 2, 2, 2, 1)$ where $N = 6$. Hence, it might be of interest to illustrate how to construct a tight fusion frame for this sequence.

Example 7.2. Let $N = 6$ and $\mathbf{L} = (4, 2, 2, 2, 1)$. The first step in our construction is identifying a skew Littlewood-Richardson tableaux corresponding to our TFF sequence.

1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	1	1
3	3	3	3	3	3	1	1	1	2	2
4	4	4	4	4	4	1	1	1	1	1
1	2	2	1	2	1	2	2	2	2	2
2	2	2	2	2	1	1	1	1	1	1

The above tableaux shows the existence of projections $P_1, \ldots, P_5$ in $\mathbb{R}^6$ with

$$\sum_{i=1}^{5} P_i = \frac{11}{6}\mathbf{I}, \quad \operatorname{rank} P_1 = 4,\ \operatorname{rank} P_2 = \operatorname{rank} P_3 = \operatorname{rank} P_4 = 2,\ \operatorname{rank} P_5 = 1. \tag{7.1}$$

By Theorem 4.2 and Corollary 4.4, the tableaux also contains information on the eigenvalues of the intermediate partial sums of projections in (7.1).

sum of projections	eigenvalue list
P_1	$(1,1,1,1,0,0)$
$P_1 + P_2$	$(\frac{11}{6}, \frac{9}{6}, 1, 1, \frac{3}{6}, \frac{1}{6})$
$P_1 + P_2 + P_3$	$(\frac{11}{6}, \frac{11}{6}, \frac{11}{6}, 1, \frac{5}{6}, \frac{4}{6})$
$P_1 + \ldots + P_4$	$(\frac{11}{6}, \frac{11}{6}, \frac{11}{6}, \frac{11}{6}, \frac{11}{6}, \frac{5}{6})$
$P_1 + \ldots + P_5$	$(\frac{11}{6}, \frac{11}{6}, \frac{11}{6}, \frac{11}{6}, \frac{11}{6}, \frac{11}{6})$

Equipped with this information and a symbolic computation program such as Mathematica we can construct an explicit tight fusion frame in $\mathbb{R}^6$ associated with the sequence $(4, 2, 2, 2, 1)$. The matrix below shows an orthonormal basis (column) vectors for the corresponding ranges of projections P_i, $i = 1, \ldots, 5$.

$$\left(\begin{array}{cccc|cc|cc|cc|c}
1 & 0 & 0 & 0 & \frac{5}{6} & 0 & -\sqrt{\frac{5}{72}} & 0 & \sqrt{\frac{5}{72}} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{3} & -\frac{1}{2\sqrt{2}} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{3} & 0 & \frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{6} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{12}} & -\sqrt{\frac{5}{12}} \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & 0 & -\frac{\sqrt{11}}{6} & 0 & -\sqrt{\frac{55}{72}} & 0 & \sqrt{\frac{55}{72}} & 0 & 0
\end{array}\right)$$

A direct calculation shows that: (i) columns are orthonormal to each other in every block, and (ii) rows are orthogonal with norms $\sqrt{11/6}$. This proves the existence of a TFF (7.1).

It is worth noting that the Example 7.2 can not be obtained using the spectral tetris construction (STC). The STC has been recently introduced by Casazza et al. [6] who gave an algorithmic way of constructing sparse fusion frames. Among other things, the authors of [6] have shown that the ranks $\mathbf{L}$ of spectral tetris fusion frames must necessarily satisfy $\mathbf{L} \preccurlyeq \mathbf{L}'$, where $\mathbf{L}'$ is a sequence of ranks of the reference fusion frame. Moreover, in the tight case this condition is also sufficient, and hence [6, Theorem 3.3] characterizes possible ranks obtained by the STC in the case when the frame bound $\alpha \geq 2$. Combining this with Naimark complements, see Theorem 2.6, this yields TFFs also in the case $1 < \alpha < 2$. In particular, we have $\mathrm{TFF}(11/6, 6) = \mathrm{TFF}(11/5, 5)$. A direct calculation of the reference fusion frame corresponding to eigenvalues $(11/5, 11/5, 11/5, 11/5, 11/5)$ yields a TFF sequence $(3, 3, 3, 2)$. This happens to be another maximal element of $\mathrm{TFF}(11/6, 6)$ which is not comparable with $(4, 2, 2, 2, 1)$ with respect to the majorization relation $\preccurlyeq$. Hence, the above example can not be obtained by the STC even when paired with Naimark duality.

List of maximal TFF sequences for $N \leq 9$ and $\alpha \leq 2$.

$N = 3$	
α	max elements
1	(3)
4/3	(1, 1, 1, 1)
5/3	(2, 1, 1, 1)
2	(3, 3)

$N = 4$	
α	max elements
1	(4)
5/4	(1, 1, 1, 1, 1)
6/4	(2, 2, 2)
7/4	(3, 1, 1, 1, 1), (2, 2, 2, 1)
2	(4, 4)

$N = 5$	
α	max elements
1	(5)
6/5	(1, 1, 1, 1, 1, 1)
7/5	(2, 2, 1, 1, 1)
8/5	(3, 2, 1, 1, 1), (2, 2, 2, 2)
9/5	(4, 1, 1, 1, 1, 1), (3, 2, 2, 2)
2	(5, 5)

$N = 6$	
α	max elements
1	(6)
7/6	(1, 1, 1, 1, 1, 1, 1)
8/6	(2, 2, 2, 2)
9/6	(3, 3, 3)
10/6	(4, 2, 2, 2)
11/6	(5, 1, 1, 1, 1, 1, 1), (4, 2, 2, 2, 1), (3, 3, 3, 2)
2	(6, 6)

$N = 7$	
α	max elements
1	(7)
8/7	(1, 1, 1, 1, 1, 1, 1, 1)
9/7	(2, 2, 2, 1, 1, 1)
10/7	(3, 3, 1, 1, 1, 1), (3, 2, 2, 2, 1)
11/7	(4, 3, 1, 1, 1, 1), (4, 2, 2, 2, 1)
12/7	(5, 2, 2, 1, 1, 1), (4, 3, 3, 1, 1), (3, 3, 3, 3)
13/7	(6, 1, 1, 1, 1, 1, 1, 1), (5, 2, 2, 2, 2), (4, 3, 3, 3)
2	(7, 7)

$N=8$	
α	max elements
1	(8)
9/8	(1, 1, 1, 1, 1, 1, 1, 1, 1)
10/8	(2, 2, 2, 2, 2)
11/8	(3, 2, 2, 2, 2), (3, 3, 2, 1, 1, 1)
12/8	(4, 4, 4)
13/8	(5, 3, 2, 1, 1, 1), (5, 2, 2, 2, 2), (4, 4, 2, 2, 1)
14/8	(6, 2, 2, 2, 2), (5, 3, 3, 2, 1), (4, 4, 4, 2)
15/8	(7, 1, 1, 1, 1, 1, 1, 1), (6, 2, 2, 2, 2, 1), (5, 3, 3, 2, 2), (4, 4, 4, 3)
2	(8, 8)

$N=9$	
α	max elements
1	(9)
10/9	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
11/9	(2, 2, 2, 2, 1, 1, 1)
12/9	(3, 3, 3, 3)
13/9	(4, 4, 1, 1, 1, 1, 1), (4, 3, 2, 2, 2), (3, 3, 3, 3, 1)
14/9	(5, 4, 1, 1, 1, 1, 1), (5, 3, 2, 2, 2), (4, 3, 3, 3, 1)
15/9	(6, 3, 3, 3)
16/9	(7, 2, 2, 2, 1, 1, 1), (6, 3, 3, 3, 1), (5, 4, 4, 2, 1), (4, 4, 4, 4)
17/9	(8, 1, 1, 1, 1, 1, 1, 1, 1, 1), (7, 2, 2, 2, 2, 2), (6, 3, 3, 3, 2), (5, 4, 4, 4)
2	(9, 9)

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