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Riesz wavelets and generalized multiresolution analyses

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Abstract

We investigate Riesz wavelets in the context of generalized multiresolution analysis (GMRA). In particular, we show that Zalik's class of Riesz wavelets obtained by an MRA is the same as the class of biorthogonal wavelets associated with an MRA.

Keywords: Riesz wavelet; Multiresolution analysis; Generalized multiresolution analysis; Shift invariant space; Dimension function

1. Introduction

The goal of this note is, among other things, to clarify what it means for a Riesz wavelet to be generated by an MRA. There are at least two ways in which we can say that a Riesz wavelet is associated with an MRA.

Probably the most natural definition is the following which has appeared in the number of papers [1,13,15,18]. We say that a Riesz wavelet $\psi \in L^2(\mathbb{R})$ (with respect to dilation factor 2) is *associated with an MRA* if $(V_j)_{j\in\mathbb{Z}}$ given by

$$V_j = \sum_{i<j} W_i, \quad \text{where } W_i = \overline{\text{span}}\{2^{i/2}\psi(2^i x - k)\colon k \in \mathbb{Z}\}$$

is an MRA. Here,

$$\sum_{i<j} W_i := \left\{ f \in L^2(\mathbb{R})\colon f = \sum_{i<j} w_j,\ w_i \in W_i \right\},$$

where the above series converges unconditionally.

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On the other hand, Zalik proposed the following definition. We say that a Riesz wavelet ψ is *obtained by an MRA* if there is an MRA $(V_j)_{j\in\mathbb{Z}}$ such that $\psi \in V_1$.

There are several results in the subject. Kim et al. [13] in one dimension and Larson et al. [15] in higher dimensions have shown that a Riesz wavelet is a biorthogonal wavelet if and only if it is associated with a GMRA. Kim et al. [13] have also given a characterization of Riesz wavelets which are associated with an MRA in terms of certain dimension function. They have also shown that Riesz wavelets associated with an MRA always have dual (biorthogonal) Riesz wavelets (also associated with an MRA). This result is a refinement of Wang's characterization of biorthogonal wavelets associated with an MRA [18]. On the other hand, Zalik [19] has initiated investigation of the class of Riesz wavelets obtained by an MRA by giving a characterization of this class.

The natural question concerns the relation between these different notions of Riesz wavelets. It is clear that any wavelet which is associated with an MRA is also obtained by the same MRA. However, the converse turns out to be quite a delicate question which has not been addressed in the literature, yet. In this paper we will show that the converse is also true under a necessary restriction on the number of wavelets relative to the order of the MRA. In particular, the two notions of Riesz wavelets associated with an MRA and obtained by an MRA turn out to be equivalent. The proof of this result is a consequence of a theorem of de Boor, DeVore, and Ron on the intersection of a nonstationary MRA.

Theorem 1.1 [6, Theorem 4.9]. *Suppose $(\phi_j)_{j\in\mathbb{Z}}$ is a sequence of functions in $L^2(\mathbb{R}^n)$, where $Z \subset \mathbb{Z}$ satisfies* $\inf_{j\in Z} j = -\infty$. *Let $(U_j)_{j\in Z}$ be the corresponding nonstationary (not necessarily nested) MRA given by*

$$U_j = \overline{\operatorname{span}}\{T_{2^j k}\phi_j \colon k \in \mathbb{Z}^n\}.$$

Then $Y = \bigcap_{j\in Z} U_j$ is a linear subspace of $L^2(\mathbb{R}^n)$ of dimension $\leqslant 1$.

The paper is organized as follows. In Section 2 we show some basic properties of the dimension function of shift invariant spaces. In the next section we generalize Theorem 1.1 and use it to show our main result, Theorem 3.7. Finally, in the last section we present examples illustrating the optimality of the main result.

2. Shift invariant spaces

In this section we recall some facts about shift invariant (SI) spaces that will be used in the sequel. Define the *dilation operator* $D_C f(x) = |\det C|^{1/2} f(Cx)$, where C is an $n \times n$ nonsingular matrix, and the *translation operator* $T_y f(x) = f(x-y)$, $y \in \mathbb{R}^n$. We use the following definition of the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-2\pi i\langle x,\xi\rangle}\, \mathrm{d}x.$$

Definition 2.1. Suppose that Γ is a lattice, i.e., $\Gamma = P\mathbb{Z}^n$, where P is an $n \times n$ nonsingular matrix. We say that a closed subspace $W \subset L^2(\mathbb{R}^n)$ is *shift invariant* (SI) with respect to the lattice Γ, if $f \in W$

implies $T_\gamma f \in W$ for all $\gamma \in \Gamma$. Given a (countable) family $\Phi \subset L^2(\mathbb{R}^n)$ and the lattice Γ we define the SI system $E^\Gamma(\Phi)$ and SI space $S^\Gamma(\Phi)$ by

$$E^\Gamma(\Phi) = \{T_\gamma\varphi\colon \varphi \in \Phi,\ \gamma \in \Gamma\}, \qquad S^\Gamma(\Phi) = \overline{\operatorname{span}} E^\Gamma(\Phi). \tag{2.1}$$

When $\Gamma = \mathbb{Z}^n$ we often drop the superscript Γ, and we simply say that W is SI.

The *dimension function* of a SI space W is a Γ^*-periodic function $\dim_W^\Gamma : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$, which measures the size of W over the fibers of $\mathbb{R}^n/\Gamma^*$. Here, Γ^* is the dual lattice, i.e.,

$$\Gamma^* = \bigl\{\eta \in \mathbb{R}^n\colon \langle \eta, \gamma\rangle \in \mathbb{Z} \text{ for } \gamma \in \Gamma\bigr\}. \tag{2.2}$$

That is, if $\Gamma = P\mathbb{Z}^n$ then $\Gamma^* = (P^*)^{-1}\mathbb{Z}^n$. The precise definition of the dimension function of a space W in terms of the range function is given by Proposition 2.6. Alternatively, the dimension function of a SI space W can be introduced by Stone's theorem as the multiplicity function of the projection valued measure coming from the representation of the lattice Γ acting on W via translations, see [1–3,11]. However, for our purposes, the following proposition can serve as a definition of the dimension function, where the notion of range function occurs only implicitly.

Proposition 2.2. *Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is a (countable) family and $\Gamma = P\mathbb{Z}^n$ is a lattice. Then the dimension function of $W = S^\Gamma(\Phi)$ is given by*

$$\dim_W^\Gamma(\xi) = \dim \overline{\operatorname{span}}\bigl\{\bigl(\hat\varphi(\xi+\eta)\bigr)_{\eta\in\Gamma^*} \in \ell^2\bigl(\Gamma^*\bigr)\colon \varphi \in \Phi\bigr\}. \tag{2.3}$$

Moreover, if the system $E^\Gamma(\Phi)$ is a tight frame with constant 1 for the space $W = S^\Gamma(\Phi)$ then

$$\dim_W^\Gamma(\xi) = |\det P|^{-1} \sum_{\varphi\in\Phi}\sum_{\eta\in\Gamma^*} \bigl|\hat\varphi(\xi+\eta)\bigr|^2. \tag{2.4}$$

Proof. The proof of Proposition 2.2 in the case $\Gamma = \mathbb{Z}^n$ can be found, for example, in [5, Proposition 3.1], [7, Proposition 1.5, Theorem 2.5(ii)], and [16]. The general case follows by a change of variables, see also Lemma 2.3. Indeed, suppose that $E^\Gamma(\Phi)$ is a tight frame with constant 1 for $W = S^\Gamma(\Phi)$, $\Gamma = P\mathbb{Z}^n$. Then $D_P E^\Gamma(\Phi) = E^{\mathbb{Z}^n}(D_P\Phi)$ is a tight frame for $D_P W$ which is SI with respect to $\mathbb{Z}^n$. By (2.4) applied for $\mathbb{Z}^n$ and Lemma 2.3,

$$\begin{aligned}\dim_W^\Gamma\bigl(\bigl(P^*\bigr)^{-1}\xi\bigr) &= \dim_{D_P W}^{\mathbb{Z}^n}(\xi) = \sum_{\varphi\in\Phi}\sum_{\eta\in\mathbb{Z}^n}\bigl|\widehat{D_P\varphi}(\xi+\eta)\bigr|^2 = |\det P|^{-1}\sum_{\varphi\in\Phi}\sum_{\eta\in\mathbb{Z}^n}\bigl|\hat\varphi\bigl(\bigl(P^*\bigr)^{-1}(\xi+\eta)\bigr)\bigr|^2 \\ &= |\det P|^{-1}\sum_{\varphi\in\Phi}\sum_{\eta\in\Gamma^*}\bigl|\hat\varphi\bigl(\bigl(P^*\bigr)^{-1}\xi+\eta\bigr)\bigr|^2,\end{aligned}$$

which shows (2.4) in the general case. □

Lemma 2.3. *Suppose $W \subset L^2(\mathbb{R}^n)$ is SI with respect to the lattice Γ. Let C be any $n \times n$ nonsingular matrix. Then $D_C W$ is SI with respect to the lattice $C^{-1}\Gamma$ and the following identity holds*:

$$\dim_{D_C W}^{C^{-1}\Gamma}(\xi) = \dim_W^\Gamma\bigl(\bigl(C^*\bigr)^{-1}\xi\bigr) \quad \textit{for a.e. } \xi. \tag{2.5}$$

Proof. Suppose $\Phi \subset L^2(\mathbb{R}^n)$ generates W, i.e., $S^\Gamma(\Phi) = W$. Then $D_C\Phi$ generates $D_C W$, i.e., $S^{C^{-1}\Gamma}(D_C\Phi) = D_C W$. By (2.3),

$$\begin{aligned}\dim_{D_C W}^{C^{-1}\Gamma}(\xi) &= \dim \overline{\operatorname{span}}\bigl\{\bigl(\widehat{D_c\varphi}(\xi+\eta)\bigr)_{\eta\in(C^{-1}\Gamma)^*} \in \ell^2\bigl(\bigl(C^{-1}\Gamma\bigr)^*\bigr)\colon \varphi\in\Phi\bigr\}\\ &= \dim \overline{\operatorname{span}}\bigl\{\bigl(\hat\varphi\bigl(\bigl(C^*\bigr)^{-1}(\xi+\eta)\bigr)\bigr)_{\eta\in C^*\Gamma^*} \in \ell^2\bigl(C^*\Gamma^*\bigr)\colon \varphi\in\Phi\bigr\}\\ &= \dim \overline{\operatorname{span}}\bigl\{\bigl(\hat\varphi\bigl(\bigl(C^*\bigr)^{-1}\xi+\eta\bigr)\bigr)_{\eta\in\Gamma^*} \in \ell^2\bigl(\Gamma^*\bigr)\colon \varphi\in\Phi\bigr\}\\ &= \dim_W^{\Gamma}\bigl(\bigl(C^*\bigr)^{-1}\xi\bigr),\end{aligned}$$

since $(C^{-1}\Gamma)^* = C^*\Gamma^*$. □

Lemma 2.4. *Suppose $W \subset L^2(\mathbb{R}^n)$ is SI with respect to the lattice Γ. Let $\Gamma' \subset \Gamma$ be any sublattice of Γ. Then W is SI with respect to the lattice Γ', and, moreover,*

$$\dim_W^{\Gamma'}(\xi) = \sum_{d\in(\Gamma')^*/\Gamma^*} \dim_W^{\Gamma}(\xi+d) \quad \textit{for a.e. } \xi, \tag{2.6}$$

where the sum runs over representatives of distinct cosets of $(\Gamma')^*/\Gamma^*$.

Proof. Let C be any $n\times n$ matrix such that $\Gamma' = C\Gamma$, $\Gamma = P\mathbb{Z}^n$. It is well known that the orders of the quotient groups Γ/Γ' and $(\Gamma')^*/\Gamma^*$ are the same and equal to $|\det C|$. Let $\Phi \subset L^2(\mathbb{R}^n)$ be any family whose translates by Γ generate a tight frame $E^\Gamma(\Phi)$ with constant 1 for $W = S^\Gamma(\Phi)$. Then

$$E^\Gamma(\Phi) = \bigcup_{d\in\Gamma/\Gamma'} E^{\Gamma'}(T_d\Phi) = E^{\Gamma'}\biggl(\bigcup_{d\in\Gamma/\Gamma'} T_d\Phi\biggr),$$

and consequently by (2.4),

$$\begin{aligned}\dim_W^{\Gamma'}(\xi) &= \bigl|\det(CP)\bigr|^{-1} \sum_{\varphi\in\bigcup_{d\in\Gamma/\Gamma'} T_d\Phi}\ \sum_{\eta\in(\Gamma')^*} \bigl|\hat\varphi(\xi+\eta)\bigr|^2\\ &= \bigl|\det P\bigr|^{-1} \sum_{\varphi\in\Phi}\ \sum_{d\in(\Gamma')^*/\Gamma^*}\ \sum_{\eta\in\Gamma^*} \bigl|\hat\varphi(\xi+\eta+d)\bigr|^2 = \sum_{d\in(\Gamma')^*/\Gamma^*} \dim_W^{\Gamma}(\xi+d). \qquad \Box\end{aligned}$$

Lemma 2.5. *Suppose $U, V \subset L^2(\mathbb{R}^n)$ are SI with respect to the lattice Γ, and $\dim_U^\Gamma(\xi)$, $\dim_V^\Gamma(\xi)$ are finite for a.e. ξ. Then $W = \overline{U+V}$ is also SI with respect to Γ, and*

$$\dim_W^\Gamma(\xi) = \dim_U^\Gamma(\xi) + \dim_V^\Gamma(\xi) - \dim_{U\cap V}^\Gamma(\xi) \quad \textit{for a.e. } \xi. \tag{2.7}$$

Proof. The proof follows immediately from the range function interpretation of the dimension function, see Proposition 2.6. □

Let $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$, where $\mathbb{T}^n = [-1/2, 1/2)^n$, be the isometric isomorphism given by

$$\mathcal{T}f : \mathbb{T}^n \to \ell^2\bigl(\mathbb{Z}^n\bigr), \quad \mathcal{T}f(\xi) = \bigl(\hat f(\xi+k)\bigr)_{k\in\mathbb{Z}^n}. \tag{2.8}$$

We recall that a *range function* is a mapping

$$J : \mathbb{T}^n \to \bigl\{\text{closed subspaces of } \ell^2\bigl(\mathbb{Z}^n\bigr)\bigr\}.$$

We also need the following fundamental description of SI spaces, see [5, Result 1.5], or [7, Proposition 1.5]. For simplicity, we state Proposition 2.6 for SI spaces with respect to the standard lattice $\mathbb{Z}^n$.

Proposition 2.6. *A closed subspace $V \subset L^2(\mathbb{R}^n)$ is SI (with respect to $\mathbb{Z}^n$) if and only if*

$$V = \{ f \in L^2(\mathbb{R}^n)\colon\ \mathcal{T} f(\xi) \in J(\xi) \text{ for a.e. } \xi \in \mathbb{T}^n \},$$

where J is a measurable range function. The correspondence between V and J is one-to-one under the convention that the range functions are identified if they are equal a.e.

Furthermore, the dimension function of V is given by

$$\dim_V(\xi) = \dim_V^{\mathbb{Z}^n}(\xi) = \dim J(\xi) \quad \text{for a.e. } \xi \in \mathbb{T}^n.$$

3. GMRAs

In this section we recall already known results about Riesz wavelets and GMRAs, extend a result of de Boor, DeVore, and Ron mentioned in Section 1, and show our main result. We start by recalling the notion of a GMRA, which has been studied by a number of authors [1–4,9,15].

Definition 3.1. Let A be an $n \times n$ integer expansive dilation matrix which is fixed throughout this section. A *generalized multiresolution analysis* (GMRA) is a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ satisfying:

$$V_j \subset V_{j+1}, \qquad D_A V_j = V_{j+1},$$
$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n), \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\},$$
$$T_k V_0 = V_0 \quad \text{for all } k \in \mathbb{Z}^n.$$

The space V_0 is often called a *core space.*

Given a Riesz wavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ we define the sequence $(V_j(\Psi))_{j\in\mathbb{Z}}$ by

$$V_j = V_j(\Psi) = \overline{\operatorname{span}}\{D_{A^i} T_k \psi\colon\ i < j,\ k \in \mathbb{Z}^n,\ \psi \in \Psi\}. \tag{3.1}$$

It is easy to see that the sequence $(V_j)_{j\in\mathbb{Z}}$ satisfies all the properties of GMRA except possibly the last one, i.e., that the core space V_0 is SI. In the case when $(V_j(\Psi))_{j\in\mathbb{Z}}$ forms a GMRA, we say that Ψ is *associated* with a GMRA, or Ψ *generates* a GMRA.

It turns out that the core space V_0 is SI if and only if Ψ is a biorthogonal wavelet, i.e., there exists a Riesz wavelet $\Phi = \{\phi^1, \ldots, \phi^L\}$ such that

$$\langle D_{A^j} T_k \psi^l, D_{A^{j'}} T_{k'} \phi^{l'} \rangle = \delta_{j,j'} \delta_{k,k'} \delta_{l,l'} \quad \text{for all } j, j' \in \mathbb{Z},\ k, k' \in \mathbb{Z}^n,\ l, l' = 1, \ldots, L.$$

This was shown by Kim et al. [13] in one dimension and by Larson et al. [15] in higher dimensions.

Theorem 3.2. *Let $\Psi = \{\psi^1, \ldots, \psi^L\}$ be a Riesz wavelet. The following are equivalent*:

(i) *the sequence $(V_j(\Psi))_{j\in\mathbb{Z}}$ forms a GMRA,*
(ii) *Ψ is a biorthogonal wavelet,*
(iii) *there exists an orthonormal wavelet $\Phi = \{\phi^1, \ldots, \phi^L\}$ which is associated with the same GMRA as Ψ, i.e., $V_0(\Phi) = V_0(\Psi)$.*

Conversely, Baggett et al. [1] have characterized those GMRAs that can be generated by orthonormal wavelets, and thus by Riesz wavelets by Theorem 3.2.

Theorem 3.3. *Let $(V_j)_{j\in\mathbb{Z}}$ be a GMRA. The following are equivalent*:

(i) *there exists an orthonormal wavelet $\Psi = \{\psi^1,\dots,\psi^L\}$ which is associated with $(V_j)_{j\in\mathbb{Z}}$,*
(ii) *there exists a Riesz wavelet $\Psi = \{\psi^1,\dots,\psi^L\}$ which is associated with $(V_j)_{j\in\mathbb{Z}}$,*
(iii) *the dimension function $\dim_{V_0}(\xi)$ of the core space V_0 is finite for a.e. ξ and it satisfies the consistency equation*

$$\sum_{d\in\mathcal{D}} \dim_{V_0}\bigl((A^*)^{-1}(\xi+d)\bigr) = \dim_{V_0}(\xi) + L, \quad \textit{for a.e. } \xi, \tag{3.2}$$

where $\mathcal{D}$ is the set of $|\det A|$ representatives of different cosets of $\mathbb{Z}^n/A^*\mathbb{Z}^n$.

Following Zalik [19] we can introduce the notion of wavelets obtained by a GMRA.

Definition 3.4. We say that a GMRA $(V_j)_{j\in\mathbb{Z}}$ is *admissible* of order L, i.e., the core space V_0 satisfies the consistency equation (3.2). We say that a Riesz wavelet $\Psi = \{\psi^1,\dots,\psi^L\}$ is *obtained* by a GMRA if there is an admissible GMRA $(V_j)_{j\in\mathbb{Z}}$ of order L such that

$$\psi^1,\dots,\psi^L \in V_1. \tag{3.3}$$

Clearly, every Riesz wavelet which is associated with a GMRA is also obtained by the same GMRA. The converse to this, see Theorem 3.7, is much less obvious and requires hard work.

We start by generalizing Theorem 1.1 on the intersection of a nonstationary MRA to the case of general dilations and higher multiplicities.

Theorem 3.5. *Suppose $(\Phi_j)_{j\in Z}$ is a sequence of finite subsets of $L^2(\mathbb{R}^n)$ of cardinality $\leqslant L$, where $Z\subset\mathbb{Z}$ satisfies $\inf_{j\in Z} j = -\infty$. Let $(U_j)_{j\in Z}$ be the corresponding nonstationary* (*not necessarily nested*) *GMRA given by*

$$U_j = S^{A^{-j}\mathbb{Z}^n}(\Phi_j) = \overline{\operatorname{span}}\bigl\{T_{A^{-j}k}\phi\colon\ \phi\in\Phi_j,\ k\in\mathbb{Z}^n\bigr\}. \tag{3.4}$$

Then $Y = \bigcap_{j\in Z} U_j$ is a linear subspace of $L^2(\mathbb{R}^n)$ of dimension $\leqslant L$.

To show Theorem 3.5 we need a lemma describing linear independence of a finite set of measurable functions in terms of their values on some dense subset of $\mathbb{R}^n$.

Lemma 3.6. *Suppose D is a countable and dense subset of $\mathbb{R}^n$. Suppose $f_1,\dots,f_m:\mathbb{R}^n\to\mathbb{C}$ are any measurable functions. The following are equivalent*:

(i) *$f_1,\dots,f_m$ are linearly dependent, i.e., there exist $(c_1,\dots,c_m)\in\mathbb{C}^m\setminus\{0\}$ such that*

$$\sum_{l=1}^m c_l f_l(x) = 0 \quad \textit{for a.e. } x\in\mathbb{R}^n,$$

(ii) *for almost every* $x \in \mathbb{R}^n$ *there exist* $(c_1(x), \ldots, c_m(x)) \in \mathbb{C}^m \backslash \{0\}$ *such that*

$$\sum_{l=1}^{m} c_l(x) f_l(x+d) = 0 \quad \textit{for all } d \in \mathcal{D}.$$

Proof. The implication (i) ⇒ (ii) is trivial.

Suppose (ii) holds. Define $f : \mathbb{R}^n \to \mathbb{C}^m$ by $f(x) = (f_1(x), \ldots, f_m(x))$. Following the terminology of Helson [5,7,10], we define the "range function" $J : \mathbb{R}^n \to \{\text{subspaces of } \mathbb{C}^m\}$ by

$$J(x) = \operatorname{span}\{f(x+d) \colon d \in D\}.$$

Let $P(x)$ denote the orthogonal projection onto $J(x)$. It is clear that $J(x)$ is measurable, i.e., for any $v, w \in \mathbb{C}^m$ the scalar function $p_{v,w}(x) = \langle P(x)v, w\rangle$ is measurable. Clearly,

$$p_{v,w}(x) = p_{v,w}(x+d) \quad \text{for all } d \in D. \tag{3.5}$$

Let $x, y \in \mathbb{R}^n$ be two Lebesgue points of the function $p_{v,w}$, where $v, w \in \mathbb{C}^m$. Recall that if $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ then a point $x \in \mathbb{R}^n$ is said to be a *Lebesgue point* of g if

$$\lim_{|B| \to 0,\ x \in B} \frac{1}{|B|} \int_B g(z)\, \mathrm{d}z = g(x)$$

with the limit taken over balls B. The Lebesgue Differentiation theorem asserts that almost every point $x \in \mathbb{R}^n$ is a Lebesgue point of $g \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Let $(B_i)_{i \in \mathbb{N}}$ be a sequence of balls such that $x \in B_i$ and $|B_i| \to 0$ as $i \to \infty$. By the density of D there exists a sequence $(d_i)_{i \in \mathbb{N}} \subset D$ such that $y - d_i \in B_i$ for all $i \in \mathbb{N}$. By the Lebesgue Differentiation theorem and (3.5),

$$p_{v,w}(x) = \lim_{i \to \infty} \frac{1}{|B_i|} \int_{B_i} p_{v,w}(z)\, \mathrm{d}z = \lim_{i \to \infty} \frac{1}{|B_i|} \int_{B_i} p_{v,w}(z + d_i)\, \mathrm{d}z = p_{v,w}(y).$$

Hence, $p_{v,w}(x) = \text{const}$ for a.e. x.

Since $v, w \in \mathbb{C}^m$ are arbitrary $J(x) = \text{const}$ for a.e. x. By (ii), $J(x) \neq \mathbb{C}^m$ for a.e. x, and hence there exists a hyperplane $H \subset \mathbb{C}^m$ such that $J(x) \subset H$ for a.e. x. Therefore, $f(x) \in H$ for a.e. x, and (i) holds. □

Proof of Theorem 3.5. Let $f_1, \ldots, f_m$ be arbitrary functions in Y, where $m = L + 1$. It suffices to show that $f_1, \ldots, f_m$ axe linearly dependent. Let $j \in Z$ and let $\Phi_j = \{\phi^1, \ldots, \phi^L\}$. As a consequence of [5, Theorem 1.7] and a change of variables $g \in U_j$ if and only if

$$\hat{g}(\xi) = \sum_{l=1}^{L} \tau^l(\xi) \hat{\phi}^l(\xi) \quad \text{for all } \xi \in \mathbb{R}^n,$$

for some measurable and $(A^*)^j \mathbb{Z}^n$-periodic functions $\tau^1, \ldots, \tau^L$, i.e.,

$$\tau^l(\xi) = \tau^l\bigl(\xi + (A^*)^j k\bigr) \quad \text{for all } k \in \mathbb{Z}^n.$$

Therefore,

$$\hat{f}_k(\xi) = \sum_{l=1}^{L} \tau^l_k(\xi) \hat{\phi}^l(\xi) \quad \text{for } k = 1, \ldots, m, \tag{3.6}$$

for some measurable and $(A^*)^j\mathbb{Z}^n$-periodic functions $\tau_k^l(\xi)$. Let $(c_1(\xi),\dots,c_m(\xi))$ be a nonzero solution to the undetermined system of linear equations

$$\begin{cases} c_1(\xi)\tau_1^1(\xi)+\cdots+c_m(\xi)\tau_m^1(\xi)=0,\\ \cdots\\ c_1(\xi)\tau_1^L(\xi)+\cdots+c_m(\xi)\tau_m^L(\xi)=0. \end{cases} \tag{3.7}$$

By (3.6) and (3.7) we have

$$\sum_{k=1}^m c_k(\xi)\hat{f}_k(\xi)=0$$

and by the periodicity of $\tau_k^l(\xi)$,

$$\sum_{k=1}^m c_k(\xi)\hat{f}_k\bigl(\xi+\bigl(A^*\bigr)^j k\bigr)=0 \quad \text{for all } k\in\mathbb{Z}^n. \tag{3.8}$$

Let $D=\bigcup_{j\in Z}(A^*)^j\mathbb{Z}^n$. Since $\inf_{j\in Z} j=-\infty$, D is dense in $\mathbb{R}^n$. Since A (and thus A^*) preserves the lattice $\mathbb{Z}^n$ we can find universal $c_1(\xi),\dots,c_m(\xi)$ yielding (3.8) for all $j\in Z$. Therefore,

$$\sum_{k=1}^m c_k(\xi)\hat{f}_k(\xi+d)=0 \quad \text{for all } d\in D.$$

By Lemma 3.6, $\hat{f}_1,\dots,\hat{f}_m$ are linearly dependent which completes the proof of Theorem 3.5. □

We are now ready to state our main result.

Theorem 3.7. *Suppose $\Psi=\{\psi^1,\dots,\psi^L\}$ is a Riesz wavelet, and $(V_j)_{j\in\mathbb{Z}}$ is an admissible GMRA of order L, i.e.,* $\dim_{V_0}(\xi)<\infty$ *for a.e. ξ and* (3.2) *holds. The following are equivalent*:

(i) *Ψ is obtained by the GMRA $(V_j)_{j\in\mathbb{Z}}$, i.e.,* (3.3) *holds,*

(ii)
$$W_j+V_j=V_{j+1} \quad \textit{and} \quad W_j\cap V_j=\{0\} \quad \textit{for all } j\in\mathbb{Z}, \tag{3.9}$$
where
$$W_j:=\overline{\operatorname{span}}\bigl\{D_{A^j}T_k\psi:\ k\in\mathbb{Z}^n,\ \psi\in\Psi\bigr\}, \tag{3.10}$$

(iii) *Ψ is associated with the GMRA $(V_j)_{j\in\mathbb{Z}}$, i.e.,*
$$V_j(\Psi):=\sum_{i<j}W_i=V_j \quad \textit{for all } j\in\mathbb{Z}. \tag{3.11}$$

The implications (iii) ⇒ (ii) ⇒ (i) are trivial. It remains to show the difficult implication (i) ⇒ (iii). We will present two different proofs of this implication. The first proof works only for GMRAs of finite height, i.e., those for which the dimension function of the core space V_0 is essentially bounded, and uses Theorem 3.5. This is the most interesting situation, since it covers Riesz wavelets obtained by an MRA (also MRAs with higher than one multiplicities). However, the second proof works for general GMRAs of possibly infinite height.

Proof of Theorem 3.7 for finite height GMRAs. Suppose Ψ is obtained by the GMRA $(V_j)_{j\in\mathbb{Z}}$. For $j \in \mathbb{Z}$ let W_j be given by (3.10). Since $W_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$ we have that

$$\sum_{j<0} W_j \subset V_0. \tag{3.12}$$

Here, $\sum_{j<0} W_j$ consists of all functions $f \in L^2(\mathbb{R}^n)$ which can be represented as $f = \sum_{j<0} w_j$, where the series converges unconditionally and $w_j \in W_j$.

To complete the proof of Theorem 3.7 it remains to show that we have the equality in (3.12). Indeed, suppose that

$$V_0(\Psi) := \sum_{j<0} W_j = V_0.$$

By applying the dilation operator

$$V_j(\Psi) = D_{A^j}\big(V_0(\Psi)\big) = D_{A^j}(V_0) = V_j \quad \text{for all } j \in \mathbb{Z}.$$

Therefore, the sequence $(V_j(\Psi))_{j\in\mathbb{Z}}$ is exactly the same as the GMRA $(V_j)_{j\in\mathbb{Z}}$, and hence Ψ is associated with the GMRA $(V_j)_{j\in\mathbb{Z}}$.

To show the equality in (3.12), note that W_j is SI with respect to the lattice $A^{-j}\mathbb{Z}^n$ and its dimension function

$$\dim_{W_j}^{A^{-j}\mathbb{Z}^n}(\xi) = L \quad \text{for a.e. } \xi, \tag{3.13}$$

since $D_{A^j}(E^{\mathbb{Z}^n}(\Psi)) = E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)$ forms a Riesz basis for W_j. Define a nonstationary GMRA $(U_j)_{j\leqslant 0}$ by

$$U_j = V_0 \ominus \left(\sum_{j\leqslant i<0} W_i\right). \tag{3.14}$$

Since $\sum_{j\leqslant i<0} W_i$ is SI with respect to $A^{-j}\mathbb{Z}^n$, hence its orthogonal complement is also SI with respect to the same lattice, and therefore $U_j = V_0 \cap (\sum_{j\leqslant i<0} W_i)^{\perp}$ is also SI with respect to $A^{-j}\mathbb{Z}^n$.

We claim that for all $j \leqslant 0$ the dimension function of U_j coincides with the dimension function of V_j, i.e.,

$$\dim_{U_j}^{A^{-j}\mathbb{Z}^n}(\xi) = \dim_{V_j}^{A^{-j}\mathbb{Z}^n}(\xi) \quad \text{for a.e. } \xi. \tag{3.15}$$

Indeed, (3.15) holds for $j = 0$. By induction hypothesis suppose that (3.15) holds for some $j \leqslant 0$. Then we need to show (3.15) for $j-1$. Since

$$\begin{aligned}(W_{-1} + W_{-2} + \cdots + W_j) \oplus U_j = V_0 &= (W_{-1} + W_{-2} + \cdots + W_j + W_{j-1}) \oplus U_{j-1} \\ &= (W_{-1} + W_{-2} + \cdots + W_j) + (W_{j-1} \oplus U_{j-1}),\end{aligned}$$

and

$$(W_{-1} + W_{-2} + \cdots + W_j) \cap U_j = \{0\} = (W_{-1} + W_{-2} + \cdots + W_j) \cap (W_{j-1} \oplus U_{j-1}),$$

we conclude by Lemma 2.5 that the dimension functions of U_j and $W_{j-1} \oplus U_{j-1}$ are the same, i.e.,

$$\dim_{U_j}^{A^{-j+1}\mathbb{Z}^n}(\xi) = \dim_{W_{j-1}\oplus U_{j-1}}^{A^{-j+1}\mathbb{Z}^n}(\xi) = \dim_{W_{j-1}}^{A^{-j+1}\mathbb{Z}^n}(\xi) + \dim_{U_{j-1}}^{A^{-j+1}\mathbb{Z}^n}(\xi) = L + \dim_{U_{j-1}}^{A^{-j+1}\mathbb{Z}^n}(\xi). \tag{3.16}$$

Observe that the consistency equation (3.2) can be written by Lemmas 2.3 and 2.4,

$$\begin{aligned}\dim_{V_1}^{\mathbb{Z}^n}(\xi) &= \dim_{D_A(V_0)}^{\mathbb{Z}^n}(\xi) = \dim_{V_0}^{A\mathbb{Z}^n}\big(\big(A^*\big)^{-1}\xi\big)\\ &= \sum_{d\in(A^*)^{-1}\mathbb{Z}^n/\mathbb{Z}^n} \dim_{V_0}^{\mathbb{Z}^n}\big(\big(A^*\big)^{-1}\xi+d\big) = \dim_{V_0}^{\mathbb{Z}^n}(\xi)+L.\end{aligned}$$

Hence, for $j\in\mathbb{Z}$ by the above and Lemma 2.3,

$$\begin{aligned}\dim_{V_j}^{A^{-j+1}\mathbb{Z}^n}\big(\big(A^*\big)^{-1}\xi\big) &= \dim_{D_{A^j}(V_0)}^{A^{-j+1}\mathbb{Z}^n}\big(\big(A^*\big)^{-1}\xi\big) = \dim_{V_0}^{A\mathbb{Z}^n}\big(\big(A^*\big)^{-j-1}\xi\big)\\ &= \dim_{V_0}^{\mathbb{Z}^n}\big(\big(A^*\big)^{-j}\xi\big)+L = \dim_{V_j}^{A^{-j}\mathbb{Z}^n}(\xi)+L.\end{aligned} \tag{3.17}$$

On the other hand, by Lemma 2.4 and the induction hypothesis,

$$\begin{aligned}\dim_{U_j}^{A^{-j+1}\mathbb{Z}^n}(\xi) &= \sum_{d\in(A^*)^{j-1}\mathbb{Z}^n/(A^*)^j\mathbb{Z}^n} \dim_{U_j}^{A^{-j}\mathbb{Z}^n}(\xi+d)\\ &= \sum_{d\in(A^*)^{j-1}\mathbb{Z}^n/(A^*)^j\mathbb{Z}^n} \dim_{V_j}^{A^{-j}\mathbb{Z}^n}(\xi+d) = \dim_{V_j}^{A^{-j+1}\mathbb{Z}^n}(\xi).\end{aligned} \tag{3.18}$$

Combining (3.16)–(3.18), and Lemma 2.3,

$$\begin{aligned}\dim_{U_{j-1}}^{A^{-j+1}\mathbb{Z}^n}(\xi) &= \dim_{U_j}^{A^{-j+1}\mathbb{Z}^n}(\xi)-L = \dim_{V_j}^{A^{-j+1}\mathbb{Z}^n}(\xi)-L\\ &= \dim_{V_j}^{A^{-j}\mathbb{Z}^n}\big(A^*\xi\big) = \dim_{D_A V_{j-1}}^{A^{-j}\mathbb{Z}^n}\big(A^*\xi\big) = \dim_{V_{j-1}}^{A^{-j+1}\mathbb{Z}^n}(\xi),\end{aligned}$$

we obtain (3.15) for $j-1$. Therefore, by induction, (3.15) holds for all $j\leqslant 0$.

Suppose that our GMRA $(V_j)_{j\in\mathbb{Z}}$ has a finite height $\tilde{L}$. Therefore,

$$\dim_{V_j}^{A^{-j}\mathbb{Z}^n}(\xi) = \dim_{V_0}^{\mathbb{Z}^n}\big(\big(A^*\big)^{-j}\xi\big)\leqslant\tilde{L}\quad\text{for a.e. }\xi.$$

By (3.15) this implies for all $j\leqslant 0$, $\dim_{U_j}^{A^{-j}\mathbb{Z}^n}(\xi)\leqslant\tilde{L}$. This means that for each $j\geqslant 0$ we can find a generating set Φ_j of cardinality at most $\tilde{L}$ such that $U_j=S^{A^{-j}\mathbb{Z}^n}(\Phi_j)$, see [5, Theorem 3.5]. By Theorem 3.5, this implies that $Y=\bigcap_{j\leqslant 0}U_j$ is a linear subspace of $L^2(\mathbb{R}^n)$ of dimension $\leqslant\tilde{L}$. By (3.14),

$$Y = V_0\ominus\Big(\sum_{j<0}W_j\Big), \tag{3.19}$$

which represents a "defect" from the equality in (3.12). It remains to show that the defect space Y is trivial, i.e., $Y=\{0\}$.

This can be most easily seen by considering an auxiliary defect space $\tilde{Y}$ defined by

$$\tilde{Y} = V_0\cap\Big(\sum_{j\geqslant 0}W_j\Big). \tag{3.20}$$

In other words, $\tilde{Y}$ is the unique complementary space of $\sum_{j<0}W_j$ inside V_0, which is, in addition, contained in $\sum_{j\geqslant 0}W_j$, i.e.,

$$\Big(\sum_{j<0}W_j\Big)+\tilde{Y}=V_0,\qquad \Big(\sum_{j<0}W_j\Big)\cap\tilde{Y}=\{0\},\quad\text{and}\quad \tilde{Y}\subset\sum_{j\geqslant 0}W_j.$$

The defect space $\tilde{Y}$ must have the same dimension as Y. Moreover, since V_0 and $\sum_{j\geqslant 0} W_j$ are SI with respect to $\mathbb{Z}^n$, $\tilde{Y}$ is SI, too. Since the trivial space is the only finitely-dimensional SI space, the defect space $\tilde{Y}$ must be trivial, and consequently,

$$\sum_{j<0} W_j = V_0.$$

This completes the proof of Theorem 3.7 in the finite height case. □

Proof of Theorem 3.7 for infinite height GMRAs. The proof follows along the lines of the argument in the finite height case. As before, we could define a nonstationary GMRA $(U_j)_{j\leqslant 0}$, the defect space Y by (3.19), and the auxiliary SI defect space $\tilde{Y}$ by (3.20). The defect spaces Y and $\tilde{Y}$ have the same (possibly infinite) dimension. Even though, a priori, Y (and thus $\tilde{Y}$) may not be finite-dimensional, we will see that $\tilde{Y}$ has to be in a certain sense "arbitrarily small", and thus trivial.

Note first that if $(V_j)_{j\in\mathbb{Z}}$ is an admissible GMRA of order L,

$$\int_{\mathbb{T}^n} \dim_{V_0}^{\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi = \frac{L}{|\det A| - 1}, \tag{3.21}$$

where $\mathbb{T}^n$ is a fundamental domain, say $\mathbb{T}^n = [-1/2, 1/2)^n$. Indeed, (3.21) follows from the fact that for any orthonormal wavelet $\tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^L\}$ associated with $(V_j)_{j\in\mathbb{Z}}$, we have

$$\dim_{V_0}^{\mathbb{Z}^n}(\xi) = D_{\tilde{\Psi}}(\xi) := \sum_{\psi\in\tilde{\Psi}} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\hat{\psi}\left(\left(A^*\right)^j(\xi + k)\right)\right|^2,$$

see [7,8,12,15,17].

Since for any $j < 0$,

$$(W_{-1} + W_{-2} + \cdots + W_j) + \tilde{Y} \subset V_0 \quad \text{and} \quad (W_{-1} + W_{-2} + \cdots + W_j) \cap \tilde{Y} = \{0\},$$

therefore,

$$\dim_{\tilde{Y}}^{A^{-j}\mathbb{Z}^n}(\xi) + \sum_{i=j}^{-1} \dim_{W_i}^{A^{-j}\mathbb{Z}^n}(\xi) \leqslant \dim_{V_0}^{A^{-j}\mathbb{Z}^n}(\xi).$$

By Lemma 2.4 and (3.13), $\dim_{W_i}^{A^{-j}\mathbb{Z}^n}(\xi) = |\det A|^{i-j} L$, and thus

$$\dim_{\tilde{Y}}^{A^{-j}\mathbb{Z}^n}(\xi) \leqslant \dim_{V_0}^{A^{-j}\mathbb{Z}^n}(\xi) - L\frac{|\det A|^{-j} - 1}{|\det A| - 1}.$$

Therefore, integrating the above over $(A^*)^j(\mathbb{T}^n)$ we have

$$\int_{\mathbb{T}^n} \dim_{\tilde{Y}}^{\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi \leqslant \int_{\mathbb{T}^n} \dim_{V_0}^{\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi + L\frac{|\det A|^{j} - 1}{|\det A| - 1}, \tag{3.22}$$

since for any SI space V (with respect to $\mathbb{Z}^n$) and $j \leqslant 0$ we have

$$\int_{(A^*)^j(\mathbb{T}^n)} \dim_V^{A^{-j}\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi = \int_{(A^*)^j(\mathbb{T}^n)} \sum_{d\in(A^*)^j\mathbb{Z}^n/\mathbb{Z}^n} \dim_V^{\mathbb{Z}^n}(\xi + d)\,\mathrm{d}\xi = \int_{\mathbb{T}^n} \dim_V^{\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi.$$

Therefore, by letting $j \to -\infty$ in (3.22) we see that by (3.21),

$$\int_{\mathbb{T}^n} \dim_{\tilde{Y}}^{\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi = 0,$$

and consequently the defect space $\tilde{Y}$ is trivial. This completes the proof of Theorem 3.7 in the general case. □

As an immediate corollary of Theorems 3.2 and 3.7 we have.

Corollary 3.8. *Suppose a Riesz wavelet $\Psi = \{\psi^1, \dots, \psi^L\}$, $L = |\det A| - 1$, is obtained by an MRA, i.e., there is an MRA $(V_j)_{j\in\mathbb{Z}}$ such that $\psi^1, \dots, \psi^L \in V_1$. Then Ψ is a biorthogonal wavelet associated with the same MRA $(V_j)_{j\in\mathbb{Z}}$.*

Remarks. (i) The assumption in Theorem 3.7 that $(V_j)_{j\in\mathbb{Z}}$ is an admissible GMRA of order L can be relaxed a bit, but not too much. Indeed, it follows from the second proof of Theorem 3.7 that it suffices to assume (instead of admissibility) that a GMRA $(V_j)_{j\in\mathbb{Z}}$ satisfies (3.21) or even less,

$$\int_{\mathbb{T}^n} \dim_{V_0}^{\mathbb{Z}^n}(\xi)\,\mathrm{d}\xi \leqslant \frac{L}{|\det A| - 1}. \tag{3.23}$$

In that case, any of the conditions (i)–(iii) would imply that $(V_j)_{j\in\mathbb{Z}}$ is an admissible GMRA of order L. Therefore, there are no Riesz wavelets Ψ that are obtained by a GMRA of a strictly lower order than the cardinality of Ψ. However, there are examples of Riesz wavelets Ψ that are obtained by a GMRA of a strictly higher order than the cardinality of Ψ, but which are not associated with a GMRA, see Example 4.4. Therefore, the assumption of admissibility in Theorem 3.7 is necessary.

(ii) Corollary 3.8 can be combined with Zalik's characterization of wavelets obtained by an MRA [19, Proposition 2.2] to yield several interesting results about such wavelets, see [14].

4. Examples

One could think that the implication (ii) ⇒ (iii) in Theorem 3.7 is immediate and follows from basic functional analysis. A slightly counterintuitive Example 4.1 shows that this is false.

Example 4.1. Let $(e_i)_{i\in\mathbb{Z}}$ be the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Define the spaces $(V_i)_{i\in\mathbb{Z}}$ and $(W_i)_{i\in\mathbb{Z}}$ by

$$W_i = \operatorname{span}\{e_i\}, \qquad V_i = \overline{\operatorname{span}}\Big\{\dots, e_{i-2}, e_{i-1}, \sum_{j\geqslant i} c^j e_j\Big\},$$

where $0 < c < 1$. Clearly, $(V_i)_{i\in\mathbb{Z}}$ satisfies GMRA like properties, i.e.,

$$V_i \subset V_{i+1}, \quad D(V_i) = V_{i+1}, \quad \bigcap_{i\in\mathbb{Z}} V_i = \{0\}, \quad \overline{\bigcup_{i\in\mathbb{Z}} V_i} = \ell^2(\mathbb{Z}),$$

where D is a "dilation operator" (known otherwise as the shift operator) acting on the standard basis by $D(e_i) = e_{i+1}$. Moreover,

$$V_i + W_i = V_{i+1}, \qquad V_i \cap W_i = \{0\}, \tag{4.1}$$

and the corresponding projections of V_{i+1} onto V_i and W_i are uniformly bounded for all $i \in \mathbb{Z}$. Nevertheless,

$$V_0 = \left(\bigoplus_{i<0} W_i\right) \oplus Y, \quad \text{where } Y = \operatorname{span}\left\{\sum_{j\geqslant 0} c^j e_j\right\}, \tag{4.2}$$

and the defect space Y is nontrivial.

Example 4.1 can be easily transplanted to the setting of SI spaces using Proposition 2.6.

Example 4.2. Using the notation from Example 4.1, define the SI spaces $\tilde{V}_i$, $\widetilde{W}_i$, and $\tilde{Y}$ by

$$\tilde{V}_i = \left\{f \in L^2(\mathbb{R})\colon\ \mathcal{T}f(\xi) \in V_i \text{ for a.e. } \xi \in [-1/2, 1/2)\right\},$$
$$\widetilde{W}_i = \left\{f \in L^2(\mathbb{R})\colon\ \mathcal{T}f(\xi) \in W_i \text{ for a.e. } \xi \in [-1/2, 1/2)\right\},$$
$$\tilde{Y} = \left\{f \in L^2(\mathbb{R})\colon\ \mathcal{T}f(\xi) \in Y \text{ for a.e. } \xi \in [-1/2, 1/2)\right\},$$

where $\mathcal{T}f$ is given by (2.8). Clearly, $(\tilde{V}_i)_{i\in\mathbb{Z}}$ satisfies GMRA like properties, i.e.,

$$\tilde{V}_i \subset \tilde{V}_{i+1}, \quad \tilde{D}(\tilde{V}_i) \subset \tilde{V}_{i+1}, \quad \bigcap_{i\in\mathbb{Z}} \tilde{V}_i = \{0\}, \quad \overline{\bigcup_{i\in\mathbb{Z}} \tilde{V}_i} = L^2(\mathbb{R}),$$

where $\tilde{D}$ is a "dilation operator" mapping $\hat{f}(\xi)$ to $\hat{f}(\xi - 1)$ in the Fourier domain. Hence, $\tilde{D}$ is just, the modulation operator given by $\tilde{D}f(x) = \mathrm{e}^{2\pi i x} f(x)$. It is then obvious that $\tilde{V}_i$ and $\widetilde{W}_i$ satisfy the analogues of (4.1) and (4.2), where the defect space $\tilde{Y}$ is nontrivial.

Example 4.2 would be more satisfying if the "dilation operator" $\tilde{D}$ were the usual dyadic dilation operator. However, the author is unaware whether such construction is possible.

The following simple example shows that the assumption of the admissibility of a GMRA in Theorem 3.7 is necessary, although it can be relaxed a bit by (3.23). In particular, Example 4.2 shows that the restriction on the number of wavelets being equal to the order of a GMRA is unavoidable.

Example 4.3. Let $(V_i)_{i\in\mathbb{Z}}$ be an MRA associated to the Haar wavelet $\psi = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$, i.e.,

$$V_i = \left\{f \in L^2(\mathbb{R})\colon\ f|_{[k2^{-i},(k+1)2^{-i})} \text{ is constant for all } k \in \mathbb{Z}\right\}.$$

Define the "misnumbered" GMRA $(V_i')_{i\in\mathbb{Z}}$ by $V_i' = V_{i+1}$. Clearly, $(V_i')_{i\in\mathbb{Z}}$ is an admissible GMRA of order 2 in the sense of Definition 3.4. In fact, $(V_i')_{i\in\mathbb{Z}}$ is an MRA of order 2, with $\mathbf{1}_{[0,1/2)}$ and $\mathbf{1}_{[1/2,1)}$ being two scaling functions of V_0'. Then ψ is obtained by $(V_i')_{i\in\mathbb{Z}}$, in the sense that $\psi \in V_1'$, but it cannot be associated with $(V_i')_{i\in\mathbb{Z}}$, since by Theorem 3.3, L would have to be 2. Naturally, in this example, the Haar wavelet ψ can be replaced by any orthonormal dyadic wavelet in $L^2(\mathbb{R})$.

A more elaborated Example 4.4 gives an example of a Riesz wavelet ψ which cannot be obtained by any admissible GMRA of order 1, but yet, it is obtained by some MRA of higher order.

Example 4.4. Let ψ be a discontinuous perturbation of the Haar function,

$$\psi = \mathbf{1}_{[0,1/2-\varepsilon)} - \mathbf{1}_{[1/2+\varepsilon,1)}, \quad \text{where } 0 < \varepsilon < 1/2.$$

Zalik [19, Example 3.2] has shown that for sufficiently small $\varepsilon > 0$ (say $\varepsilon = 1/32$), ψ is a Riesz wavelet which is not obtained by an MRA. Let $(V_i)_{i\in\mathbb{Z}}$ be the classical Haar MRA as in Example 4.3. Define the GMRA $(V_i')_{i\in\mathbb{Z}}$, by $V_i' = V_{i+4}$ for $i \in \mathbb{Z}$. Clearly, $(V_i')_{i\in\mathbb{Z}}$ is an admissible GMRA of order 16; more precisely, an MRA of order 16. Moreover, the perturbed Haar wavelet ψ with $\varepsilon = 1/32$ is obtained by the MRA $(V_i')_{i\in\mathbb{Z}}$ (of order 16), that is, $\psi \in V_1'$.

However, ψ cannot be obtained by any admissible GMRA of order 1. On the contrary, suppose otherwise. By Theorem 3.7, ψ must be associated with an admissible GMRA $(V_i'')_{i\in\mathbb{Z}}$ order 1. Moreover, $\dim_{V_0''}(\xi) \geqslant 1$ for a.e. ξ, since $\hat{\psi}(\xi) \neq 0$ for a.e. ξ. On the other hand, $\int_0^1 \dim_{V_0''}(\xi)\,d\xi = 1$, hence $\dim_{V_0''}(\xi) = 1$ for a.e. ξ, which means that $(V_i'')_{i\in\mathbb{Z}}$ is an MRA. This is a contradiction with a fact due to Zalik that ψ is not obtained by an MRA.

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