Math 95
October 23rd, 2006
Midterm Exam 2A

T F $-2^2 = 4$

FALSE! Order of operations. Square 2 first, then multiply by -1. Note that $(-2)^2 = 4$, the parentheses make all the difference.

T F $|a - b| = |b - a|$ for and real numbers a and b.

True. We've seen that $a - b = -(b - a)$, i.e. a-b and b-a are opposites, which means their absolute values are the same.

T F The graph of a rational function can cross it's vertical asymptote.

False. A rational function f has a vertical asymptote at $x = a$ if $f(a)$ is undefined because of division by zero. Therefore the point $(a, f(a))$ cannot be on the graph.

T F To multiply rational expressions you must have a common denominator.

False. Just multiply the numerators and the denominators straight across.

T F If $x > 2$ or $x < -3$, then we can say $2 < x < -3$.

False. $2 < x < -3$ means $2 < x$ AND $x < -3$. There are no numbers that are more than 2 and less than -3 whereas 5 (for example) is a number that is more than 2 or less than -3

T F $\{2, 4, 6, 8, 10\} \cup \{4, 6, 12, 14\} = \{2, 4, 6, 8, 10, 12, 14\}$

True. Just remember that $\cup$ stands for Union, which means join together.

T F $(x + y)^2 = x^2 + y^2$

False! Doing the multiplication shows that $(x + y)^2 = x^2 + 2xy + y^2$

T F The domain of any rational function is all real numbers.l

False. Let $f(x) = \frac{1}{x}$, the domain is $x \neq 0$.

T F $\frac{1}{a} - \frac{1}{b} = \frac{b - a}{ab}$

True. The common denominator is ab hence

$$\frac{1}{a} - \frac{1}{b} = \left(\frac{b}{b}\right)\frac{1}{a} - \left(\frac{a}{a}\right)\frac{1}{b} = \frac{b - a}{ab}$$

T F $\frac{x + 3}{x + 2} = \frac{3}{2}$

FALSE!!! $\frac{x + 3}{x + 2}$ is completely simplified!

1. Solve the following equations for x. Where applicable, write the solution sets in interval notation.

   (a) $|3x-8| \geq 7$

   This inequality is saying that the number $2x+3$ is more than 15 units away from 0. Hence we can rewrite $|3x-8| \geq 7$ as

   $$3x-8 \geq 7$$

   or

   $$3x-8 \leq -7$$

   Solving each equation we get that $x \geq 5$ or $x \leq \frac{1}{3}$. The union of these two solution sets in interval notation is $\left(-\infty, \frac{1}{3}\right] \cup [5, \infty)$.

   (b) $|2x-3| = 11$

   This means that the number $2x-3$ is 11 units away from 0 so either

   $$2x-3 = 11$$

   or

   $$2x-3 = -11$$

   Solving each equation we see that $x = 7$ or $x = -4$. Remember that for absolute value equations we should always check our solutions in the original equation. Both solutions work.

   (c) $-5x \leq 20$ and $3x > -18$

   We need to solve each equation and then take the **intersection** of both solution sets.

   The first inequality gives us $x \geq -4$ (remember to switch the direction of the inequality when you divide by a negative number) and the second inequality gives us $x > -6$.

   Numbers that are at least -4 **and** greater than -6 are in the interval $[-4, \infty)$.

   (d) $(x-3)(x+8) = -30$ Remember that the first step is to put the equation in standard form, everything on one side of the of the equal sign and 0 on the other:

   $$\begin{aligned} (x-3)(x+8) &= -30 \\ x^2+5x-24 &= -30 \\ x^2+5x+6 &= 0 \\ (x+3)(x+2) &= 0 \end{aligned}$$

   Using the zero product principle the last line implies that $x = -3$ or $x = -2$.

2. Compute and completely simplify your answer:

(a) $\dfrac{x+5}{7} \div \dfrac{4x+20}{9}$

$$\begin{aligned} \frac{x+5}{7} \div \frac{4x+20}{9} &= \frac{x+5}{7} \cdot \frac{9}{4x+20} \\ &= \frac{x+5}{7} \cdot \frac{9}{4(x+5)} \\ &= \frac{1}{7} \cdot \frac{9}{4} \\ &= \frac{9}{28} \end{aligned}$$

(b) $\dfrac{8x^3+27}{x^4-1} \cdot \dfrac{x^4+x^2-2}{4x^2+12x+9}$

The numerator and denominator of each fraction can be factored:

$8x^3+27$ is a sum of cubes: $(2x)^3+3^3$.
x^4-1 is a difference of squares: $(x^2)^2-1$.
x^4+x^2-2 is a closet quadratic; let $t=x^2$ and factor t^2+t-2.
$4x^2+12x+9$ is a perfect square: $(2x)^2+2(2x)(3)+3^2$.

$$\begin{aligned} \frac{8x^3+27}{x^4-1} \cdot \frac{x^4+x^2-2}{4x^2+12x+9} &= \frac{(2x+3)\left((2x)^2+(2x)(3)+3^2\right)}{(x^2-1)(x^2+1)} \frac{(x^2+2)(x^2-1)}{(2x+3)^2} \\ &= \frac{(4x^2+6x+9)(x^2+2)}{(x^2+1)(2x+3)} \end{aligned}$$

(c) $\dfrac{x+4}{x^2-x-2} - \dfrac{2x+3}{x^2+2x-8}$

To subtract we need to find the least common denominator. To do this we need to completely factor each denominator:

$$x^2-x-2=(x-2)(x+1)$$

and

$$x^2+2x-8=(x+4)(x-2)$$

therefore the least common denominator is $(x-2)(x+1)(x+4)$.

$$\frac{x+4}{x^2-x-2} - \frac{2x+3}{x^2+2x-8} = \frac{(x+4)}{(x+4)}\frac{(x+4)}{(x-2)(x+1)} - \frac{(x+1)}{(x+1)}\frac{(2x+3)}{(x+4)(x-2)}$$

$$= \frac{x^2+8x+16}{(x+4)(x-2)(x+1)} - \frac{2x^2+5x+3}{(x+4)(x-2)(x+1)}$$

$$= \frac{x^2+8x+16-(2x^2+5x+3)}{(x+4)(x-2)(x+1)}$$

$$= \frac{-x^2+3x+13}{(x+4)(x-2)(x+1)}$$

3. Consider the graph of the rational function f below:

(a) What is $f(1)$?
$f(1) = -1$

(b) What are the vertical asymptotes of f?
$x = 2$ and $x = -2$

(c) When x gets very very large, what can you say about $f(x)$?
$f(x)$ gets closer and closer to 3.

(d) What is the range of f?
The range is the set of outputs, or all values for y where the graph lies to the right or left or on the $y-$axis.
So the range is $(-\infty, 0] \cup (3, \infty)$.

(e) What is the domain of f?
Remember that the domain is the set of all possible inputs, all the values for x where the graph lies above, below or on the $x-$axis.
So the domain is $x \neq 2$ or $x \neq -2$. In interval notation: $(-\infty, -2) \cup (-2, 2,) \cup (2, \infty)$

4. The function

$$f(x) = -\frac{1}{4}x^2 + 3x + 17$$

models the number of people, $f(x)$, in millions, receiving food stamps x years after 1990.

(a) How many people received food stamps in 1994?

x is the number of years after 1990, so compute $f(4)$:

$$\begin{aligned} f(4) &= -\frac{1}{4} \cdot 4^2 + 12 + 17 \\ &= -\frac{16}{4} + 29 \\ &= -4 + 29 \\ &- 25 \end{aligned}$$

So 25 million people received food stamps in 1995.

(b) In which years did 26 million peoples receive food stamps?

Now we know how many people received food stamps, we want to determine in what year, i.e. we know our output, we need to find the input. So solve for x in the equation:

$$-\frac{1}{4}x^2 + 3x + 17 = 26$$

We begin by putting the equation into standard form.

$$\begin{aligned} -\frac{1}{4}x^2 + 3x + 17 &= 26 \\ -\frac{1}{4}x^2 + 3x + 17 - 26 &= 0 && \textit{move 26 to the other side} \\ -\frac{1}{4}x^2 + 3x - 9 &= 0 \\ x^2 - 12x + 36 &= 0 && \textit{multiply everything by -4 to clear the denominator} \\ (x-6)(x-6) &= 0 && \textit{factor!} \end{aligned}$$

So $x = 6$ which means that 26 million people received food stamps in the year 1996.

5. What is the domain of $f(x) = \dfrac{(x-1)^2}{8x^2+10x+3}$? State your answer any way you'd like (interval notation, set notation, or short-hand).

To find the domain we just need to figure out what values for x would make the denominator 0, which means we need to solve the equation

$$8x^2 + 10x + 3 = 0$$

which means we need to factor $8x^2 + 10x + 3$.

Since $8x^2 + 10x + 3$ isn't a perfect square probably the "ac" method is best:
$ac = 24$
$6 \cdot 4 = 24$ and $6 + 4 = 10$

$$\begin{aligned} 8x^2 + 10x + 3 &= 8x^2 + 4x + 6x + 3 \\ &= (8x^2 + 4x) + (6x + 3) \\ &= 4x(2x+1) + 3(2x+1) \\ &= (2x+1)(4x+3) \end{aligned}$$

Hence the solutions to the equation $8x^2 + 10x + 3 = 0$ are $x = -\dfrac{1}{2}$ or $x = -\dfrac{3}{4}$ and thus the domain of f is $x \neq -\dfrac{1}{2}$, $x \neq -\dfrac{3}{4}$.

6. Match the polynomial function with its graph:

   (a) $-x^3 + x^2 + 2x \longrightarrow IV$
   (b) $-x^4 + 1 \longrightarrow II$
   (c) $x^5 - 5x^3 + 4x \longrightarrow I$
   (d) $x^6 - 6x^4 + 9x^2 \longrightarrow III$

7. The x-intercept(s) of the function $y = x^2 - 2x - 8$ are

To find the x-intercepts just find the solutions to the equation $x^2 - 2x - 8 = 0$.

$$x^2 - 2x - 8 = (x-4)(x-2)$$

so the x-intercepts are 4 and 2.

Extra Credit: At the end of the day, the change machine at a laundrette contained at least \$3.20 and at most \$5.45 in nickels, dimes, and quarters. There were 3 fewer dimes than twice the number of nickels and 2 more quarters than twice the number of nickels. What was the least possible number and the greatest possible number of nickels?

HINT Let your unknown be the number of nickels and write the number of quarters and dimes in terms of that. Instead of worrying about arithmetic with decimals just remember that everything is in terms of dollars and cents, you know how to add money, I know you do!

Let n be the number of nickels so the number of dimes is $2n - 3$ and the number of quarters is $2n + 2$.

The total value of the change is $0.05n + 0.10(2n - 3) + 0.25(2n + 2)$.

We are given that the value of the change is at least \$3.20 and at most \$5.45 hence

$$\begin{array}{rcccl} 3.20 & \leq & 0.05n + 0.10(2n-3) + 0.25(2n+2) & \leq & 5.45 \\ 3.20 & \leq & 0.05n + 0.20n - 0.30 + 0.50n + 0.50 & \leq & 5.45 \\ 3.20 & \leq & 0.75n + 0.20 & \leq & 5.45 \\ 3.00 & \leq & 0.75n & \leq & 5.25 \\ \frac{3.00}{0.75} & \leq & n & \leq & \frac{5.25}{0.75} \end{array}$$

To solve for n, rather than worry about dividing by decimals, remember that this is just all money. So $\frac{3.00}{0.75}$ is just however many times 75 cents are in \$3.00, 4 and $\frac{5.25}{0.75}$ is just however many times 75 cents are in \$5.25, 7. So

$$4 \leq n \leq 7$$

There are between 4 and 7 nickels.