Mathematics 432/532 Midterm Answers

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Name ____

1. (a) Let γ and $\tau : I \to X$ be continuous paths beginning and ending at x_0. Define $\gamma \star \tau$.

   Answer: $\gamma \star \tau(u) = \begin{cases} \gamma(2u) & 0 \le u \le \frac{1}{2} \\ \tau(2u-1) & \frac{1}{2} \le u \le 1 \end{cases}$

   (b) Suppose the path τ is identically equal to x_0. Prove that $\gamma \star \tau$ is homotopic to γ. First give the idea of the proof in words or a picture, and then write down an explicit homotopy.

   Answer: Gradually slow down γ until at the end it takes all of u to go from the start to the end. If it finishes at $u(t) = \frac{1}{2} + \frac{t}{2}$, then

$$h(u,t) = \begin{cases} u\left(\frac{u}{\frac{1}{2}+\frac{t}{2}}\right) & 0 \le u \le \frac{1}{2} + \frac{t}{2} \\ x_0 & \frac{1}{2} + \frac{t}{2} \le u \le 1 \end{cases}$$

2. (a) If $A \subseteq X$ is a subspace and $r : X \to A$ is a strong deformation retraction, prove that $r_\star : \pi(A, a_0) \to \pi(X, a_0)$ is an isomorphism.

   Answer: Consider the sequence

$$\pi(A) \to \pi(X) \to \pi(A) \to \pi(X)$$

   The composition of the first two maps is the identity because $A \to X \xrightarrow{r} A$ is the identity, and the composition of the last two maps is the identity because $X \xrightarrow{r} A \to X$ is homotopic to the identity. So $\pi(A) \to \pi(X)$ is one-to-one and onto.

   (b) Calculate the fundamental group of projective space minus a point.

   Answer: Think of RP^2 as the unit disk with opposite boundary points identified, and remove the origin from this disk. The remaining points can be gradually pushed out to the boundary, so the boundary is a strong deformation retract of $RP^2 - \{0\}$. But the boundary is homeomorphic to S^1, even after we identify opposite points. By part a), the fundamental group of $RP^2 - \{0\}$ is isomorphic to $\pi(S^1) = Z$.

3. (a) Let $R \xrightarrow{\pi} S^1$ be the standard covering space given by $\pi(x) = e^{2\pi ix}$. If $\gamma : (I, 0) \to (S^1, 1)$ is a continuous path, sketch the proof that there is a lift $\tilde{\gamma} : (I, 0) \to (R, 0)$ such that $\pi \circ \tilde{\gamma} = \gamma$.

   Answer: By Lebesgue, we can subdivide I as $0 = t_0 < t_1 < \ldots < t_n = 1$ so $\gamma([t_{i-1}, t_i])$ is in an evenly covered open subset of S^1.

   We extend piece by piece. Suppose $\tilde{\gamma}$ is defined on $[0, t_{i-1}]$. Then $\gamma([t_{i-1}, t_i]) \subseteq \mathcal{U}$ where $\pi^{-1}(\mathcal{U}) = \cup\, \mathcal{U}_\alpha$. In particular, $\tilde{\gamma}(t_{i-1})$ belongs to one of the $\mathcal{U}_\alpha$, say $\mathcal{U}_\beta$. But $\pi : \mathcal{U}_\beta \to \mathcal{U}$ is a homeomorphism, so we can extend $\tilde{\gamma}$ to $[t_{i-1}, t_i]$ by $\pi^{-1} \circ \gamma$.

   (b) Explain very briefly how we use the previous result to obtain a map $\pi(S^1, 1) \to Z$.

   Answer: Given $\gamma : I \to S^1$ representing an element of $\pi(S^1)$, we lift to $\tilde{\gamma} : I \to R$ starting at $0 \in R$. This lift ends at a point over 1, and the points over 1 are the elements of Z. So map the represented element of $\pi(S^1)$ to $\tilde{\gamma}(1)$. It can be proved that homotopic elements map to the same point, essentially by the lifting property for $I \times I \to X$.

4. If $\tilde{X} \xrightarrow{\pi} X$ is a covering space, prove that $\pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is one-to-one. Hint: suppose γ and τ are closed paths in $\tilde{X}$ which map to the same element of $\pi(X, x_0)$. Then there is a homotopy $h : I \times I \to X$ from $\pi \circ \gamma$ to $\pi \circ \tau$ in X. Continue.

   Answer: Lift the homotopy to a map $\tilde{h} : I \times I \to \tilde{X}$ which sends 0×0 to $\tilde{x}_0$. Then $\tilde{h}(u, 0)$ is the unique lift of $\pi \circ \gamma$, and γ is such a lift. So $\tilde{h}(u, 0) = \gamma(u)$. Similarly $\tilde{h}(u, 1) = \tau(u)$ and $\tilde{h}(1, t) = \tilde{x}_0$. So $\tilde{h}$ is a homotopy between γ and τ and consequently they represent the same element of $\pi(\tilde{X})$.

5. Suppose $f : S^2 \to S^2$ has no fixed points. Explain why f is homotopic to the antipodal map. It suffices to describe the homotopy in words; you need not give a formula.

   Answer: Push $f(x)$ to $-x$ along the straight line joining the two, pushed out to S^2. Since $f(x)$ is not x, this line does not pass through the center. So

$$h(x, t) = \frac{(1-t)f(x) + t(-x)}{||(1-t)f(x) + t(-x)||}$$

6. Prove that any map $f : S^2 \to S^1$ is homotopic to a constant.

   Answer: Notice that S^2 is simply connected and locally pathwise connected. So we can using the extended lifting theorem to lift f to a map $\tilde{f} : S^2 \to R$. But R is convex, so $\tilde{f}$ is homotopy to a constant; compose this homotopy with $\pi : R \to S^1$ to get a homotopy between f and a constant map.

   Instead of using the general lifting theorem, it would suffice to use lifting for $f : I \times I \to S^1$, and notice that S^2 can be obtained from $I \times I$ by gluing all boundary points together. So f induces a map $I \times I \to S^1$ which then lifts.