Existence of a Universal Cover

Richard Koch

February 26, 2006

1 The Theorem

Theorem 1 *Suppose the topological space X is connected, locally pathwise connected, and semi-locally simply connected. Then X has a universal cover $\mathcal{C}$.*

Remark: All nice spaces satisfy these hypotheses, so the essential point is that every reasonable space has a universal cover.

Remark: The hypothesis that X be semi-locally simply connected is necessary. Indeed if $p \in X$, find an evenly covered neighborhood $\mathcal{U}$ of p. We claim that every loop in $\mathcal{U}$ starting at p is homotopic to a constant in X. Indeed find $q \in \mathcal{C}$ projecting to p and let $\mathcal{U}_\alpha \subseteq \mathcal{C}$ be the neighborhood of q corresponding to $\mathcal{U} \subseteq X$. We want to prove that $\pi(\mathcal{U}, p) \to \pi(X, p)$ is the zero map, but $\pi : \mathcal{U}_\alpha \to \mathcal{U}$ is a homeomorphism so it suffices to prove that $\pi(\mathcal{U}_\alpha, q) \to \pi(\mathcal{U}, p) \to \pi(X, p)$ is the zero map. This map can be written as the composition $\pi(\mathcal{U}_\alpha, q) \to \pi(\mathcal{C}, q) \to \pi(X, p)$ and this map is zero because $\mathcal{C}$ is simply connected.

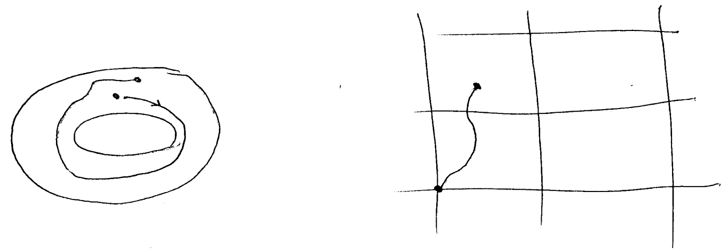
2 Motivating the Proof

If we only know X, how can we get our hands on points of its univeral cover $\mathcal{C}$?

Suppose we had a point $c \in \mathcal{C}$ over a point $x \in X$. We may as well assume that $\mathcal{C}$ has a base point c_0 over a base point $x_0 \in X$. Choose a path $\gamma : I \to \mathcal{C}$ starting at c_0 and ending at c. Then $\pi \circ \gamma$ will be a path in X starting at x_0 and ending at x. *But conversely, if we had $\pi \circ \gamma$ in X, we could uniquely lift it to $\mathcal{C}$ and thus find c.*

If we had two paths γ and τ in $\mathcal{C}$ from c_0 to c, these paths would be homotopic with fixed endpoints because $\mathcal{C}$ is simply connected, and therefore $\pi \circ \gamma$ and $\pi \circ \tau$ would be homotopic in X. *But conversely if $\pi \circ \gamma$ and $\pi \circ \tau$ were homotopic in X, then γ and τ would end at the same point of $\mathcal{C}$ by lifting the homotopy $h : I \times I \to X$ to $\mathcal{C}$.*

Putting all of this together, we discover that a point in $\mathcal{C}$ over x is uniquely determined by a homotopy class of paths in X which begin at x_0 and end at x.



Definition 1 *Define an equivalence relation $\sim$ on paths in X which start at x_0 by $\gamma \sim \tau$ if γ and τ end at the same point and are homotopic in X with fixed endpoints. Let*

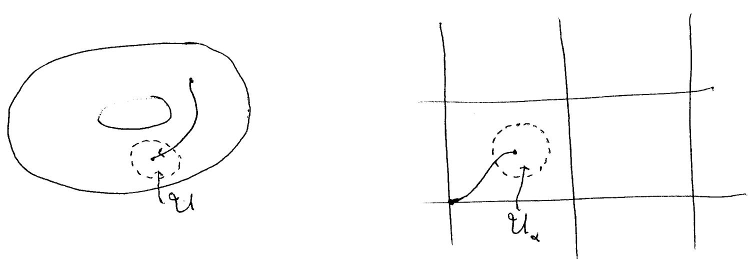
$$\mathcal{C} = \{\gamma : I \to X \mid \gamma(0) = x_0\} \ / \ \gamma \text{ and } \tau \text{ are identified if } \gamma \sim \tau$$

Define $\pi : \mathcal{C} \to X$ by $\gamma \to \gamma(1)$.

3 A Topology on $\mathcal{C}$

Next we want to define a topology on $\mathcal{C}$. It suffices to define small open sets homeomorphic under π to open sets in X, since the general open set will be a union of these.

Examine the picture below. In this picture, γ is a fixed path in X starting at x_0 and ending at x, and $\mathcal{U}$ is a path connected evenly covered open neighborhood of x in X. Also $\tilde{\gamma}$ is the lift of γ and $\tilde{\mathcal{U}}$ is the open set which projects homeomorphically to $\mathcal{U}$ and contains the end of $\tilde{\gamma}$.



When if we construct a curve by following γ to its end and then continuing along a path entirely in $\mathcal{U}$, the lift of this path will follow $\tilde{\gamma}$ to its end and then remain entirely in $\tilde{\mathcal{U}}$. So the end of the lift will belong to $\tilde{\mathcal{U}}$, which can be defined as lifts of appropriate paths.

In practice it is not necessary to restrict $\mathcal{U}$ to be path connected and evenly covered until a later stage of the argument. So:

Definition 2 *Let* $\gamma : I \to X$ *be a path starting at* x_0 *and let* $\mathcal{U}$ *be an open neighborhood of the end of* γ *in* X*. Then* $< \mathcal{U}, \gamma >$ *is the set of points of* $\mathcal{C}$ *which can be represented by paths of the form* $\gamma \star \alpha$ *where* α *is a path entirely in* $\mathcal{U}$ *starting at the end of* γ*.*

Definition 3 *A subset of* $\mathcal{C}$ *is said to be* open *if it is a union of sets of the form* $< \mathcal{U}, \gamma >$.

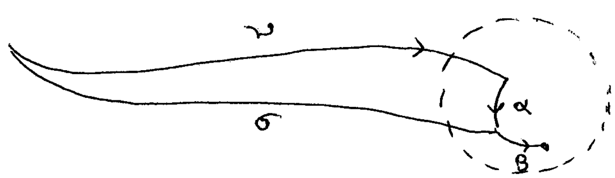
Theorem 2 *This is a topology on* $\mathcal{C}$*. Under this topology,* $\pi : \mathcal{C} \to X$ *is continuous.*

Proof: All of the axioms are trivial except the intersection action (why?). To prove that the intersection of two arbitrary open sets is open, it suffices to prove that the intersection of $< \mathcal{U}, \gamma >$ and $< \mathcal{V}, \tau >$ is open (why?). To prove this, it suffices to prove that if the equivalence class of σ is in $< \mathcal{U}, \gamma > \cap < \mathcal{V}, \tau >$, then there is an open neighborhood $\mathcal{W}$ of the end of σ such that $< \mathcal{W}, \sigma > \subseteq \Big(< \mathcal{U}, \gamma > \cap < \mathcal{V}, \tau > \Big)$ (why?). We will show that $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ has this property. By symmetry, it suffices to prove that $< \mathcal{U} \cap \mathcal{V}, \sigma > \subseteq < \mathcal{U}, \gamma >$. Since $< \mathcal{U} \cap \mathcal{V}, \sigma > \subseteq < \mathcal{U}, \sigma >$, it suffices to prove

Lemma 1 *If* $\sigma \in < \mathcal{U}, \gamma >$*, then* $< \mathcal{U}, \sigma > \subseteq < \mathcal{U}, \gamma >$

Proof: By assumption the path σ is homotopic to $\gamma \star \alpha$ where α is a path entirely in $\mathcal{U}$ starting at the end of γ. This homotopy fixes endpoints, and in particular σ and $\gamma \star \alpha$ end at the same point.

A typical element of $< \mathcal{U}, \sigma >$ has a representative of the form $\sigma \star \beta$ where β is a path entirely in $\mathcal{U}$ starting at the end of σ. Then $\sigma \star \beta$ is homotopic to $(\gamma \star \alpha) \star \beta$ by a homotopy which only moves $\gamma \star \alpha$ to σ and leaves β fixed. But $(\gamma \star \alpha) \star \beta$ is homotopic to $\gamma \star (\alpha \star \beta)$ and $\alpha \star \beta$ is a path entirely in $\mathcal{U}$. Hence $\gamma \star (\alpha \star \beta)$ represents an element of $< \mathcal{U}, \gamma >$ and so $\sigma \star \beta$ is in this set.



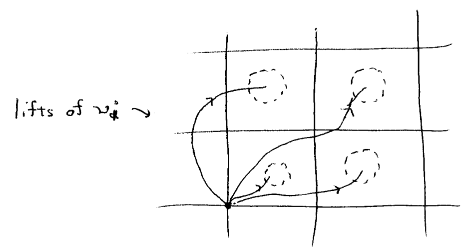
Completion of the proof of theorem 2: Finally, $\pi : \mathcal{C} \to X$ is continuous, for if $\mathcal{U}$ is open in X, then $\pi^{-1}(\mathcal{U}) = \cup \ < \mathcal{U}, \gamma >$ over all paths γ which end in $\mathcal{U}$.

4 $\mathcal{C} \to X$ is a Covering Space

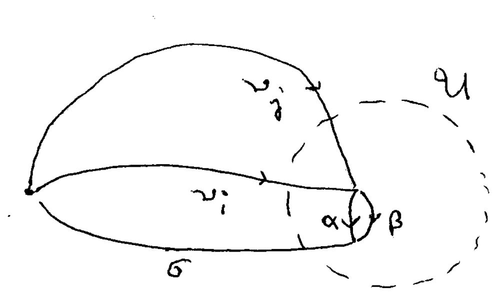
To complete the proof of our main theorem, it suffices to show that $\mathcal{C} \to X$ is a covering space, and that $\mathcal{C}$ is simply connected. The more difficult of these steps is the first one.

Given $x \in X$, choose a semi-locally simply connected open neighborhood $\mathcal{U}$ of x. This exists by hypothesis. Shrink $\mathcal{U}$ to a smaller neighborhood of x which is pathwise connected; this is possible since X is locally pathwise connected. Notice that shrinking a semi-locally simply connected set produces another such set. We will prove that $\mathcal{U}$ is evenly covered.

Separate the paths from x_0 to x into homotopy equivalence classes and let $\{\gamma_i\}$ be a complete set of representatives. We will prove that the inverse image of $\mathcal{U}$ in $\mathcal{C}$ is the union of the $< \mathcal{U}, \gamma_i >$, that each of these sets is homeomorphic to $\mathcal{U}$, and that they are disjoint.



We'll prove them disjoint first. Suppose $p \in \Big(< \mathcal{U}, \gamma_i > \ \cap \ < \mathcal{U}, \gamma_j > \Big)$. Suppose p is represented by a path σ. Since $p \in < \mathcal{U}, \gamma_i >$, there is a path α entirely in $\mathcal{U}$ such that $\sigma \sim \gamma_i \star \alpha$. Similarly there is a path β entirely in $\mathcal{U}$ such that $\sigma \sim \gamma_j \star \beta$. Notice that α and β begin and end at the same points.



Let $\overline{\beta}$ be β traced backward. Then $\sigma \star \overline{\beta} \sim \gamma_i \star \alpha \star \overline{\beta}$ and $\sigma \star \overline{\beta} \sim \gamma_j \star \beta \star \overline{\beta}$ and so

$$\gamma_i \star \alpha \star \overline{\beta} \sim \gamma_j \star \beta \star \overline{\beta}$$

But $\beta \star \overline{\beta}$ is homotopic to a constant, and so the right side of this similarity is homotopic to γ_j. Also $\alpha \star \overline{\beta}$ is a loop at x and by hypothesis such loops are homotopic to constants in X. So the left side of the above similarity is homotopic to γ_i. Thus $\gamma_i \sim \gamma_j$, contradicting the choice of the γ_i.

From here on everything is easy. We must prove $< \mathcal{U}, \gamma_i > \to \mathcal{U}$ a homeomorphism. It is onto because $\mathcal{U}$ is pathwise connected, so any $u \in \mathcal{U}$ is the end of a path α starting at x and thus the image of an element $\gamma_i \star \alpha$ in $< \mathcal{U}, \gamma_i >$. It is one-to-one, for if $\gamma_i \star \alpha$ and $\gamma_i \star \beta$ map to the same point, then α and β begin and end at the same point of $\mathcal{U}$. Since the loop $\alpha \star \overline{\beta}$ is homotopic to a constant in X, α and β are homotopic in X with fixed endpoints, and so $\gamma_i \star \alpha$ and $\gamma_i \star \beta$ are homotopic paths and represent the same point of $\mathcal{C}$.

The map $< \mathcal{U}, \gamma_i > \to \mathcal{U}$ is continuous because $\pi : \mathcal{C} \to X$ is continuous. It is an open map, and thus has continuous inverse, because every open set in $\mathcal{C}$ is a union of $< \mathcal{U}, \tau >$ with $\mathcal{U}$ pathwise connected (why?) and the image of such a $< \mathcal{U}, \tau >$ is $\mathcal{U}$.

5 $\mathcal{C}$ is Pathwise Connected

The final step of the proof is fun because it is hokus-pokus. You'll need to sit in a closet and see if you buy it.

We must prove that $\mathcal{C}$ is connected, and that every loop beginning and ending at c_0 is homotopic to a constant.

We'll prove $\mathcal{C}$ pathwise connected. Let $c \in \mathcal{C}$. Then c is an equivalence class of paths in X; let $\gamma : I \to X$ be a representative of c. For each $u \in I$, consider the path γ_u from x_0 to $\gamma(u)$ obtained by following the first part of γ to $\gamma(u)$. We want this path to be parameterized by $[0, 1]$ so write

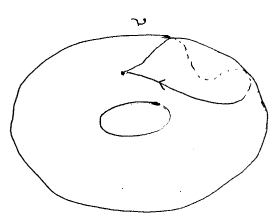
$$\gamma_u(t) = \gamma(ut) \qquad 0 \le t \le 1$$

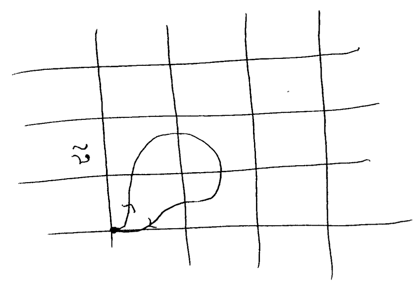
Then each γ_u represents a point of $\mathcal{C}$. We can then think of the entire collection of γ_u's as a path in $\mathcal{C}$; this path assigns to u the point of $\mathcal{C}$ represented by the path γ_u. (Please think about this slight of hand until you understand it; it is the central point of the arguments which follow.) We call this new path $\tilde{\gamma}$ because it defines a lift of γ to $\mathcal{C}$. So $\tilde{\gamma}(u)$ is the point of $\mathcal{C}$ represented by γ_u and thus by the portion of γ from x_0 to $\gamma(u)$.

Notice that γ_0 is the path which is constantly x_0; this path represents the base point $c_0 \in \mathcal{C}$. Thus $\tilde{\gamma}$ is a path in $\mathcal{C}$ from c_0 to the element c represented by γ. So $\mathcal{C}$ is pathwise connected.

6 $\mathcal{C}$ is Simply Connected

Now we prove $\mathcal{C}$ simply connected. Let $\tilde{\gamma}$ be a loop in $\mathcal{C}$ starting and ending at c_0. Project this loop to a path γ in X beginning and ending at x_0. Then for each $u \in [0,1]$ we can consider the first portion of the path γ from $\gamma(0)$ to $\gamma(u)$; earlier we called this path γ_u. This path represents a point of $\mathcal{C}$; as u varies we obtain a path in $\mathcal{C}$. As before, this path represents $\tilde{\gamma}$ and is thus another way to think about the curve $\tilde{\gamma}$ which started this paragraph.





Said another way, $\tilde{\gamma}(u)$ is a point in $\mathcal{C}$ and so represented by a curve; this curve is γ_u.

In particular, $\tilde{\gamma}(1) = \gamma_1 = \gamma$. Since $\tilde{\gamma}$ is a closed curve, it ends at c_0. So γ is one representative for c_0. But the path which is constantly x_0 is another representative, so these two paths must be homotopic. Thus γ is homotopic to a constant through closed curves in X.

We already know that $\mathcal{C} \to X$ is a covering space. On our midterm, we proved that consequently $\pi(\mathcal{C}, c_0) \to \pi(X, x_0)$ is one-to-one. Since $\tilde{\gamma} \in \pi(\mathcal{C}, c_0)$ maps to $\gamma \in \pi(X, x_0)$ which is homotopically trivial, $\tilde{\gamma}$ must itself be homotopically trivial. QED.