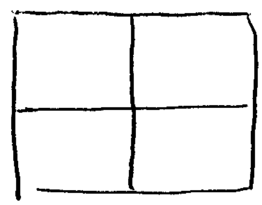
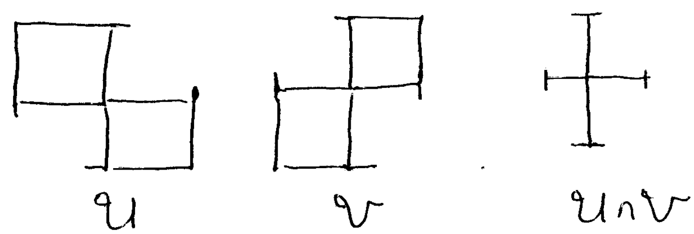
Assignment 9; Due Friday, March 17

24.4b: A picture of this set is shown below. Note that the set only contains points on the lines; internal points are missing.



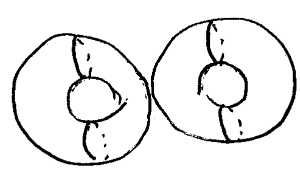
Below are choices for $\mathcal{U}$ and $\mathcal{V}$. Notice that $\mathcal{U}, \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V}$ are arcwise connected.



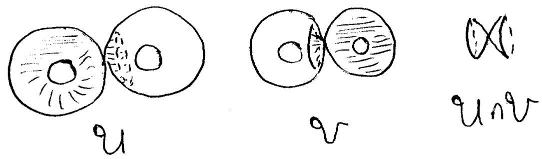
Both $\mathcal{U}$ and $\mathcal{V}$ can be strongly deformed to a bouquet of two circles, so their fundamental groups are the free groups $F(a, b)$ and $F(c, d)$. Since $\mathcal{U} \cap \mathcal{V}$ can be contracted to a point, its fundamental group is trivial. So there are no relations and the fundamental group of our set is the free group $F(a, b, c, d)$.

This is not surprising since our space is *almost* a bouquet of four circles.

24.4d: The space consists of two tori joined at a point, as below.



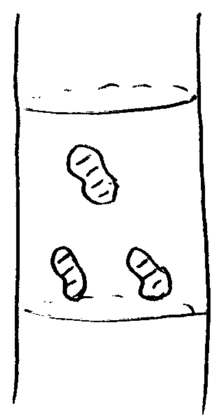
Choose $\mathcal{U}$ to be the entire first torus together with a collar about the common point in the second torus. Choose $\mathcal{V}$ similarly. See the pictures below.



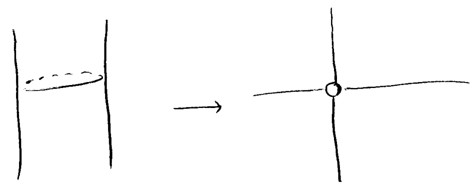
Notice that $\mathcal{U}$ can be strongly deformed to a torus, so its fundamental group is $Z \times Z$. Think of this group as the free abelian group on two generators, which we write $FA(a, b)$. Each element of this group can be written uniquely as a power of a followed by a power of b: $a^m b^n$. A similar statement holds for $\mathcal{V}$; its fundamental group is $FA(c, d)$. The set $\mathcal{U} \cap \mathcal{V}$ can be deformed to a point, so its fundamental group is trivial and there are no relations. Thus the fundamental group of the join of the two circles is the free product $FA(a, b) \star FA(c, d)$. A typical element of this group has the form

$$(a^3 b^2)(c^5 d^{-2})(a^5 b)(c^2 d^2)(ab^2)(cd) \dots$$

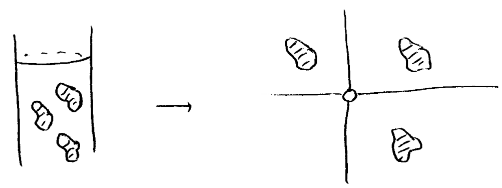
24.4e: Below is a picture when $n = 2$. The space $S^1 \times R$ is a cylinder and we are to remove k disks.



We get a different answer when $n = 2$ and when $n \geq 3$. We'll first do the calculation when $n = 2$. Recall that a cylinder $S^1 \times R$ is homeomorphic to a punctured plane $R^2 - \{0\}$ by using polar coordinates in the second space. Indeed a point in the punctured plane can be described by an angle $\theta \in S^1$ and a real $0 < r < \infty$. We can map the punctured plane to the cylinder by sending $(\theta, r) \to \theta \times \ln(r)$.

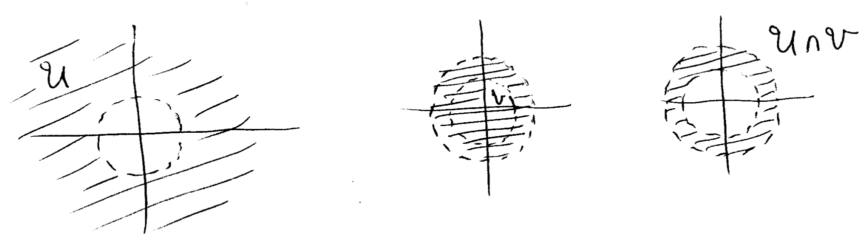


Hence removing k disks from $S^1 \times R$ is the same as removing $k+1$ disks from the plane. This space can be strongly deformed to a bouquet of $k+1$ circles, so the fundamental group is the free group on $k+1$ generators.



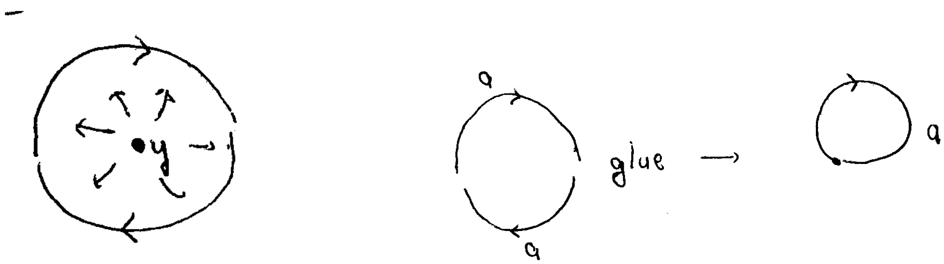
When $n \geq 3$, we can still find a homeomorphism from $S^{n-1} \times R$ to $R^n - \{0\}$, so the space of interest is still R^n minus $k+1$ disks. We will prove by induction on k that the fundamental group of this object is trivial.

Let $\mathcal{U}$ be R^n minus all $k+1$ disks, and let $\mathcal{V}$ one of these disks slightly enlarged to a bigger open disk. Notice that $\mathcal{V}$ is contractible and its fundamental group is zero.



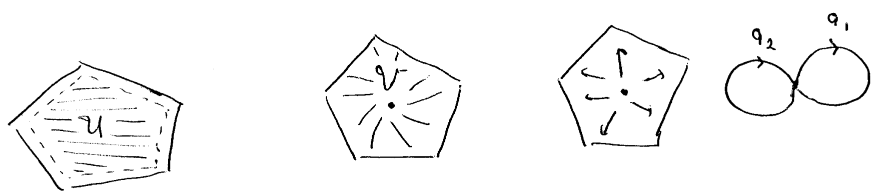
The space $\mathcal{U} \cap \mathcal{V}$ is an annulus which can be strongly deformed to a sphere S^{n-1}. Since $n \geq 3$, the fundamental group of this sphere is trivial, so $\pi(\mathcal{U} \cap \mathcal{V}) = 0$. Thus there are no relations and the Seifert-Van Kampen theorem states that $\pi(\mathcal{U} \cup \mathcal{V}) = \pi(\mathcal{U})$. But $\mathcal{U} \cup \mathcal{V}$ is R^n minus k holes and $\mathcal{U}$ is R^n minus $k+1$ holes.

24.4k: The picture below shows that $RP^2 - \{y\}$ can be strongly deformed to a circle, so its fundamental group equals the fundamental group of a circle, which is Z.

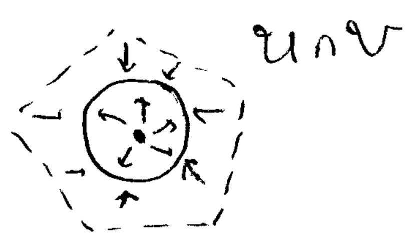


25.1b: Techniques from the end of last term show that all vertices of the polygon are glued to the same point.

Let $\mathcal{U}$ be the interior of the pentagon, and let $\mathcal{V}$ be the entire pentagon with boundary glued together, minus the center. Note that $\mathcal{U} \cap \mathcal{V}$ is the interior of the pentagon minus the center, which can be deformed to a circle. Note that $\mathcal{U}$ is contractible, and $\mathcal{V}$ can be retracted to the boundary, which is a bouquet of two circles. Hence the fundamental group of $\mathcal{U}$ is trivial and the fundamental group of $\mathcal{V}$ is $F(a_1, a_2)$.



The fundamental group of $\mathcal{U} \cap \mathcal{V}$ is Z, and the map $\pi(\mathcal{U} \cap \mathcal{V}) \to \pi(\mathcal{V})$ sends the generator of Z to $a_1^2 a_2^{-1} a_1 a_2$ So the fundamental group of this space is $F(a_1, a_2)/a_1 a_2 a_1^2 = a_2$.

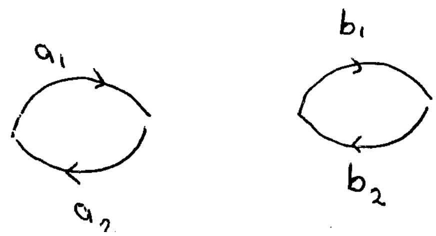


25.1f-a: Once again, techniques from last term show that all vertices are glued to the same point.

Exactly the method of the previous problem works here. The boundary of this polygon is again a bouquet of two circles a_1 and a_2, and the generator of $\pi(\mathcal{U} \cap \mathcal{V}) = Z$ maps to $a_1^2 a_2^{-1} a_1^{-2} a_2$. Hence the required fundamental group is $F(a_1, a_2)$ with relation $a_1^2 a_2^{-1} = a_2^{-1} a_1^2$. We can replace the generator a_2 by a_2^{-1} and get the relation $a_1^2 a_2 = a_2 a_1^2$.

25.1f-h: I interprete this space as the region between the inner circle and the outer circle.

This time we must be careful because it is no longer true that all vertices glue together. Indeed in both the inner and outer circle, there are two vertices and each edge goes from one to the other. Hence the inner and outer boundaries are circles. Note that a_1 does not generate an element of any fundamental group because it doesn't start and end at the same point. But $a_1 \star a_2$ does define an element of a fundamental group.



Let $\mathcal{U}$ be the entire set minus the exterior boundary, and let $\mathcal{V}$ be the entire set minus the interior boundary. Note that $\mathcal{U}$ can be deformed to the interior boundary and V can be deformed to the exterior boundary. So $\pi(\mathcal{U})$ is Z with generator $a_1 \star a_2$ and $\pi(\mathcal{V})$ is Z with generator $b_1 \star b_2$.

The set $\mathcal{U} \cap \mathcal{V}$ is the space minus both boundaries. This can be deformed to a circle. The generator of this circle maps to $(b_1 \star b_2)^2$ in $\pi(\mathcal{V})$ and maps to $(a_1 \star a_2)^3$ in $\pi(U)$. So the fundamental group is $F(a_1 \star a_2, b_1 \star b_2)$ with relation $(a_1 \star a_2)^3 = (b_1 \star b_2)^2$. If we let $C = a_1 \star a_2$ and $D = b_1 \star b_2$, then this group is $F(C, D)/C^3 = D^2$. Notice that this is exactly the group of the trefoil knot.

Extra Exercise: Clearly h^p is the identity since $\left(e^{\frac{2\pi i}{p}}\right)^p = 1$ and $\left(e^{\frac{2\pi i q}{p}}\right)^p = 1$.

Suppose $h^k(z_1, z_2) = (z_1, z_2)$ where $1 \le k < p$. If $z_1 \neq 0$, we conclude that $e^{\frac{2\pi i k}{p}} = 1$. This can only happen if p divides k, which is doesn't.

If $z_1 = 0$, then $z_2 \neq 0$ and we conclude that $e^{\frac{2\pi i k q}{p}} = 1$. This can only happen if p divides kq. Since p and q are relatively prime, we again conclude that p divides k, which is false.

Extra Exercises Continued: We show that $S^3 \to S^3/Z_p$ is a covering space as follows. Pick a point $p \in S^3$. We will show that there is an open neighborhood $\mathcal{U}$ of p in S^3 such that $\mathcal{U}, h(\mathcal{U}), h^2(\mathcal{U}), \ldots, h^{p-1}(\mathcal{U})$ are disjoint. This is the key observation.

Suppose we succeed. I claim that $\pi : \mathcal{U} \to \pi(\mathcal{U}) \subseteq S^3/Z_p$ is a homeomorphism, and the inverse image of this set is exactly the collection of sets $\mathcal{U}, h(\mathcal{U}), \ldots, h^{p-1}(\mathcal{U})$, each homeomorphic to $\pi(\mathcal{U})$. Indeed, the map $\pi : \mathcal{U} \to \pi(\mathcal{U})$ is certainly onto; it is one-to-one because if two points p and q map to the same point, then there is a k such that $q = h^k(p)$, but $\mathcal{U} \cap h^k(\mathcal{U}) = \emptyset$. The remaining assertions are easily checked.

Next we prove that there is a $\mathcal{U}$ with $\mathcal{U}, h(\mathcal{U}), h^2(\mathcal{U}), \ldots, h^{p-1}(\mathcal{U})$ disjoint. If this assertion is not correct, then for any neighborhood $\mathcal{U}$ of p we can find unequal integers m and n between 0 and $p-1$ such that $h^m(\mathcal{U}) \cap h^n(\mathcal{U}) \neq \emptyset$. Thus we can find points q and r in $\mathcal{U}$ with $h^m(q) = h^n(r)$. Without loss of generality suppose $n > m$, and notice that $q = h^{n-m}(r)$. Since p does not divide $n - m$, h^{n-m} has no fixed points, so q and r are distinct.

If no neighborhood $\mathcal{U}$ of p works, we can find a sequence of unequal pairs q_n and r_n such that $q_n \to p$ and $r_n \to p$, and $q_n = h^{k_n}(r_n)$ for k_n between 1 and $p - 1$. Since only finitely many k_n are available, we can find a subsequence of p_n and q_n such that all k_n are equal to some fixed k. Then $q_n \to p$ and $r_n \to p$ and $q_n = h^k(r_n)$. By continuity, $p = h^k(p)$, contradicting the assertion that h^k has no fixed points.

Extra Exercises Continued:

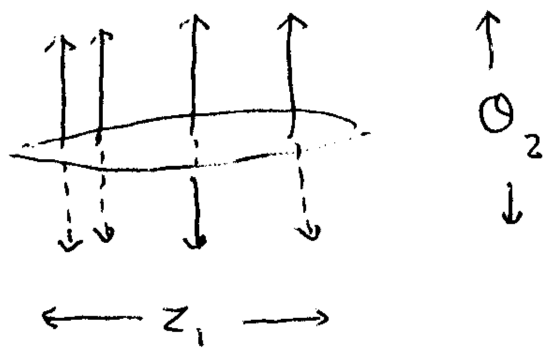
Since S^3 is simply connected, $\pi(S^3/Z_p)$ is isomorphic to the deck transformation group, and this group is obviously the set $\{id, h, h^2, \ldots, h^{p-1}\} \sim Z_p$ since each of this is a deck transformation, and there are no other deck transformations because this set already acts transitively on $\pi^{-1}(p)$ for $p \in \mathcal{U}$.

When $p = 2$ our map is $h(z_1, z_2) = (-z_1, -z_2)$, so S^3/Z_2 is obtained by gluing opposite points together, and thus equals RP^3.

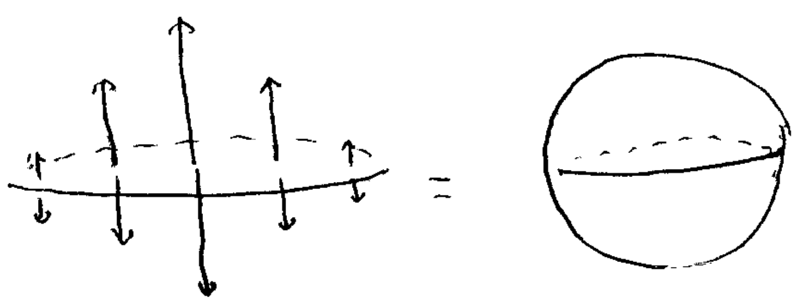
Extra Exercise Concluded: This is the fun part, and I intend to describe some initial tries and false steps before explaining the method ultimately used.

The first goal is to ignore the group Z_p and try to get the full S^3 to look like a ball B^3 with the upper hemisphere glued to the lower hemisphere.

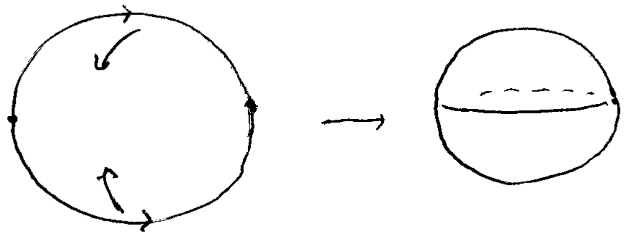
Consider $S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$. Notice that a typical $z \in C$ is given in polar coordinates by giving its radius $|z|$ and its angle θ. We can assume that $-\pi <= \theta <= \pi$, but we must then glue $-\pi$ to π. In our case it is not necessary to give both $|z_1|$ and $|z_2|$ because these quantities are related by $|z_1|^2 + |z_2|^2 = 1$. So a point in S^3 is completely described by giving z_1 with $0 \leq |z_1| \leq 1$ and then giving a second angle θ_2. We can draw this by attaching a line segment $-\pi \leq \theta_2 \leq \pi$ to each point of the z_1 disk. See the picture below.



However, when $|z_1| = 1$, then $|z_2| = 0$ and there is only one point rather than an interval of points. We can picture this by drawing the intervals $-\pi \leq \theta_2 \leq \pi$ shorter and shorter near the boundary as in the picture below.



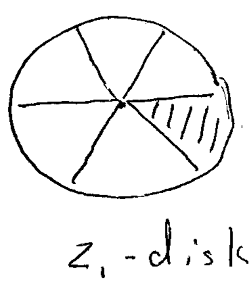
We still must remember that $\theta_2 = -\pi$ is the same point as $\theta_2 = \pi$. So to get S^3 from a ball, we must glue each point on the upper hemisphere to the point immediately below it on the lower hemisphere. This is analogous to constructing S^2 from a two-dimensional disk by gluing the semicircle at the top to the semicircle at the bottom as shown below.

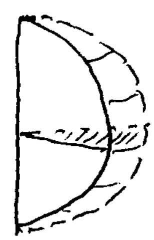


Conclusion II:

Already this construction looks like our previous construction of a lens space. To push things further, let us bring the group Z_p into the picture. We wish to draw a shaded subregion of S^3 such that every point in S^3 is equivalent under Z_p to a unique point in the shaded region, except for identifications along the boundary.

Let us concentrate on the action of Z_p on the z_1 disk. Here the group acts by rotation by $\frac{2\pi}{p}$. Every point in the disk is equivalent to a point in a shaded region below. Once we know where the z_1 component goes under the map $h \in Z_p$, the action on z_2 is completely determined. So every point in S^3 is equivalent to a unique point in the wedge below, except for boundary points. Thus we can take this wedge as our "shaded region."





I like to think of this piece as a wedge of an orange. We must examine gluing along the boundary. We already know that the points on the upper hemisphere must be glued to corresponding points on the lower hemisphere. If we do that, our wedge folds around and becomes a sort of lens with a sharp corner along the entire outside. This sharp corner corresponds to the equator of the lens, and the sides of the orange have become the top and bottom of the lens. We still must examine the manner that Z_p forces us to glue this top and bottom together (or equivalently, the sides of the orange together).

Let us concentrate on the action of the group Z_p on the equator, and thus the action on the original line over $z_1 = 0$. Notice that this center is a singular place where the z_1 action of Z_p is trivial and the z_2 action becomes significant. This Z_p action is multiplication by multiples of $e^{\frac{2\pi i q}{p}}$. To understand this multiplication, we must break the equator into p equal pieces; the multiplication rotates each piece by q units. This begins to look like a lens space.

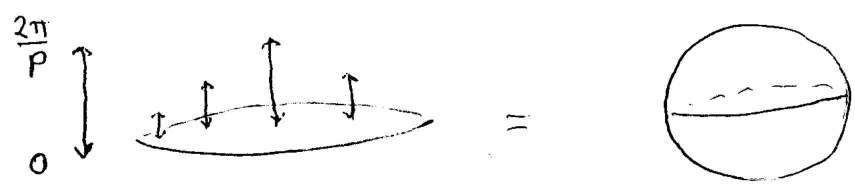
By continuing to the end, this method can probably be made to work. However, at this point I'll change to a different strategy.

Conclusion III: Instead of obtaining a shaded region by cutting the z_1 disk into wedges, let us obtain it by cutting the angle θ_2 into pieces. This is a little easier to do if we think of θ_2 as moving from 0 to 2π. Thus we break this into subintervals

$$[0, 2\pi] = \left[0, \frac{2\pi}{p}\right] \cup \left[\frac{2\pi}{p}, \frac{2\pi \cdot 2}{p}\right] \cup \ldots \cup \left[\frac{2\pi \cdot (p-1)}{p}, \frac{2\pi \cdot p}{p}\right]$$

For most points, the action of Z_p is completely determined by its action on the second component, so we can get a shaded region by allowing any z_1, but suitably restricting θ_2.

Thus our shaded region can be the set of all (z_1, θ) with z_1 in the unit disk and θ in the interval $\left[0, \frac{2\pi}{p}\right]$. There is still a restriction that θ collapses at the boundary where $|z_1| = 1$ and so $|z_2| = 0$. Notice that this fundamental region looks like a ball B^3, so we are definitely making progress.

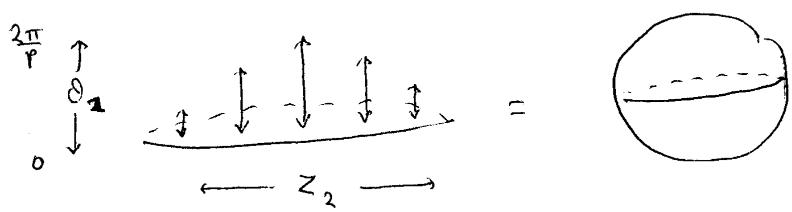


We need only glue boundaries of this set, which involves gluing the top to the bottom. Usually if we apply $h^k \in Z_p$ to a point in the shaded region, we get a point in a completely different region, so we can ignore h^k. But sometimes if we take a boundary point of the shaded region and apply h^k, we get another boundary point of the shaded region. In that special case, we must glue the two boundary points together.

Indeed, notice that h^k maps $\theta_2 = 0$ to $\theta_2 = \frac{2\pi qk}{p}$. If this second angle is $\frac{2\pi}{p}$ modulo multiples of 2π, then h^k glues the bottom of the ball to the top of the ball, while simultaneously rotating z_1 by k clicks. So this looks very promising.

But the q is annoying. A slight modification will simplify matters.

Conclusion IV; the Real Conclusion Instead of letting z_1 belong to a disk and describing z_2 by giving θ_2, let us allow z_2 to belong to a disk and describe z_1 by giving θ_1. To get a shaded region, let us divide the θ_1 interval into p pieces just as we earlier divided the θ_2 interval. Our picture is exactly the same as before.



We now ask how h^k could map a boundary point of the shaded region to another boundary point. Since h acts on θ_1 by $\theta_1 \rightarrow \theta_1 + \frac{2\pi}{p}$, the only maps h^k which send points in the shaded region to other such points are h and $h^{-1} = h^{p-1}$. Restrict attention to the first map h. It maps the lower boundary $\theta_1 = 0$ to the upper boundary $\theta_1 = \frac{2\pi}{p}$. We can ignore h^{p-1} because it just does this same gluing in reverse.

But notice that when the lower boundary is glued to the upper one, h is simultaneously rotating z_2 by $\frac{2\pi q}{p}$. So the rule is that we glue the lower hemisphere to the upper one while simultaneously rotating about the z-axis counterclockwise by q clicks. Equivalently, we glue the upper hemisphere to the lower one while simultaneously rotating about the z axis clockwise by q clicks. This is exactly the description of the lens space $L(p, q)$. Whew.