Assignment 8; Due Friday, March 10

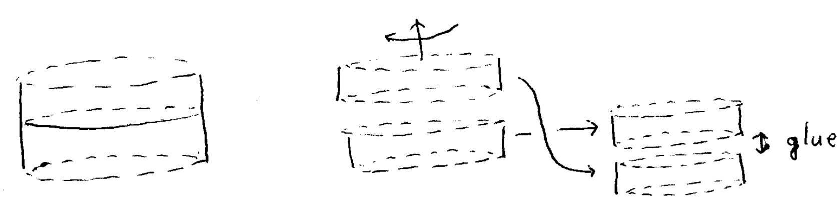
Exercise 1: If p is in the interior of the ball, p is equivalent only to itself, so the interior of the ball automatically has a neighborhood homeomorphic to an open subset of R^3.

There are many ways to deal with points on the boundary of the ball. Here is one of them. Remove the equator from the ball but keep all other points. Map the remaining set as follows:

$$(x, y, z) \to \left(x, y, \frac{z}{\sqrt{1 - x^2 - y^2}}\right)$$

Notice that when (x, y, z) is on the boundary, z satisfies $z = \pm\sqrt{1 - x^2 - y^2}$ and thus the image has the form $(x, y, \pm 1)$. Consequently, our map is a homeomorphism from the closed ball minus the equator to $\{(x, y, z) \mid x^2 + y^2 < 1 \text{ and } -1 \le z \le 1\}$.

This new set looks like a muffin. Cut it into two pieces along the xy-plane. Rotate the top piece about z by θ. Then glue the top face of the top muffin to the bottom face of the bottom muffin as indicated below. In the end, points on the top and bottom of the original muffin are now in the middle of a new muffin, which forms an open Euclidean neighborhood of these points.



Exercise 2: We will use the following theorem of Kronecker:

Theorem 1 *Identify angles with points in S^1. Suppose θ is an irrational multiple of 2π. Then the set $\{0, \pm\theta, \pm 2\theta, \pm 3\theta, \ldots\}$ is dense in S^1.*

Proof: Let A be this set together with all limit points of the set. Then A is closed in S^1. If the complement is not empty, it is open and thus a disjoint union of intervals $\tau_0 < \tau < \tau_1$ where τ_0 and τ_1 are in A.

Notice that A is invariant under the map $\tau \to \tau + m\theta$ for an integer m. Hence the complement is also invariant under this map, so if (τ_0, τ_1) is one of the intervals, then so is $(\tau_0 + m\theta, \tau_1 + m\theta)$. Each of these intervals is a maximal connected open set in the complement of A because its endpoints belong to A. If m and n are distinct integers, then the intervals $(\tau_0 + m\theta, \tau_1 + m\theta)$ and $(\tau_0 + n\theta, \tau_1 + n\theta)$ are either disjoint or identical, for otherwise their union would be a larger connected open interval. But if the two sets are identical, then they have the same endpoints, so $\tau_0 + m\theta$ and $\tau_0 + n\theta$ agree up to a multiple of 2π. But then $\tau_0 + m\theta = \tau_0 + n\theta + k(2\pi)$ and $\theta = \frac{k}{m-n}(2\pi)$ contradicting the assumption that θ is an irrational multiple of 2π.

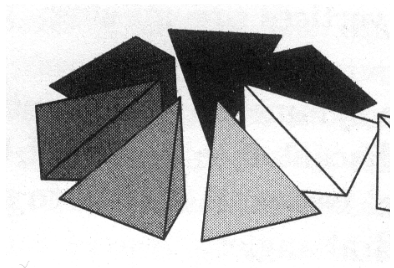
It follows that the sets $(\tau_0 + m\theta, \tau_1 + m\theta)$ form an infinite family of disjoint intervals in S^1, all of the same length. This is certainly impossible, so A must be everything in S^1. QED.

We use this result to solve the exercise. Consider the points on the equator equivalent to $1 \in S^1$. By Kronecker's theorem, these points are dense. They all represent the same point, say p, in the quotient space. On the other hand, there are only countably many such points, and thus the quotient space has a point $q \neq p$ with representatives in the equator.

If the quotient space were Hausdorff, then we could find disjoint open neighborhoods of q and p. Call the inverse images of these open sets $\mathcal{U}$ and $\mathcal{V}$. Then all points representing p belong to $\mathcal{U}$, and points representing q are in $\mathcal{V}$. Since the points representing p are dense in the equator, one of these points must belong to $\mathcal{V}$, contradicting the assumption that $\mathcal{U}$ and $\mathcal{V}$ are disjoint.

Exercise 3: It remains to show that points coming from the equator in the quotient space have neighborhoods homeomorphic to open sets in R^3. Fix one such point, p.

Flatten the sphere to a lens shape, as shown in the following picture from Thurston's book. This lens can be rotated arbitrarily; rotate so the point p is in the interior of an edge around the equator, rather than at an endpoint of such an edge.

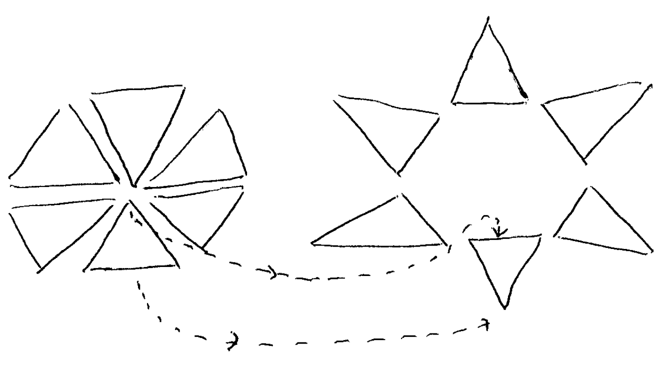


Thurston shows this lens broken into (nonregular) tetrahedra. For $L(p, q)$ there should be p such tetrahedra.

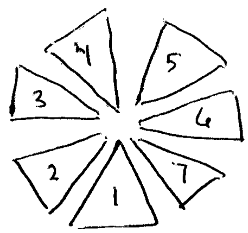
The gluing operation which forms $L(p, q)$ is then easily visualized. The top of each tetrahedron should be glued to the bottom of the tetrahedron that is q steps away, as determined by a clockwise action from above. In particular, the equatorial edge of each tetrahedron is glued to the equatorial edge of the tetrahedron q steps away. Ultimately, the equatorial edges of all tetrahedra are glued together.

Notice that a similar situation occurs at the vertical edges of these tetrahedra at the center of the lens. All of the p tetrahedra are glued to this edge; they form a Euclidean neighborhood of the edge.

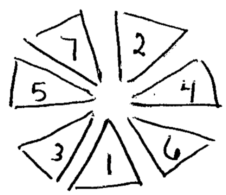
Now unglue the left and right edges of the tetrahedra. Then turn each tetrahedron ninety degrees. The equatorial edges are now vertical, and the old central vertical edges are now horizontal. The left and right edges that were glued together to form the lens are now at the top and bottom, and the top and bottom original edges are now at the left and right of the rotated tetrahedra.



Glue the new rotated tetrahedra together along their left and right edges (that is, their old top and bottom edges) as described three paragraphs back. The tetrhahedra need to be rearranged to make that happen. In the end, a new lens is formed, and the old equatorial edges, which are now vertical, glue together in the center of this lens to form a Euclidean neighborhood of this edge. See the picture below.

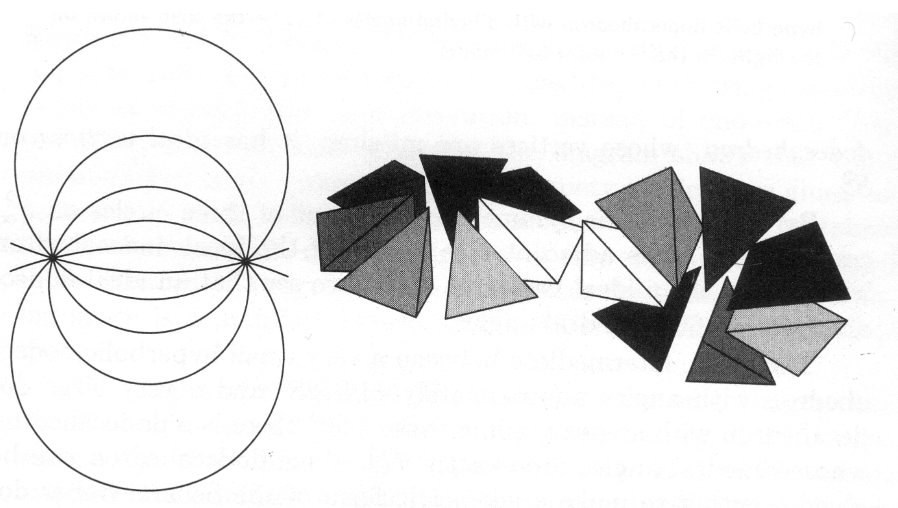


L(7,2)



L(7,2)

Thurston shows this another way. He doesn't turn the tetrahedra; instead he keeps the white tetrahedron fixed and glues the others around it using top and bottom sides:



Exercise 4: As in the hint, let $\mathcal{U}$ be the open ball without boundary points, and let $\mathcal{V}$ be all points in the lens space except the center of the ball. Then $\mathcal{U} \cap \mathcal{V}$ contains all points in the open ball except the origin. This space can be strongly deformation retracted to the sphere of radius $\frac{1}{2}$. Since the ball minus the origin can be pushed out to its boundary, $\mathcal{V}$ can be strongly deformation retracted to the boundary modulo the equivalence relation on this boundary.

The standard diagram

$$\begin{tikzcd}
& \pi(\mathcal{U}) \arrow[dr] & \\
\pi(\mathcal{U} \cap \mathcal{V}) \arrow[ur] \arrow[dr] & & \pi(\mathcal{U} \cup \mathcal{V}) \\
& \pi(\mathcal{V}) \arrow[ur] &
\end{tikzcd}$$

then becomes

$$\begin{tikzcd}
& 0 \arrow[dr] & \\
\pi(S^2) \arrow[ur] \arrow[dr] & & \pi(L(p,q)) \\
& \pi(\partial B^3/\sim) \arrow[ur] &
\end{tikzcd}$$

and an immediate consequence is that $\pi(L(p,q)) = \pi(\partial B^3/\sim)$. So from now on we ignore the interior of the ball and concentrate on the boundary. The gluing operation glues portions of this boundary to other portions, forming a complicated surface (with singularities). We must find the fundamental group of this surface.

Notice that every point in $B^3/\sim$ has a representative in the upper hemisphere. Let $\mathcal{U}$ be all points in $\partial B^3/\sim$ represented by points above the equator, and let $\mathcal{V}$ be all points in $\partial B^3/\sim$ represented by points in the upper hemisphere not equal to the north pole. Notice that $\mathcal{U}$ is contractible to a point, and $\mathcal{V}$ can be strongly deformed to the equator modulo the equivalence relation. Notice that $\mathcal{U}\cap\mathcal{V}$ consists of points above the equator and below the north pole; this can be strongly deformed to a circle.

The equator modulo the equivalence relation is just a circle; each point on this circle has p representatives in $\mathcal{V}$. The map

$$\mathcal{U}\cap\mathcal{V} \to \mathcal{V} \to \text{(this circle)}$$

maps the circle around itself p times. So the standard diagram

$$\begin{tikzcd} & \pi(\mathcal{U}) \arrow[dr] & \\ \pi(\mathcal{U}\cap\mathcal{V}) \arrow[ur] \arrow[dr] & & \pi(\mathcal{U}\cup\mathcal{V}) \\ & \pi(\mathcal{V}) \arrow[ur] & \end{tikzcd}$$

reduces to

$$\begin{tikzcd} & 0 \arrow[dr] & \\ \overset{Z}{\overset{\shortparallel}{\pi(S^1)}} \arrow[ur] \arrow[dr, "n \to pn"'] & & \pi(\partial B^3/\sim) \\ & \underset{Z}{\underset{\shortparallel}{\pi(S^1)}} \arrow[ur] & \end{tikzcd}$$

and this diagram immediately implies that the fundamental group of $\partial B^3/\sim$ is Z_p.