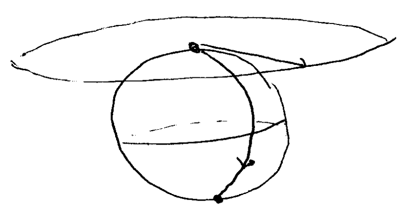
Assignment 7; Due Friday, March 3

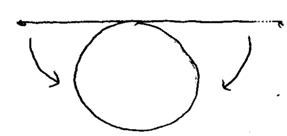
22.3a Let $p \in X$. As open neighborhood of p choose $\mathcal{U} = X$. If γ is a loop in at p which is in $\mathcal{U}$, then γ is homotopic to the constant map because every loop in X is homotopic to the constant map.

22.3b M is semi-locally simply connected because every point p has a neighborhood homeomorphic to a unit ball. But every loop in the unit ball is homotopic to a constant *in the ball* and thus trivially homotopic to a constant if we even permit the homotopy to leave the ball.

Every manifold is also locally path connected since open balls are path connected. Hence M satisfies all requirements for the existence of a universal cover. The universal cover is a manifold, for is $\tilde{x} \in \tilde{M}$ then find an evenly covered open neighborhood of $\pi(\tilde{x}) \in M$ which is homeomorphic to an open set in R^n. Then $\pi^{-1}(\mathcal{U}) = \cup\, \mathcal{U}_\alpha$. Suppose $\tilde{x} \in \mathcal{U}_\beta$. Then $\mathcal{U}_\beta$ is homeomorphic to an open set in R^n.

22.3c The author suggests that we use the mapping $f : D^n \to S^n$ pictured below. Each line in the disk is mapped to a great circle on the sphere; the mapping sends the entire boundary of the disk to the south pole. It is possible to find equations for this map and thereby show that it makes sense in all dimensions $n \geq 1$.





Let $\tilde{S^n}$ be the universal cover of S^n. We know that $\tilde{S^1} = R$, while S^n is simply connected for $n \geq 2$ and in these cases $\tilde{S^n} = S^n$. But suppose for a moment that we didn't know these things.

Since D^n is simply connected, our general lifting theorem guarantees a lift $\tilde{f} : D^n \to \tilde{S^n}$. In particular, we can restrict f and $\tilde{f}$ to the boundary of the disk, and get maps

$$f : \partial D^n \to \{\text{south pole of } S^n\}$$

and

$$\tilde{f} : \partial D^n \to \tilde{S^n}$$

At this point, there is a difference between the cases $n = 1$ and $n \geq 2$. If $n = 1$, the space $\partial D^1 = S^0$ is not connected. But if $n \geq 2$, the space ∂D^n is connected, and we can apply our earlier result that lifts are unique. Since f maps the boundary of D^n to a single point, $\tilde{f}$ must be constant on the boundary of D^n because one possible lift on the boundary is constant. Since $\tilde{f}$ is constant on the boundary, it induces a map on the quotient space $D^n/\sim$ obtained by gluing all boundary points together. Our picture shows that this space is homeomorphic to S^n. So we obtain a map $\tilde{f} : S^n \to \tilde{S^n}$ covering $f : S^n \to S^n$. Notice that the map $f : S^n \to S^n$ is the identity map since $D^n/\sim$ is homeomorphic to S^n.

We conclude that the identity map can be written as the composition $\pi \circ \tilde{f} : S^n \to \tilde{S^n} \to S^n$. So $id_\star : \pi(S^n) \to \pi(S^n)$ is a composition of the maps $\pi(S^n) \to \pi(\tilde{S^n}) \to \pi(S^n)$. Since the group in the middle is the zero group, both of these maps are identically zero and so $\pi(S^n) = 0$.

23.1b We will do this by playing around rather than by taking a systematic approach. The relation A^4 says that $A^4 = E$ and so the powers of A are E, A, A^2, A^3. The relation A^2B^{-2} says that $A^2 = B^2$. But then the powers of B are $E, B, A^2, A^2B, A^4, \ldots$. The negative powers of B also have this form, because $A^2 = B^2$ implies $A^{-2} = B^{-2}$, but $A^{-2} = A^2$ and so $A^2 = B^{-2}$. Then $A^2B = B^{-2}B = B^{-1}$, etc.

The relation $A^3BA^{-1}B^{-1}$ says that $A^3BA^{-1}B^{-1} = E$; multiplying both sides by B and then by A gives $A^3B = BA$. Using this rule over and over, we can write any product of A's and B's as an expression in which all the A's come first, follow by any B's.

Putting this altogether shows that every element of the group can be written as $E, A, A^2, A^3, B, AB, A^2B, A^3B$. I'm ready to guess the group. I believe it is the set of unit quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$. Indeed let $A = i$ and notice that E, A, A^2, A^3 are $1, i, -1, -i$. Let $B = j$ and notice that $B^2 = -1 = A^2$. Finally notice that $A^3BA^{-1}B^{-1} = (-i)j(-i)(-j) = (-k)k = 1$.

We have been playing around, but we can now make this rigorous. Map $F(A, B)$ to the group H of unit quaternions by mapping A to i and B to j. Since each relation is true in the group of unit quaternions, this induces a map of the quotient group of $F(A, B)$ by the normal subgroup generated by the relations to the quaternions. By definition this quotient group is $G(S, R)$ where $S = \{A, B\}$ and $R = A^4, A^2B^{-1}, A^3BA^{-1}B^{-1}$. So we obtain a homomorphism $G(S, R) \to H$. This map is onto since clearly i and j generate H. It is one-to-one, since we have already concluded that every element can be written as one of $E, A, A^2, A^3, B, AB, A^2B, A^3B$ and each of these is distinct in H.

23.1c The relation $ABA^{-1}B^{-1}$ says that A and B commute, so every element can be written A^kB^l for integers k and l. Since $A^4 = E = B^2$, we can assume that $0 \leq k \leq 3$ and $0 \leq l \leq 1$ and it immediately follows that the group is $Z_4 \times Z_2$.

23.1d The relations $xyx^{-1}y^{-1}$ on pairs of generators quarantees that all generators commute. Since any word is a product of generators, any two words commute. We certainly have a map $G \to AG$ by just mapping generators to generators; this map is well defined since if a relation holds on the left, it also holds on the right.

The kernel of this map consists is the subgroup of G consisting of all products of commutators $g_1 g_2 g_1^{-1} g_2^{-1}$ where $g_1, g_2 \in G$. Indeed, all such elements map to the identity since AG is abelian.

23.1f This theorem is more of an observation than a theorem requiring a formal proof. The first point, for instance, says that we are adding a relation r, but it really isn't needed because $r = e$ is already true before adding it. The second result says that if r isn't really needed because $r = 1$ is already true when we omit it, when it is save to omit the relation r.

The third result says that we can add a new symbol x that is not one of the generators if we add the relation wx^{-1}, that is, if we add the relation $x = w$. This relation says that whenever we have a word which involves x, we can write down an equivalent word with x replaced by w, and thus get an equivalent word which only involves the previous generating symbols.

The last result is equally easy.

You must be annoyed that I assigned this problem. It's significance is that Tietze proved the converse. If two finite presentations define isomorphic groups, then one presentation can be converted into the other by a finite sequence of Tietze transformations. This is not obvious. Unfortunately, there is no systematic way to obtain an efficient sequence of transformations, so in practice it is difficult to prove isomorphism this way.

Extra Problem 1: As in the hint, obtain d by lifting using the above diagram. This lift exists because X is connected and locally pathwise connected, so $\tilde{X}$ also has these properties.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_1) \arrow[d] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, dashed, "d"] \arrow[r] & (X, x_0) \end{tikzcd}$$

Define $d_1 : (\tilde{X}, \tilde{x}_1) \to (\tilde{X}, \tilde{x}_0)$ by reversing the above diagram and lifting. Composing the two lifts, we find that $d_1 \circ d : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_0)$ is a lift of the identity map. Since lifts are unique and the identity map is one possible lift, this composition must be the identity map. Similarly $d \circ d_1$ is the identity. So d is a homeomorphism (with inverse d_1) and therefore a deck transformation.

Extra Problem 2: There are several ways to do this. Here is one method.

Let γ represent an element in $\pi(X, x_0)$. Since $\tilde{X} \to X$ is a covering space, we can lift γ to $\tilde{\gamma} : I \to \tilde{X}$. This path ends at a point $\tilde{x}_1 \in \pi^{-1}(x_0)$. By lifting homotopies, we discover that homotopic paths, lift to paths which end in the same point $\tilde{x}_1$.

By the previous exercise, there is a deck transformation of $\tilde{X} \to X$ which maps $\tilde{x}_0$ to $\tilde{x}_1$. By uniqueness of lifts, this deck transformation is unique.

Denote the deck transformation obtained in this way from γ by d_γ. We obtain a map

$$\pi(X, x_0) \to \Gamma(\tilde{X} \to X)$$

I claim this map is a group homomorphism. Indeed, suppose γ and τ represent elements of $\pi(X, x_0)$. Lift γ to $\tilde{\gamma} : I \to \tilde{X}$, a path which begins at $\tilde{x}_0$ and ends at $\tilde{x}_1 = d_\gamma(\tilde{x}_0)$. Lift τ to $\tilde{\tau} : I \to \tilde{X}$, a path which also begins at $\tilde{x}_0$. Then $\gamma \star \tau$ is lifted by following $\tilde{\gamma}$ to $\tilde{x}_1 = d_\gamma(\tilde{x}_0)$, and then following $d_\gamma(\tilde{\tau})$ from $d_\gamma(\tilde{x}_0)$ to $d_\gamma(d_\tau(\tilde{x}_0))$ and therefore $\gamma \star \tau$ maps to $d_\gamma \circ d_\tau$.

It is clear that this group homomorphism is onto. To complete the proof it suffices to show that its kernel is $\pi(\tilde{X}, \tilde{x}_0)$, for then the quotient group $\pi(X, x_0)/\pi(\tilde{X}, \tilde{x}_0)$ will map isomorphically to $\Gamma(\tilde{X} \to X)$.

An element of $\pi(X, x_0)$ maps to the identity in $\Gamma(\tilde{X} \to X)$ just in case its lift ends at $\tilde{x}_0$ and thus represents a loop in $\tilde{X}$. But in this case the lifted loop would induce an element in $\pi(\tilde{X}, \tilde{x}_0)$ which would map to $\gamma \in \pi(X, x_0)$ and so γ would belong to the subgroup $\pi(\tilde{X}, \tilde{x}_0)$.

Conversely, if γ belongs to this subgroup, then a loop homotopic to γ has a lift which ends at $\tilde{x}_0$ and so γ maps to the identity in $\Gamma(\tilde{X} \to X)$.

Extra Problem 3: We define a group law on $\tilde{G}$ by the following lift:

$$\begin{tikzcd}
 & & (\tilde{G}, \tilde{e}) \arrow[d] \\
(\tilde{G} \times \tilde{G}, \tilde{e} \times \tilde{e}) \arrow[r] \arrow[urr, dashed] & (G \times G, e \times e) \arrow[r, "\text{multiply}"] & (G, e)
\end{tikzcd}$$

The only difficulty is the proof that this lift exists. We must prove that the composition $\tilde{G} \times \tilde{G} \to G \times G \to G$ maps $\pi(\tilde{G} \times \tilde{G})$ into the image of $\pi(\tilde{G}, \tilde{e}) \to \pi(G, e)$. But a representative of an element of $\pi(\tilde{G} \times \tilde{G})$ has the form $(\gamma(t), \tau(t))$ where γ and τ are closed curves in $\tilde{G}$. This element is sent to the path $\pi(\gamma) \circ \pi(\tau)$ by the first map.

The solution to exercise 15.16c in assignment 4 shows that in a topological group, the sum $\gamma \star \tau$ of two paths in the fundamental group can also be represented by their group theoretical product $\gamma \circ \tau$. So $\pi(\gamma) \circ \pi(\tau)$ is a representative of $\pi(\gamma) \star \pi(\tau)$ in the fundamental group of G. This representative is certainly in the image of the map $\pi(\tilde{G}, \tilde{e}) \to \pi(G, e)$ because it is the sum of two elements in the image. QED.

Extra Problem 4: Since $S^3 \to SO(3)$ is a two-folder cover and S^3 is simply connected and so the universal cover, the fundamental group of $SO(3)$ is Z_2. Therefore the fundamental group of $SO(3) \times SO(3)$ is $Z_2 \times Z_2$.

The subgroups of this group are $\{0\}, Z_2 \times 0, 0 \times Z_2, \{(0,0),(1,1)\}$, and $Z_2 \times Z_2$. Clearly these correspond to the covering spaces $Sp(1) \times Sp(1), SO(3) \times Sp(1), Sp(1) \times SO(3)$, ?, and $SO(3) \times SO(3)$.

Extra Problem 5: Note that $||q_1 h q_2^{-1}|| = ||q_1|| \; ||h|| \; ||q_2^{-1}|| = ||h||$ since q_1 and q_2 have norm one. So our map preserves distances. Also note that the map is linear in h for fixed q_1 and q_2. By linear algebra, such maps belong to $O(4)$, the full group of orthogonal transformations. The determinant of such a map is ± 1.

But determinant is continuous and the space S^3 of unit quaternions is connected, so the determinant of our maps must always be one. Thus our maps belong to $SO(4)$.

Suppose that $h \to q_1 h q_2^{-1}$ is the identity map. Setting $h = 1$ gives $q_1 q_2^{-1} = 1$, so $q_1 = q_2$. But in an earlier exercise we proved that $v \to q v q^{-1}$ is the identity on the set V of all quaternions of the form $a_1 i + a_2 j + a_3 k$ only if $q = \pm 1$. So $q_1 = \pm 1$ and $(q_1, q_2) = \pm(1,1)$.

Finally we show that all rotations can be obtained from $Sp(1) \times Sp(1)$. Suppose $A \in SO(4)$ is a rotation. Then $A(1)$ is a unit vector $q \in H$ and thus a unit quaternion. Let $R \in SO(4)$ be the rotation given by $(q_1, q_2) = (q, 1)$. Notice that $A(1) = R(1)$ and thus $R^{-1}A$ leaves 1 fixed. If we can prove that this element is given by some pair (q_1', q_2'), then we are done because A will be given by $R(q_1', q_2') = (q, 1) \circ (q_1', q_2')$.

But since $R^{-1}A$ leaves 1 fixed, it must be rotation in the three-dimensional plane V perpendicular to 1. Last week we proved that every element in this $SO(3)$ is given by a map $v \to q' v q'^{-1}$. Notice that this corresponds to the pair (q', q'^{-1}), so we are done.