Assignment 6; Due Friday, February 24

The Fundamental Group of the Circle

Theorem 1 *Let* $\gamma : I \rightarrow S^1$ *be a path starting at* 1. *This path can be lifted to a path* $\tilde{\gamma} : I \rightarrow R$ *starting at* 0.

Proof: Find a covering $\{\mathcal{W}_\alpha\}$ of S^1 by evenly covered open sets. Actually there is such a cover with two open sets, but we don't need to know that.

Then $\{\gamma^{-1}(\mathcal{W}_\alpha)\}$ is an open cover of I. By the theorem of Lebesgue, there is a $\delta > 0$ such that every subset of I of diameter less than δ is inside one of these sets. Choose a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ so $|t_i - t_{i-1}| < \delta$ for each i. Then for each i, $\gamma([t_{i-1}, t_i])$ is in some evenly covered $\mathcal{W}_\alpha$.

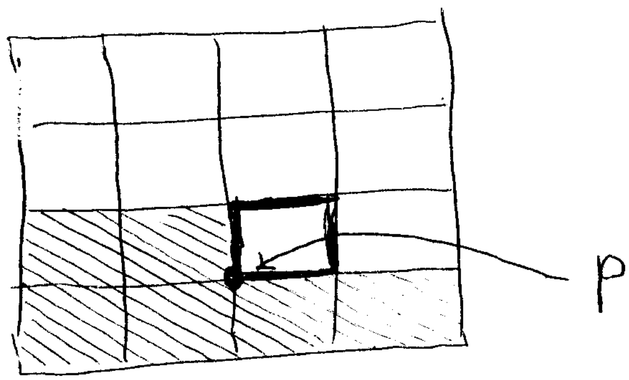
We now inductively define $\tilde{\gamma}$. Suppose $\tilde{\gamma}$ has already been defined on $[0, t_{i-1}]$. We extend it to $[t_{i-1}, t_i]$. Notice that this inductive step even covers the start of the induction, because $\tilde{\gamma}(0)$ has been defined to be 0 and needs to be extended to $[0, t_1]$.

There is an evenly covered $\mathcal{W}_\alpha \subseteq S^1$ such that $\gamma([t_{i-1}, t_i]) \subseteq \mathcal{W}_\alpha$. Let us call this open set $\mathcal{U}$ to avoid a collision of indices. Then $\pi^{-1}(\mathcal{U}) = \cup\, \mathcal{U}_\alpha$. The path $\tilde{\gamma}$ is already defined at t_{i-1}; this point belongs to one of the $\mathcal{U}_\alpha$, say $\mathcal{U}_\beta$. But $\pi : \mathcal{U}_\beta \rightarrow \mathcal{U}$ is a homeomorphism. Define $\tilde{\gamma}$ on $[t_{i-1}, t_i]$ to be $\pi^{-1} \circ \gamma : [t_{i-1}, t_i] \rightarrow \mathcal{U} \rightarrow \mathcal{U}_\beta$.

Remark: We proved that this lift is unique, not just for I but for any connected Y. Thus if we are given a loop $\gamma : I \rightarrow S^1$ beginning and ending at 1, there is a unique lift to a path $\tilde{\gamma} : I \rightarrow R$ which begins at 0 and ends at some point projecting to 1. Since $\pi(x) = e^{2\pi i x}$ equals one exactly when x is an integer, $\tilde{\gamma}(1) \in Z$.

Theorem 2 *Let* $h : I \times I \rightarrow S^1$ *be a continuous map such that* $h(0 \times 0) = 1$. *This map can be lifted to a map* $\tilde{h} : I \times I \rightarrow R$ *such that* $\tilde{h}(0 \times 0) = 0$.

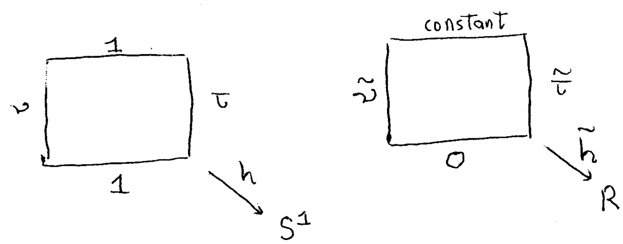
Proof: The proof proceeds exactly as the proof of theorem one. This time we apply Lebesgue's result to find a subdivision of the square $I \times I$ into subsquares so h maps each subsquare to some evenly covered open $\mathcal{U} \subseteq S^1$. Then we inductively define $\tilde{h}$ subsquare by subsquare. Suppose $\tilde{h}$ has already been defined on the shaded region at the top of the next page, and we want to define $\tilde{h}$ on the outlined square which comes next.



This outlined square is mapped by h to an evenly covered $\mathcal{U}$ in S^1. Then $\pi^{-1}(\mathcal{U}) = \cup\, \mathcal{U}_\alpha$. The extension $\tilde{h}$ is already defined at the point p; actually it has already been defined at other points but we ignore these for a moment. Thus $\tilde{h}(p)$ belongs to some $\mathcal{U}_\alpha$, say $\mathcal{U}_\beta$. Define $\tilde{h}$ on the outlined square to be $\pi^{-1} \circ h$: outlined square $\rightarrow \mathcal{U} \rightarrow \mathcal{U}_\beta$.

To complete the proof, we must show that the new extension $\tilde{h}$ to the outlined square and the previous extension $\tilde{h}$ on the shaded region agree on the bottom and left side of the shaded square. Both both extensions are lifts of h to these lines and the lifts agree at p. By uniqueness of lifts, the lifts agree on the entire bottom or left.

Remark: Suppose γ and $\tau : I \to S^1$ are loops which induce the same element of $\pi(S^1, 1)$. Then these loops are homotopic by a homotopy $h : I \times I \to S^1$. Lift this homotopy to $\tilde{h} : I \times I \to R$ using the previous theorem. In the picture on the left below, we show values of the original h on the boundaries of the square. On the right we show values of $\tilde{h}$ on this square; these values are determined by uniqueness of lifts. In particular, $\tilde{h}$ is constant on the top of the square, so $\tilde{\gamma}(1) = \tilde{\tau}(1)$. It follows that the map $\gamma \to \tilde{\gamma}(1)$ induces a well-defined map $\pi(S^1, 1) \to Z$.



Theorem 3 *The map $\pi(S^1, 1) \to Z$ is one-to-one.*

Proof: Suppose $\tilde{\gamma}(1) = \tilde{\tau}(1)$. Then $\tilde{\gamma}$ and $\tilde{\tau} : I \to R$ are paths which start at the same point and end at the same point. Since R is simply connected, these paths are homotopic with fixed endpoints; call the homotopy $\tilde{h}$. This homotopy can be made explicit: $\tilde{h}(u, t) = (1 - t)\tilde{\gamma}(u) + t\tilde{\tau}(u)$. But then $\pi \circ \tilde{h}$ is a homotopy from γ to τ, so γ and τ induce the same element of $\pi(S^1, 1)$.

Theorem 4 *The map $\pi(S^1, 1) \to Z$ is onto.*

Proof: If $n \in Z$, let $\tilde{\gamma}(u) = nu : I \to R$. Define $\gamma(u) = \pi \circ \tilde{\gamma} : I \to S^1$. Then γ defines an element of $\pi(S^1, 1)$ which clearly maps to n.

Classifying Covering Spaces; Projective Space

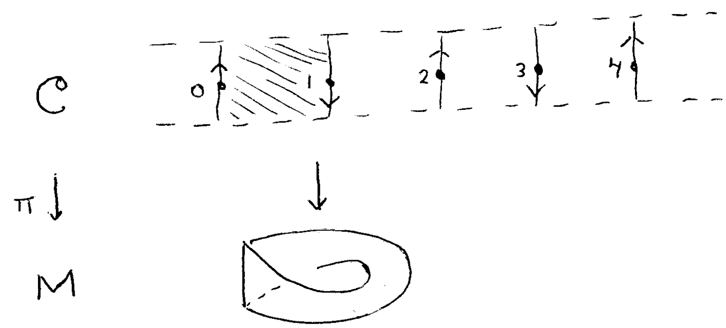
Since RP^n is S^n with opposite points identified, the natural map $\pi : S^n \to RP^n$ is a two-fold covering space. But S^n is simply connected. We proved that if $\pi : \tilde{X} \to X$ is the universal covering space, then $\pi(X, x_0)$ is set theoretically equal to $\pi^{-1}(x_0)$. In our case this set has two points, so $\pi(RP^n, x_0)$ is a group with two elements. There is only one such group up to isomorphism, so $\pi(RP^n) = Z_2$.

Classifying Covering Spaces; the Mobius Band

We begin our discussion of the Mobius band by constructing its universal cover, $\mathcal{C}$. Let $\mathcal{C}$ be the infinite strip

$$\mathcal{C} = \{(x, y) \mid -1 < y < 1\}$$

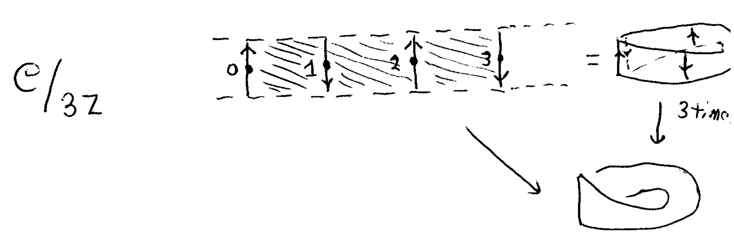
and define a group $\Gamma = Z$ acting on this strip by $(x, y) \to (x + n, (-1)^n y)$. The generator of this group translates right by one unit while flipping the y-axis. If we form equivalence classes $\mathcal{C}/\Gamma$, then every point is equivalent to a point (x, y) with $0 \le x \le 1$. The left and right boundary lines of this set are equivalent by $(0, y) \sim (1, -y)$. Clearly the set of equivalence classes with the quotient topology is a Mobius band, as pictured below.



Incidentally, this universal cover $\mathcal{C}$ is homeomorphic to an open disk.

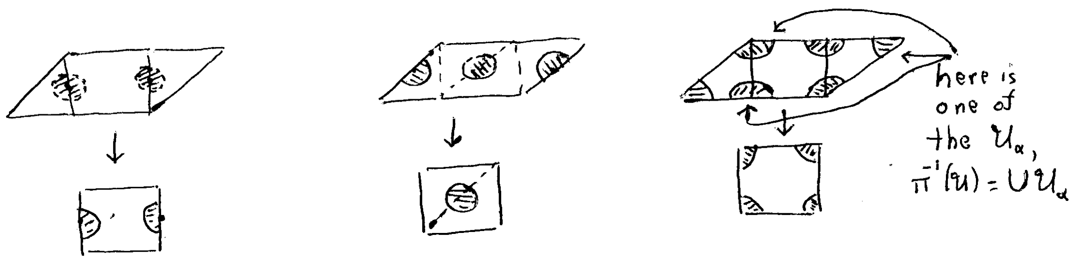
Let $p \in M$ be the element in the Mobius band corresponding to $(0, 0)$. The inverse image of this point in the strip is clearly $\{(n, 0) \mid n \in Z\}$. The map $d_n : (x, y) \to (x+n, (-1)^n y)$ is clearly a deck transformation because it induces the identity map on M; indeed (x, y) and $(x + n, (-1)^n y)$ are equivalent. This particular deck transformation maps $(0, 0)$ to $(n, 0)$. Since a deck transformation is completely determined by its operation on one point, and the d_n map $(0, 0)$ to all possible points, they form the complete list of deck transformations. Clearly this deck transformation group is isomorphism to Z, which is comforting because the fundamental group of the Mobius band is Z.

The subgroups of Z are $\Gamma_G = ZN$ for $N = 0, 1, 2, \ldots$. Each corresponds to a covering space of the Mobius band, constructed by forming the set of equivalence classes $\mathcal{C}/\Gamma_G$. When $N = 0$ the only element of Γ_G is the identity map, so two points of $\mathcal{C}$ are equivalent only if they are equal and $\mathcal{C}/\Gamma_G = \mathcal{C}$. Otherwise every point of $\mathcal{C}$ is equivalent to a point (x, y) with $0 \le x \le N$ and we obtain an N-fold covering space as pictured below. Notice that the left and right ends are glued together via $(0, y) \to (N, (-1)^N y)$, so the covering space is a cylinder if N is even and a Mobius band is N is odd.

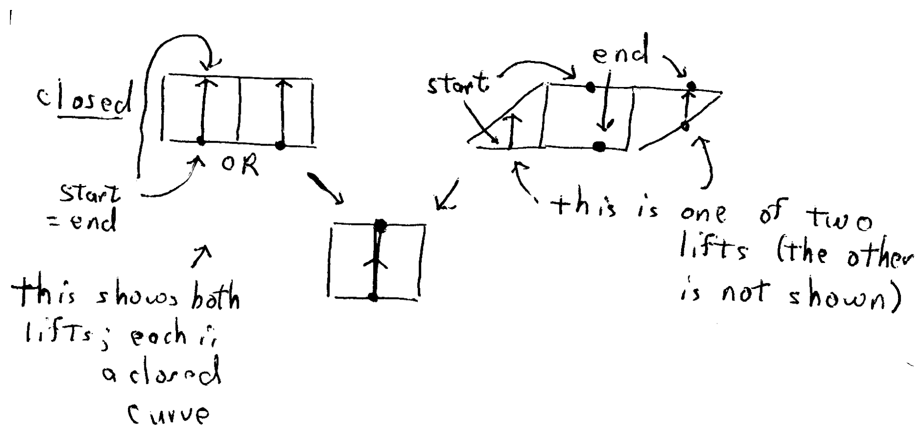


Two Covering Spaces of the Torus

The pictures below show that the more complicated of the two spaces is a covering space, and evenly covered disks on the torus are covered twice on the moving space. The pictures illustrate particularly difficult cases when boundary points of the fundamental region still lie in evenly covered open sets.



We must prove that these covering spaces are different. Consider the path below on the torus, and its lift to the two covering spaces. In both cases there are two possible lifts depending on the starting point. Both lifts are shown in the example on the left, but only one lift is shown in the example on the right.



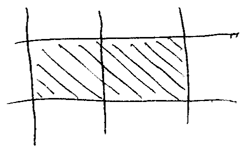
Notice that the lift on the left is a closed curve, but the lift on the right is not a closed curve. So the two covering spaces are not isomorphic.

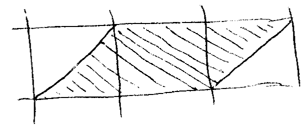
Fundamental Groups of These Covering Spaces We know that $\pi(S^1 \times S^1) = Z \times Z$. If $\mathcal{C} = R^2$ is the universal cover and $\pi : \mathcal{C} \to S^1 \times S^1$, then this fundamental group can be identified with $\pi^{-1}(1 \times 1)$, which is the integer lattice in our pictures.

If $\tilde{X} \to S^1 \times S^1$ is a covering space, then $\mathcal{C}$ is also the universal cover of $\tilde{X}$ and the fundamental group of $\tilde{X}$ can be identified with the inverse image of the base point $\tilde{x}_0 \in \tilde{X}$ In our particular example, $\tilde{X}$ is described by a shaded fundamental region and we will take

$\tilde{x}_0 = 0 \times 0$. Then the fundamental group of $\tilde{X}$ consists of all lattice points equivalent to 0×0 under the equivalence relation whose equivalence classes are described by the fundamental region.

In the example on the left, the equivalence relation is $(x, y) \sim (x + 2m, y + n)$ and the points which map to the base point are all lattice points $(2m, n)$. This yields the subgroup $2Z \times Z$ of $Z \times Z$.





In the example on the right, the equivalence relation is generated by $(x, y) \sim (x+2, y)$ and $(x, y) \sim (x + 1, y + 1)$ and the fundamental group consists of all lattice points equivalence to 0×0 under this equivalence relation. These points consist of all $(2m, n)$ with n even, and all $(2m + 1, n)$ with n odd. The set of such points forms our subgroup of $Z \times Z$. This group is generated by $(2, 0)$ and $(1, 1)$.

Subgroups of $Z \times Z$

It is useful to resist the temptation of treating both copies of Z symmetrically. Instead we analyze using the sequence

$$0 \to Z \xrightarrow{i} Z \times Z \xrightarrow{r} Z \to 0$$

where $i(m) = (m, 0)$ and $r(m, n) = n$.

If $G \subseteq Z \times Z$ is a subgroup, restrict r to this subgroup. The image is a subgroup of Z and so has the form NZ for a unique $N = 0, 1, 2, \ldots$ The kernel of this r is the set of all $(m, 0) \in G$; this kernel is a subgroup of Z and thus MZ for a unique $M = 0, 1, 2, \ldots$ Thus we have an exact sequence

$$0 \to MZ \to G \to NZ \to 0$$

Look at the generator N of the group on the right. This generator comes from an element of G, but the first component of this element need not be zero. Suppose it comes from $(A, N) \in G \subseteq Z \times Z$. We can modify this element by adding any element of the kernel of r, so we can assume that $0 \leq A < M$. Thus the subgroup G uniquely determines three numbers M, N, and A.

Now I claim that conversely these three numbers determine G. This is a special case of the general theory of semidirect products described in the handout sheet *Semidirect Products and the Fundamental Group of the Klein Bottle.* In the language of that handout sheet, a choice of A determines a splitting $s : NZ \to G$ of the sequence, and this splitting then determines G. In the handout, it was proved that for a split sequence

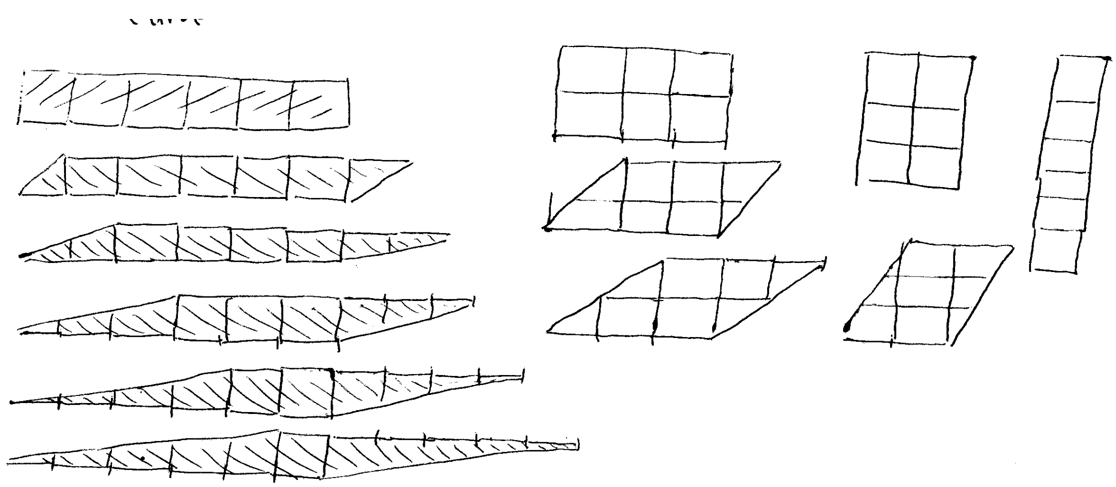
$$0 \to H \to G \to K \to 0$$

every element of G can be uniquely written $g = hk$ where $h \in H$ and $k \in K$. In our case, every element can be written as a sum of a multiple of $(M, 0)$ and a multiple of (A, N). Thus

$$G = \{(kM + lA, lN) \mid k, l \in Z\}$$

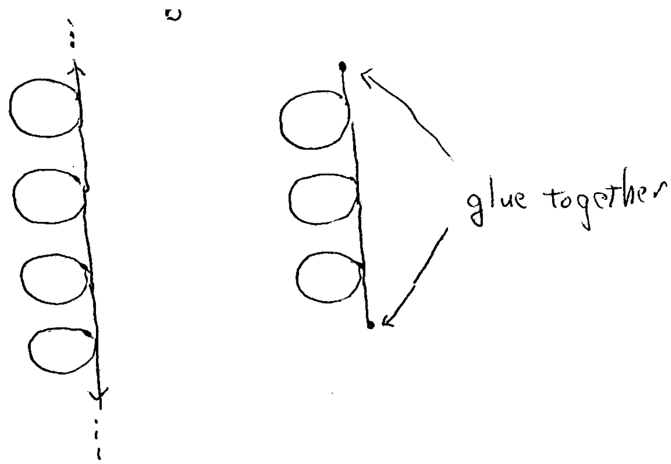
Notice that something special happens when M and/or N is zero. If both are zero, the group is $\{0\}$ and the corresponding covering space is the universal cover R^2. If $N = 0$, the group is $G = Z(M, 0)$ and the covering space is a cylinder corresponding to the shaded region consisting of all (x, y) with $0 \le x \le M$. If $M = 0$, the group is $G = Z(A, N)$ and the covering space is a cylinder as illustrated below.

In all other cases the covering space is a torus. This torus covers MN times, and there are exactly as many different n-fold covers as there are factorizations $n = MN$ and choices of A with $0 \le A \le M$. All examples with $n = 6$ are shown below. Note that $6 = 6 \times 1 = 3 \times 2 = 2 \times 3 = 1 \times 6$; there are 6 possible A in the first case, 3 in the second, 2 in the third, and 1 in the fourth.



Incidentally, the lack of symmetry in the final answer may be puzzling. If we perform the same analysis switching the roles of the first and second components, we get a different list of twelve shaded regions. You might want to try this. The point is that the shaded region is not unique, but the subgroup G of $Z \times Z$ is.

Covering Spaces of a Bouquet of Circles



The groups of these four covering spaces are subgroups of the free group on a and b. Each element of this group is a word, something like $a^3b^{-1}a^5b^2a$. The four groups are

- all words with the sum of the exponents of b equal to zero
- all words with the sum of the exponents of b congruent to to zero modulo 3
- all words with the sum of the exponents of a equal to zero and the sum of the exponents of b equal to zero
- all words with the sum of the exponents of a equal to zero modulo 5 and the sum of the exponents of b equal to zero modulo 4

The Commutator Subgroup of $F(a, b)$ I gave this exercise because I was curious if there was a direct inductive proof that all words with the sum of the exponents of a equal to zero and the sum of the exponents of b equal to zero can be written as a product of commutators. But I'm going to try to prove the result indirectly rather than directly.

Notice that the deck transformation group of this particular covering space is clearly all maps $(x, y) \to (x + m, y + n)$ and in particular is transitive on $\pi^{-1}(p)$ where p is the base point where our two circles meet. By an exercise which admittedly occurs next week, it follows that the subgroup G of $F(a,b)$ corresponding to this covering space is normal and the quotient group $F(a,b)/G$ is isomorphic to this deck transformation group $Z \times Z$ and in particular is abelian. Consequently, all commutators $cdc^{-1}d^{-1}$ must belong to G, since otherwise the elements of $F(a,b)/G$ induced by c and d would not commute.

Let A be the element of $F(a,b)/G$ induced by a and let B be the element induced by b. Since $aba^{-1}b^{-1}$ is in G, A and B commute in this quotient group. Since every element of $F(a,b)$ is a word in a and b, every element of the quotient group can be written A^kB^l for integers k and l. The isomorphism $F(a,b)/G \to Z \times Z$ certainly sends A^kB^l to (k,l).

It is possible that G contains other elements which are not products of commutators. Let H be the subgroup of products of commutators. If H is strictly smaller than G, then $F(a,b)/H$ will be larger than $F(a,b)/G = Z \times Z$ and the natural map $F(a,b)/G \to F(a,b)/H$ will not be onto. But this map is certainly onto since both groups are generated by the images of a and b in the respective group.

The Universal Covering Group

Suppose $\tilde{X} \to X$ is an arbitrary covering space. Here $\tilde{X}$ need not be the universal cover. One of our most important theorems says that a continuous $f : (Y, y_0) \to (X, x_0)$ can be lifted to $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if Y is simply connected and locally pathwise connected.

Apply this theorem to $\tilde{G} \to G$ when $Y = \tilde{G} \times \tilde{G}$ and the diagram is:

$$\begin{tikzcd}
 & & (\tilde{G}, \tilde{e}) \arrow[d] \\
(\tilde{G} \times \tilde{G}, \tilde{e} \times \tilde{e}) \arrow[r] \arrow[urr, dashed] & (G \times G, e \times e) \arrow[r, "\text{multiply}"] & (G, e)
\end{tikzcd}$$

The hypotheses hold, because if $\tilde{G}$ is simply connected then so is $\tilde{G} \times \tilde{G}$, and if G is locally arcwise connected then so is $\tilde{G}$ and thus so is $\tilde{G} \times \tilde{G}$.

The lifted map defines a product on $\tilde{G}$. Since the diagram is commutative $\pi(\tilde{g}_1 \circ \tilde{g}_2) = \pi(\tilde{g}_1) \circ \pi(\tilde{g}_2)$.

We claim that $\tilde{e}$ is a unit in $\tilde{G}$. Indeed consider the diagram below. The composition of the maps on the top row is a lift of the composition of maps on the bottom row. But the composition on the bottom row is the identity map $g \to g$, and lifts are unique, and one possible lift on the top row is the identity map $\tilde{g} \to \tilde{g}$, so the composition on the top is the identity and therefore $\tilde{g} \circ \tilde{e} = \tilde{g}$. Similarly $\tilde{e} \circ \tilde{g} = \tilde{g}$, so $\tilde{e}$ is an identity element.

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\tilde{g} \to \tilde{g} \times \tilde{e}"] \arrow[d] & \tilde{G} \times \tilde{G} \arrow[r, "\text{product}"] \arrow[d] & \tilde{G} \arrow[d] \\
G \arrow[r, "g \to g \times e"] & G \times G \arrow[r, "\text{product}"] & G
\end{tikzcd}$$

The diagram below has a unique lift, which we will denote $\tilde{g} \to \tilde{g}^{-1}$. Here we are omitting base points out of laziness; the base points are e and $\tilde{e}$.

$$\begin{tikzcd}
\tilde{G} \arrow[r, dashed] \arrow[d] & \tilde{G} \arrow[d] \\
G \arrow[r, "g \to g^{-1}"] & G
\end{tikzcd}$$

The diagram below shows that $\tilde{g} \to \tilde{g} \circ \tilde{g}^{-1}$ is a lift of the map $g \to g \circ g^{-1}$. But the bottom map is also $g \to e$ and thus a possible lift is $\tilde{g} \to \tilde{e}$. Since lifts are unique, $\tilde{g} \circ \tilde{g}^{-1} = \tilde{e}$ and so $\tilde{g}^{-1}$ is indeed an inverse in $\tilde{G}$.

$$\begin{tikzcd}
\tilde{G} \arrow[r, "\tilde{g} \to \tilde{g} \times \tilde{g}^{-1}"] \arrow[d] & \tilde{G} \times \tilde{G} \arrow[r, "\text{multiply}"] \arrow[d] & \tilde{G} \arrow[d] \\
G \arrow[r, "g \to g \times g^{-1}"] & G \times G \arrow[r, "\text{multiply}"] & G
\end{tikzcd}$$

Finally the product on $\tilde{G}$ is associative, for the diagram below shows that $\left(\tilde{g}_1 \circ \tilde{g}_2\right) \circ \tilde{g}_3$ is a lift of $\left(g_1 \circ g_2\right) \circ g_3$, but since $\left(g_1 \circ g_2\right) \circ g_3 = g_1 \circ \left(g_2 \circ g_3\right)$, another possible lift is $\tilde{g}_1 \circ \left(\tilde{g}_2 \circ \tilde{g}_3\right)$; uniqueness of lifts then implies that multiplication is associate on $\tilde{G}$.

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \times \tilde{G} \arrow[r, "\text{multiply} \times \text{id}"] \arrow[d] & \tilde{G} \times \tilde{G} \arrow[r, "\text{multiply}"] \arrow[d] & \tilde{G} \arrow[d] \\
G \times G \times G \arrow[r, "\text{multiply} \times \text{id}"] & G \times G \arrow[r, "\text{multiply}"] & G
\end{tikzcd}$$

Exercise on Quaternions

This is just a calculation. We have

$$(a_0 + a_1 i + a_2 j + a_3 k)(a_0 - a_1 i - a_2 j - a_3 k)$$

$$= (a_0^2 - a_1^2 i^2 - a_2^2 j^2 - a_3^2 k^2) + i(a_0 a_1 + a_1 a_0 - a_2 a_3 + a_3 a_2) + \dots$$

and the first term is $a_0^2 + a_1^2 + a_2^2 + a_3^2$ and all other terms are zero.

To prove the second result, notice that $\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1$. Then

$$||q_1 q_2||^2 = (q_1 q_2)(\overline{q_1 q_2}) = q_1 q_2 \overline{q}_2 \overline{q}_1 q_1 ||q_2||^2 \overline{q}_1 = ||q_2||^2 q_1 \overline{q}_1 = ||q_1||^2\ ||q_2||^2$$

Exercise on S^3 as a Group

The group axioms follow from associativity in the quaternions, etc. The unit quaternions consist of all (a_0, a_1, a_2, a_3) for which $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$, and this is the equation of the three dimensional sphere S^3.

Exercise on S^3 Acting on R^3

If $q = a_0 + a_1 i + a_2 j + a_3 k$, then $\overline{q} = a_0 - a_1 i - a_2 j - a_3 k$. So $q = -\overline{q}$ exactly when $a_0 = 0$. Such a quaternion has the form (a_1, a_2, a_3) and thus can be considered to live in R^3.

If q is a unit quaternion, then $q\overline{q} = 1$, so $q^{-1} = \overline{q}$. Thus the map is $v \rightarrow qv\overline{q}$. The conjugate of this element is the product of conjugates in the reverse order, so

$$\overline{qv\overline{q}} = \overline{\overline{q}}\ \overline{v}\ \overline{q} = q(-v)\overline{q} = -qv\overline{q}$$

and therefore $qv\overline{q}$ belongs to V.

$S^3 \to SO(3)$ is two-to-one

Suppose $qvq^{-1} = v$ for all $vinV$. Then $qv = vq$ for all $v \in V$. In particular, $qi = iq$, so $(a_0+a_1i+a_2j+a_3k)i = i(a_0+a_1i+a_2j+a_3k)$ and so $a_0i-a_1-a_2k+a_3j = a_0i-a_1+a_2k-a_3j$. So $a_2 = a_3 = 0$. Similarly $a_1 = 0$, so $q = a_0$. But $||q|| = 1$, so $q = \pm 1$.

$S^3 \to SO(3)$ is onto

We have

$$(\cos\theta + i\sin\theta)\ i\ \overline{(\cos\theta + i\sin\theta)} = (\cos\theta + i\sin\theta)\ i\ (\cos\theta - i\sin\theta) = i$$

and

$$(\cos\theta + i\sin\theta)\ j\ (\cos\theta - i\sin\theta) = (\cos\theta + i\sin\theta)(j\cos\theta + k\sin\theta) =$$

$$(\cos^2\theta - \sin^2\theta)j + (2\cos\theta\sin\theta)k = (\cos 2\theta)j + (\sin 2\theta)k.$$

Similarly $(\cos\theta + i\sin\theta)\ k\ (\cos\theta - i\sin\theta) = -(\sin\theta\cos\theta)j + (\cos^2\theta - \sin^2\theta)k$ which equals $-(\sin 2\theta)j + (\cos 2\theta)k$.

These formulas show that $\cos\theta + i\sin\theta$ maps to rotate about the i-axis by 2θ. The remaining assertions are proved similarly.

But we can obtain any rotation as a product of rotations about the three axes. Suppose our rotation takes a point p to the north pole. This determines the rotation up to rotation about the z-axis. But we can get that rotation as a product of axis rotations as follows. First rotate p about the z-axis until the image of p is in the xz plane. Then rotate about the y-axis until the image is the north pole. Then follow by an arbitrary rotation about the z-axis.