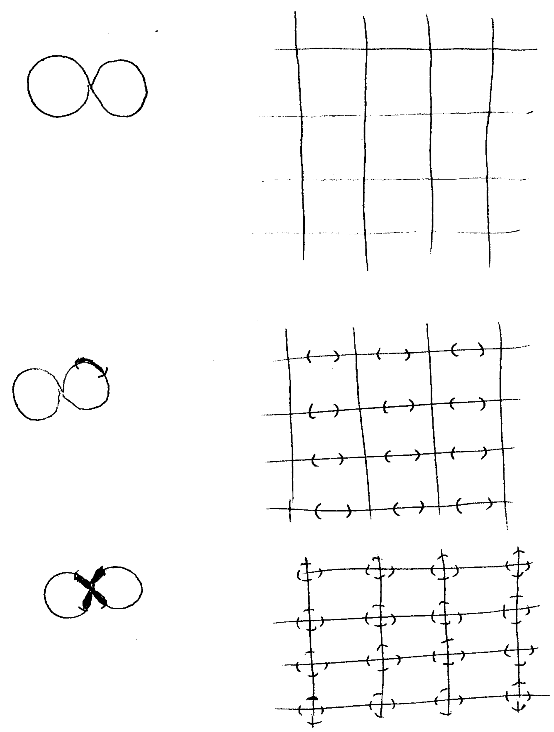
Assignment 5; Due Friday, February 10

17.9b The set X is just two circles joined at a point, and the set $\tilde{X}$ is a grid in the plane, without the interiors of the small squares. The picture below shows that the interiors of the circles are evenly covered, and that the junction where the circles meet is evenly covered. This ends the argument.



Let me put this exercise in context by telling you a few things we'll prove later.

Let $\tilde{X}$ be a covering space of X. Then the map $\pi : \tilde{X} \to X$ induces a map

$$\pi_\star : \pi(\tilde{X}) \to \pi(X)$$

It is not difficult to show that this map is one-to-one, so we can think of $\pi(\tilde{X})$ as a subgroup of $\pi(X)$. We are going to prove that this sets up a one-to-one correspondence between subgroups of $\pi(X)$ and covering spaces of X. The full group corresponds to the trivial cover $\pi = id : X \to X$ and the trivial group corresponds to the unique simply-connected covering space.

We will prove that the deck transformation group Γ of a covering $\pi : (\tilde{(X)}, \tilde{x_0}) \to (X, x_0)$ is transitive on $\pi^{-1}(x_0)$ if and only if $\pi(\tilde{X})$ is a normal subgroup of $\pi(X)$. In this case, we will prove that Γ is isomorphic to the quotient group. For example, $id : X \to X$ corresponds to the full group $\pi(X)$ and the quotient group is $\{1\}$; indeed the only deck transformation of X is the identity map. Similarly, the universal cover corresponds to the identity subgroup, and the quotient is the full group $\pi(X)$, which is isomorphic to the deck transformation group.

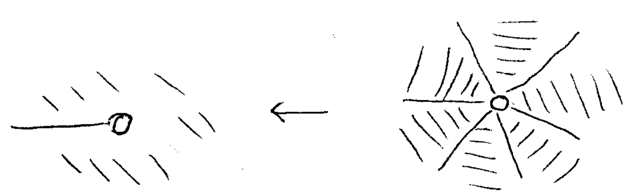
The fundamental group of a join of two circles is the free group on two generators $F(a, b)$. I claim that the fundamental group of the above grid is the group generated by all commutators $g_1 g_2 g_1^{-1} g_2^{-1}$, and the quotient group is thus the free abelian group generated by a and b. This abelianized group is just $Z \times Z$. Sure enough, the group of deck transformations is clearly $Z \times Z$ because a deck transformation of the grid is just a translation $(x, y) \to (x + m, y + n)$.

17.9c These problems probably aren't fair because they assume some knowledge of complex analysis. But heck, all of mathematics is interconnected.

The map z^n is much easier to understand in polar coordinates. It is

$$(r \cos\theta, r \sin\theta) = r(\cos\theta + i \sin\theta) = re^{i\theta} \to \Big(re^{i\theta}\Big)^n = r^n e^{in\theta} = (r^n \cos n\theta, r^n \sin n\theta)$$

This map does some stretching and shrinking in the radial direction, which isn't very important because $r^n : R^+ \to R^+$ is a homeomorphism. In the angular direction it winds around n times. Thus the map defines an n-fold covering space. In the picture below, the inverse image of $\mathcal{U} = C^\star - \{\text{negative } x\text{-axis}\}$ is covered by n open wedges.



The map $\sin(z) : C \to C$ is not a covering map. The easiest way to see this is to notice that the derivative of the map is $\cos(z)$, and this derivative has a zero at, for instance $\frac{\pi}{2}$. So $\sin(z)$ could not be a local homeomorphism near $\frac{\pi}{2}$. Once we notice this, we can give an easy argument independent of complex analysis. Notice that $\sin(\frac{\pi}{2}) = 1$ and $\sin(\frac{\pi}{2} - t) = \sin(\frac{\pi}{2} + t)$. So there could not be an open neighborhood of $\frac{\pi}{2}$ carried homeomorphically to an open neighborhood of C by $\sin(z)$.

The derivative of $(1 - z)^m z^n$ is $-m(1 - z)^{m-1} z^n + n(1 - z)^m z^{n-1}$, which equals

$$\Big(- mz + n(1 - z)\Big)(1 - z)^{m-1} z^{n-1} = \Big(n - (m + n)z\Big)(1 - z)^{m-1} z^{n-1}$$

This expression is zero if $z = 0$ or $z = 1$ or $z = \frac{n}{m+n}$. The points $z = 0$ and $z = 1$ have been omitted to form $\mathcal{U}$, but $\frac{n}{m+n}$ belongs to $\mathcal{U}$ unless $\frac{n}{m+n} = 0$ or $\frac{n}{m+n} = \infty$ or $\frac{n}{m+n} = 1$, that is, $n = 0$ or $m = -n$ or $m = 0$. By complex analysis, a map is not a local homeomorphism near a spot where its derivative is zero. So our map cannot be a covering map unless it is $(1-z)^n$ or z^n or $\left(\frac{z}{1-z}\right)^n$. But z^n doesn't map $\mathcal{U}$ to $C^\star - \{1\}$ unless $n = \pm 1$ because it maps all nth roots of unity to 1. Similar arguments show that our map is a covering map if and only if it is one of $z, \frac{1}{z}, 1-z, \frac{1}{1-z}, \frac{z}{1-z}$, or $\frac{1-z}{z}$.

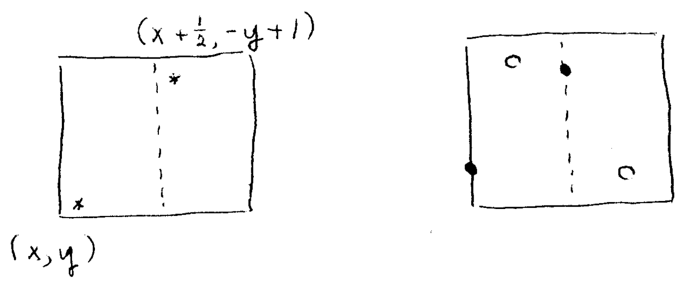
17.9ef Notice that the map a sends (x, y) to $(x, y+1)$. The map b sends (x, y) to

$$(x + \frac{1}{2}, -y + 1).$$

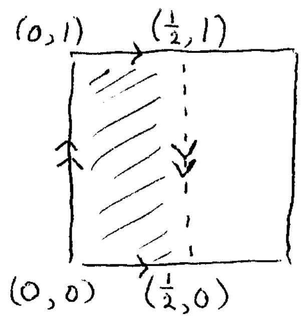
To show that $ba = a^{-1}b$ it suffices to show that $aba = b$. But $ba(x, y) = b(x, y+1) = (x + \frac{1}{2}, -(y+1)+1) = (x + \frac{1}{2}, -y)$ and therefore $aba(x, y) = a(x + \frac{1}{2}, -y) = (x + \frac{1}{2}, -y+1) = b(x, y)$. It follows that whenever we find b followed by a, we can replace this with a^{-1} followed by b. So we can get an equivalence expression with all a's on the left and all b's on the right: $a^k b^l$. Moreover, l is either even or odd, as the book says in a peculiar way.

In trying to come to grips with this problem, let us figure out where $(x, y))$ is mapped by the various group elements. The map a^k maps (x, y) to $(x + k, y)$. The map b maps (x, y) to $(x + \frac{1}{2}, -y + 1)$ and b^2 maps it to $(x + 1, -(-y + 1) + 1) = (x + 1, y)$. So if we restrict ourselves to products of a and b^2, we exactly get transformations of the integer lattice, and consequently every point is equivalent to a point in the unit rectangle with corners $(0, 0), (0, 1), (1, 1), (1, 0)$.

We still must come to grips with the action of b. It maps points on the left half of some rectangle to points on the right half of some other rectangle. Consequently (x, y) is mapped to $(x + \frac{1}{2}, -y + 1)$. Let us restrict to the previous "unit" rectangle. Here are pictures of some equivalences induced by the map b:



It now follows that every point in the plane is equivalent to a point in the shaded rectangle below, and points on the boundary are equivalent as indicated. Clearly the set of equivalence classes is the Klein bottle.



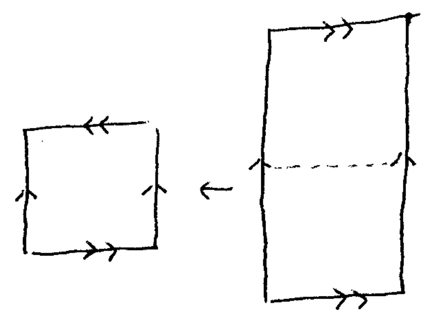
17.9h We will find a one-to-one correspondence between $p^{-1}(x)$ and $p^{-1}(y)$ where x and y are any two points in X.

Let γ be a path from x to y in X. If $q \in \pi^{-1}(x)$, there is a unique lift of γ to a path $\tilde{\gamma}$ in $\tilde{X}$ starting at q. This path ends at an element of $p^{-1}(y)$; call this element $\psi(q)$. We claim $\psi : p^{-1}(x) \to p^{-1}(y)$ is one-to-one and onto.

It is one-to-one, for if $\psi(q_1) = \psi(q_2)$, then there would be lifts $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ of γ starting at q_1 and q_2 respectively and ending at the same point. But then $\gamma \star \gamma^{-1}$ could be lifted in two different ways, as $\tilde{\gamma_1} \star \tilde{\gamma_1}^{-1}$ and as $\tilde{\gamma_1} \star \tilde{\gamma_2}^{-1}$, contradicting uniqueness of lifts.

It is onto, for if $q_1 \in p^{-1}(y)$, lift γ^{-1} to a path in $\tilde{X}$ starting at q_1. This lift ends at a point $q \in p^{-1}(x)$; clearly $\psi(q) = q_1$ because we can lift γ by tracing our lift of γ^{-1} backward.

17.9i This was done in class. See the picture below.



17.9j Recall that RP^2 can be obtained from S^2 by glueing opposite points together. In this solution, when $p \in S^2$ then $-p$ is this opposite point.

Let $\gamma : (I, 0) \to (RP^2, x_0)$ be a parameterization of the simple closed curve in RP^2. Let x_1 and $x_2 = -x_1$ be the two points in S^2 which project to x_0. Let γ_1 be the lift of γ to a path in S^2 starting at x_1. Notice that $\gamma_2 = -\gamma_1$ is then the lift of γ to a path starting at x_2. The inverse image of γ is clearly the union of the images of γ_1 and γ_2.

It may happen that $\gamma_1(1) = \gamma_1(0)$. In that case γ_1 and γ_2 are both closed curves, and they are easily proved disjoint simple closed curves.

The other possibility is that $\gamma_1(1) = -\gamma_1(0) = \gamma_2(0)$. In that case $\gamma_1 \star \gamma_2$ is a simply closed curve and so the inverse image is one curve rather than two.

17.9l This was essentially done in class. The space RP^n is obtained from S^n by glueing opposite points together, so $\pi : S^n \to RP^n$ is a two-fold cover. Since S^n is simply connected, the map $\pi(RP^n, x_0) \to \pi^{-1}(x_0)$ is one-to-one and onto. This image set has two elements, so $\pi(RP^n, x_0) = Z_2$.

17.9q By definition, X is locally path-connected if whenever x is a point and $\mathcal{U}$ is an open neighborhood of x, there is an open $\mathcal{V}$ such that $x \in \mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{V}$ is path connected.

Suppose $\tilde{X} \to X$ is a covering space and let $x \in \tilde{X}$. Choose an evenly covered open neighborhood $\mathcal{V} \subseteq X$ of $\pi(x)$. Then $\pi^{-1}(\mathcal{V}) = \cup \mathcal{V}_\alpha$. Also x is in one of these open sets; say $x \in \mathcal{V}_\beta$.

Now suppose $\mathcal{U}$ is an open neighborhood of x in $\tilde{X}$. Replace $\mathcal{V}_\beta$ by $\mathcal{V}_\beta \cap \mathcal{U}$ and replace $\mathcal{V}$ by $\pi(\mathcal{V}_\beta \cap \mathcal{U})$. This new $\mathcal{V}$ is evenly covered, and this time $\pi^{-1}(\mathcal{V}) = \cup \mathcal{V}_\alpha$ and $x \in \mathcal{V}_\beta \subseteq \mathcal{U}$.

But X is locally arcwise connected, so we can find an arcwise connected open set $\mathcal{W}$

such that $\pi(x) \in \mathcal{W} \subseteq \mathcal{V}$ and then, since $\pi : \mathcal{V}_\beta \to V$ is a homeomorphism, there is a corresponding arcwise connected open set $x \in \tilde{\mathcal{W}} \subseteq \mathcal{V}_\beta \subseteq \mathcal{U}$.

17.9r This is almost trivial. If d_1 and d_2 are covering transformations (called *deck transformations* in class), then $d_1, d_2 : \tilde{X} \to \tilde{X}$ are homeomorphisms making a diagram commute. So $d_1 \circ d_2$ is also a homeomorphism. Clearly the diagram still commutes for this composition. Similarly d_1^{-1} is a homeomorphism, and the diagram commutes for this homeomorphism.

20.7a If such a map existed, then it would induce a continuous map $\tilde{\varphi} : S^n/\pm 1 \to S^1/\pm 1$, where ± 1 symbolizes the equivalence relation which glues opposite points together. Notice that $S^n/\pm 1 = RP^n$ and $S^1/\pm 1 = S^1$. Thus we obtain a map $\tilde{\varphi} : RP^n \to S^1$.

There would be an induced map $\tilde{\varphi}_\star : Z_2 = \pi(RP^n) \to \pi(S^1) = Z$. All such group homomorphisms are trivial, so the nonzero element of $\pi(RP^n)$ would have to map to the trivial element of $\pi(S^1)$.

But $S^1 \to S^1$ by $z \to z^2$ is a covering map. which wraps the circle twice around itself. Consider the diagram below.

$$\begin{tikzcd}
 & (S^n, \tilde{x}_0) \arrow[r, "\varphi"] \arrow[d] & (S^1, \tilde{y}_0) \arrow[d, "z \to z^2"] \\
(I, 0) \arrow[ur] \arrow[r] & (RP^n, x_0) \arrow[r] & (S^1, y_0)
\end{tikzcd}$$

In this diagram, let the map $I \to RP^n$ represent the nonzero element of $\pi(RP^n)$. The corresponding lift $I \to S^n$ must then be a path which starts at $\tilde{x}_0$ and ends at $-\tilde{x}_0$. So the composition of this map with φ must be a path in S^1 which starts at $\tilde{y}_0$ and ends at $-\tilde{y}_0$. When this element is projected down by $z \to z^2$, it becomes a closed path in S^1 and this closed path is homotopic to the identity because $\pi(RP^n) \to \pi(S^1)$ is the zero map. Our lifting theory then implies that the unique lift of this map back up to where it came from as a map $I \to S^1$ must be a closed loop. This contradiction proves the exercise.

20.7b First a word about the meaning of the exercise. The sphere S^3 can be thought of as the set of all points in $C^2 = R^4$ with absolute value one, and thus as the set of all (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$. The group Z_p then acts on this space by acting on each component separately, where Z_p acts on C as the group of rotations of a regular p-sided polygon. Similarly Z_p acts on $S^1 \subseteq R^2$, again as the group of rotations of a regular p-sided polygon.

We now want a continuous map $S^3 \to S^1$. But we don't want just any map; we want our map to commute with the action of Z_p. That means that if (z_1, z_2) maps to a point $p \in S^1$, and if we then rotate z_1 and z_2 by k polygonal clicks, then the image p should also rotate by k polygonal clicks.

A special case is when $p = 2$, so rotation by one click takes (z_1, z_2) to $(-z_1, -z_2)$ and takes p to $-p$. In this case there is no such map by the previous exercise. Now we are going to prove that in general there is no such map.

If there were such a map, then it would induce a map $S^3/\sim \to S^1/\sim$ where $\sim$ denotes the equivalence relation induced by Z_p in which (z_1, z_2) is glued to the point (w_1, w_2) obtained by rotating z_1 and z_2 simultaneously by k clicks. We would get a diagram

$$\begin{tikzcd}
& (S^3, \tilde{x}_0) \arrow[r] \arrow[d] & (S^1, \tilde{y}_0) \arrow[d, "z \to z^n"] \\
(I, 0) \arrow[ur] \arrow[r] & (S^3/\sim, x_0) \arrow[r] & (S^1, y_0)
\end{tikzcd}$$

exactly as before, and the argument is going to proceed exactly as it did earlier.

Here is the key step. Consider a map $\gamma : I \to S^3$ defining a path which starts at $\tilde{x}_0$ and ends at $\tilde{x}_1$ where both complex coordinates of $\tilde{x}_1$ are obtained from the corresponding complex coordinates of $\tilde{x}_0$ by rotating by $\frac{2\pi}{n}$. This path induces a loop in $S^3/\sim$ and the corresponding element of $\pi(S^3/\sim)$ is not the identity element because its lift does not begin and end at the same point. This element has order n in $\pi(S^3/\sim)$ because when we lift the element repeated n times, we get a loop in S^3 and all loops in S^3 are homotopic to constant maps. The element in $\pi(S^3/\sim)$ must map to zero in $\pi(S^1)$ because the latter group is isomorphic to Z and has no nonzero elements of finite order. On the other hand, the map $I \to S^3 \to S^1$ obtained by following the top of the diagram is not a loop, because by equivariance its end point is obtained from its beginning by a rotation of $\frac{2\pi}{n}$. This contradiction proves the exercise.