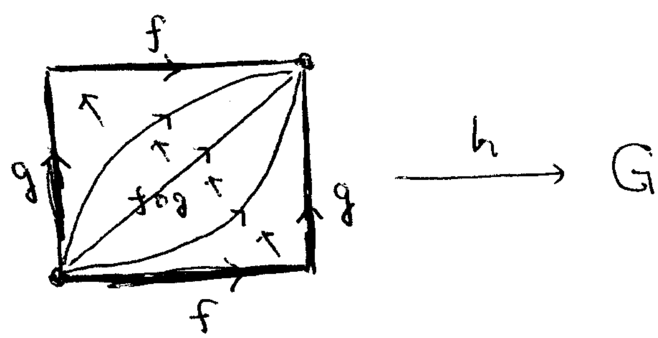
Assignment 4; Due Friday, February 3

15.6a: The isomorphism $u_f : \pi_1(X,x) \to \pi_1(X,y)$ is defined by $\gamma \to f^{-1}\star\gamma\star f$. Remember that we read such expressions from left to right. So follow f backward from y to x, and then follow the loop γ back to x, and then follow f from x to y.

If this is homotopic to $g^{-1}\star\gamma\star g$ for all γ, then multiplying both expressions by g on the left and g^{-1} on the right gives $g\star f^{-1}\star\gamma\star f\star g^{-1}\sim\gamma$. Notice that $g\star f^{-1}$ is a loop at x and thus induces an element of $\pi_1(X,x)$. In this group $(g\star f^{-1})\star\gamma = \gamma\star(f\star f^{-1})$, so $g\star f^{-1}$ is in the center of the group.

15.6b: If $\pi_1(X,x)$ is abelian, then the center of the group is everything and so $g\star f^{-1}$ is certainly in the center, so $u_f = u_g$. Conversely suppose $u_f = u_g$ for all f and g. Then $g\star f^{-1}$ is in the center of $\pi_(X,x_0)$ for all f and g. But we can arrange that $g\star f^{-1}$ is any element of $\pi_1(X,x)$ by replacing g by any loop at x followed by g. So every loop at x is in the center and the group is abelian.

15.16c: We must prove that $\pi_1(X,x)=0$ for some x. Choose $x\in\mathcal{U}\cap\mathcal{V}$. Let γ represent an element of $\pi_1(X,x)$. By exercise 14.6i, we can write $\gamma=\gamma_1\star\gamma_2\star\ldots\star\gamma_n$ where each γ_i is a path entirely in $\mathcal{U}$ or entirely in $\mathcal{V}$. We can "hook up"γ_{i-1} and γ_i if both are in $\mathcal{U}$ or both are in $\mathcal{V}$. So we can suppose that at each junction of two subcurves, the image of the junction point is in $\mathcal{U}\cap\cap V$. Since $\mathcal{U}\cap\mathcal{V}$ is arcwise connected, we can replace $\gamma_{i-1}\star\gamma_i$ by γ_{i-1} followed by a path from the end of γ_{i-1} to x, followed by the reverse of this path from x back to γ_i followed by γ_i. In the end, γ is a sum of paths which start and end at x and live entirely in $\mathcal{U}$ or entirely in $\mathcal{V}$. Each of these paths is homotopic to a constant because $\mathcal{U}$ and $\mathcal{V}$ are simply connected. So γ is homotopic to a constant.

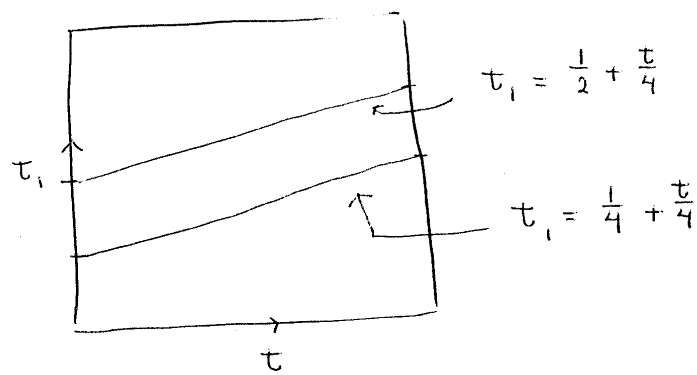


15.18f: The proof is exactly as in the above picture, replacing the map h with the map μ.

15.18g: In the previous problem we assumed that $\mu(x,x_0)=x$ for all x. In this problem we only assume that $\mu(x_0, x_0) = x_0$ and $\mu(x, x_0) : X \to X$ is homotopic to the identity by a homotopy preserving x_0. The proof is still the proof of the previous picture, where now the path along the bottom of the square and then vertically up the right side is $\mu(f(t), x_0) \star \mu(x_0, g(t))$ and the pictured homotopy in problem 15.18d makes this homotopic to $\mu(f(t), g(t))$. But the map $x \to \mu(x, x_0)$ is homotopic to the identity, so the map $t \to \mu(f(t), x_0)$ is homotopic to the map $t \to f(t)$. Similarly $\mu(x_0, g(t))$ is homotopic to $g(t)$ and so $\mu(f(t), x_0) \star \mu(x_0, g(t))$ is homotopic to $f \star g$.

In exactly the same way, $\mu(f(t), g(t))$ is homotopic to the map vertically up the left, and then over the top, which is homotopic to $g \star f$. So $f \star g$ and $g \star f$ are homotopic.

15.19a: The proof is essentially a repeat of the proof that $\pi_1(X)$ is a group, with the parameters $t_2, t_3, \ldots, t_n$ just going along for the ride. To illustrate this, we prove associativity, leaving the identity element and inverses to the reader.



If $f, g, h : I^n \to X$. then

$$(f \star g) \star h(t_1, t_2, \ldots, t_n) = \begin{cases} f(4t_1, t_2, \ldots, t_n) & 0 \le t_1 \le \frac{1}{4} \\ g(4t_1 - 1, t_2, \ldots, t_n) & \frac{1}{4} \le t_1 \le \frac{1}{2} \\ h(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \le t_1 \le 1 \end{cases}$$

and

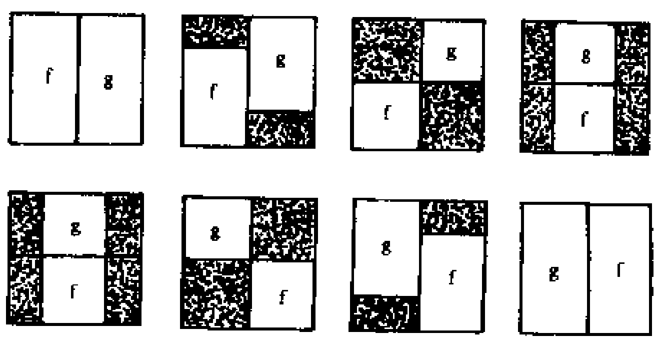
$$f \star (g \star h)(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \le t_1 \le \frac{1}{2} \\ g(4t_1 - 2, t_2, \ldots, t_n) & \frac{1}{2} \le t_1 \le \frac{3}{4} \\ h(4t_1 - 3, t_2, \ldots, t_n) & \frac{3}{4} \le t_1 \le 1 \end{cases}$$

We must prove that these maps are homotopic, for then $(f \star g) \star h = f \star (g \star h)$ in $\pi_n(X)$. The required homotopy is illustrated in the picture below, and given in equation form as

$$h(t_1, t_2, \ldots, t_n, t) = \begin{cases} f\left(\frac{t_1}{\frac{1}{4}+\frac{t}{4}}, t_2, \ldots, t_n\right) & \text{if } 0 \le t_1 \le \frac{1}{4} + \frac{t}{4} \\ g\left(\frac{t_1 - \frac{1}{4} - \frac{t}{4}}{\frac{1}{4}}, t_2, \ldots, t_n\right) & \text{if } \frac{1}{4} + \frac{t}{4} \le t_1 \le \frac{1}{2} + \frac{t}{4} \\ h(\frac{t_1 - \frac{1}{2} - \frac{t}{4}}{\frac{1}{2} - \frac{t}{4}}, t_2, \ldots, t_n) & \text{if } \frac{1}{2} + \frac{t}{4} \le t_1 \le 1 \end{cases}$$

This is indeed exactly what we wrote in class, with $t_2, \ldots, t_n$ added but with no role to play.

15.19e: The book has essentially given the proof. Notice that only t_1 and t_2 take part on the homotopy between $f \star g$ and $g \star f$; the remaining variables just go along for the ride.



Let's try to convert the pictorial proof in the book into English without writing very many equations. For this we need some notation. Notice that in π_n when we add two elements using the star operation, this star only involves the first t_1; all of the other $t_2, \ldots, t_n$ just go along for the ride. To make this clear, let us write $f \star_1 g$ to indicate that we are adding along the t_1 axis.

When we proved that $\pi_1(X, x_0)$ is a group, we proved that f is homotopic to $f \star \epsilon_{x_0}$. In our proof, f and ϵ_{x_0} were functions of only one variable, but if we like we can add other variables and let them go along for the ride. Let us apply this, thinking of t_2 as the active variable. To indicate this, we use $\star_2$, so f is homotopy to $f \star_2 \epsilon_{x_0}$. Similarly g is homotopic to $\epsilon_{x_0} \star_2 g$.

The book asserts that adding maps $f : I^n \to X$ and $g : I^n \to X$ induces a sum in homotopy classes of such maps — that is, the homotopy class of the sum depends only on the homotopy class of f and the homotopy class of g. Consequently, the element in $\pi_n(X)$

represented by $f \star_1 g$ is homotopic to the element represented by $(f \star_2 \epsilon_{x_0}) \star_1 (\epsilon_{x_0} \star_2 g)$. This homotopy is illustrated by the first three pictures in the sequence on page 133.

Next look closely at the third picture in the sequence. There are two ways to think of this picture, and it is easy to see that they give the same thing. We got this above as

$$(f \star_2 \epsilon_{x_0}) \star_1 (\epsilon_{x_0} \star_2 g)$$

but it could also be described as

$$(f \star_1 \epsilon_{x_0}) \star_2 (\epsilon_{x_0} \star_1 g).$$

Notice that in $\pi_1(X)$ we have $\gamma \star \epsilon_{x_0} \sim \gamma \sim \epsilon_{x_0} \star \gamma$. Hence in the previously displayed formula we can replace $f \star_1 \epsilon_{x_0}$ by the homotopically equivalent $\epsilon_{x_0} \star_1 f$ and we can replace $\epsilon_{x_0} \star_1 g$ by the homotopically equivalent $g \star_1 \epsilon_{x_0}$ to obtain

$$(\epsilon_{x_0} \star_1 f) \star_2 (g \star_1 \epsilon_{x_0}).$$

This gets us to the sixth picture in the sequence. Notice that this picture is algebraically the same as

$$(\epsilon_{x_0} \star_2 g) \star_1 (f \star_2 \epsilon_{x_0}).$$

Now we repeat the previous argument backward. The term $\epsilon_{x_0} \star_2 g$ is homotopic to g and the term $f \star_2 \epsilon_{x_0}$ is homotopic to f and consequently the previously displayed expression is homotopic to $g \star_1 f$.

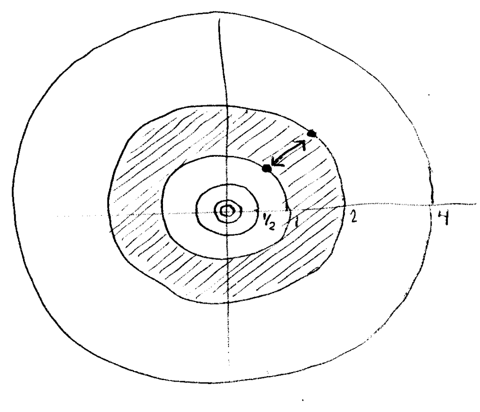
16.11b: We claim $f_\star : \pi(S^1) \to \pi(S^1)$, i.e., $f_\star : Z \to Z$, is multiplication by k, so $f_\star(n) = kn$. There are several ways to see that; here is one. It suffices to prove that $f_\star$ maps the generator $1 \in Z$ to $k \in Z$. As generator we can choose the map $\gamma : I \to S^1$ given by $\gamma(t) = e^{2\pi it} = \cos(2\pi t) + i\sin(2\pi t)$ since the lift of this map is $\tilde{\gamma} : I \to R$ by $\tilde{\gamma}(t) = t$; notice that $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1) = 1$. Then $f_\star(1)$ is represented by $\tau : I \to S^1$ given by $\tau(\gamma(t)) = (e^{2\pi it})^k = e^{2\pi ikt}$ and the lift of this path is $\tilde{\tau}(t) = kt$. Notice that $\tilde{\tau}(1) = k$, as desired.

16.11f: Suppose we remove a boundary point from D. Without loss of generality we can suppose that this point is $p_0 = (1, 0)$. Then there is a deformation retract from $D - \{p_0\}$ to the point $p_1 = (-1, 0)$; indeed we can define a homotopy from the identity map to this retract via $h(p, t) = (1 - t)p + tp_1$. Notice that $p \neq (1, 0)$ and thus this line segment stays in $D - \{p_0\}$. It follows that $\pi(D - \{p_0\})$ is isomorphic to $\pi(\{p_1\}) = 0$ and so $D - \{p_0\}$ is simply connected.

Next remove a point p_0 not in the boundary from D. Then there is a retract r from $D - \{p_0\}$ to the boundary S^1 defined as follows; to compute $r(p)$, draw a line from p_0 to p and extend this line to the boundary; let $r(p)$ be the spot where the line segment meets the boundary. Notice that boundary points are fixed under this map. It follows that $r_\star : \pi(D - \{p_0\}) \to \pi(S^1)$ is onto, so $\pi(D - \{p_0\})$ is not the zero group and thus $D - \{p_0\}$ is not simply connected.

Let $f : D \to D$ be a homeomorphism. Then if $p \in D$, $f : D - \{p\} \to D - \{f(p)\}$ is also a homeomorphism. In particular, both are simply connected or neither are simply connected. So p is in the boundary if and only if $f(p)$ is in the boundary.

16.11g (ii, iii, iv): ii) Using the picture below and the hint,



we find that the quotient space is homeomorphic to a torus, so the fundamental group is $Z \times Z$. One way to see this homeomorphism is to cut the shaded circular region along the line from $(1, 0)$ to $(2, 0)$. The resulting region is homeomorphism to a rectangle, where this cut becomes the sides of the rectangle, and the inner concentric circle of radius 1 becomes the bottom and the outer concentric circle of radius 2 becomes the top. We need to glue these sides together, giving us a torus.

iii) Every point is still equivalent to a point in the shaded region, but now we are to glue the point with angle θ on the innter concentric circle to the point with angle $2\pi - \theta$ on the outer concentric circle. When we cut and deform to a rectangle, we are to glue the sides of the rectangle as usual, but glue the top and bottom with arrows pointing in opposite directions. This gives a Klein bottle, so the fundamental group is $Z \rtimes_\varphi Z$.

iv) If $p = (x, y) = x+iy$, then $\psi(p) = -(x-iy) = -x+iy = (-x, y)$. So we are supposed to fold the left half of the plane to the right half. This space is then $\{(x, y) \mid x \geq 0\} - \{(0, 0)\}$, which is clearly simply connected. So the fundamental group is the trivial group.