Assignment 3; Due Friday, January 27

15.11a: What a hint! The injective map $S^1 \to D^2$ obtained by sending the circle to the boundary of the disk induces $\pi_1(S^1) \to \pi_1(D^2)$, i.e., $Z \to \{0\}$, and this group homomorphism cannot be one-to-one.

15.11b: Let $f : S^1 \to S^1$ be the map defined in polar coordinates by

$$f(\theta) = \begin{cases} 2\theta & \text{if } 0 \le \theta \le \pi \\ \\ 4\pi - 2\theta & \text{if } \pi \le \theta \le 2\pi \end{cases}$$

This map moves the top hemisphere counterclockwise around the complete circle, and then moves backward retracing the circle back to the identity. The map is clearly onto.

However, the induced map $\pi_1(S^1) \to \pi_1(S^1$, i.e. $Z \to Z$, sends everything to zero because it sends the generator to $\gamma \star \gamma^{-1}$ where γ goes around the complete circle, and this map is homotopic to a constant.

This exercise and the next two weren't to be turned in. Just for the record:

15.11d: This result was proved in class. Let $h : X \times I \to Y$ be a homotopy from f to g relative to x_0 and let $\gamma : I \to X$ be a closed path starting and ending at x_0. Then $h \circ \gamma$ is a homotopy from $f \circ \gamma$ to $g \circ \gamma$ relative to $\{0, 1\}$ in $[0, 1]$, so these elements represent the same element of $\pi_1(Y)$. But by definition these elements of $\pi_1(Y)$ are $f_\star(\gamma)$ and $g_\star(\gamma)$.

15.11e: Since $r \circ i : A \to X \to\to A$ is the identity, the map

$$r_\star \circ i_\star : \pi_1(A) \to \pi_1(X) \to \pi_1(A)$$

is the identity. It immediately follows that $i_\star : \pi_1(A) \to \pi_1(X)$ is one-to-one and $r_\star : \pi_1(X) \to \pi_1(A)$ is onto.

15.11g: Consider the map

$$r \circ i \circ r \circ i : A \to X \to A \to X.$$

This map induces

$$r_\star \circ i_\star \circ r_\star \circ i_\star : \pi_1(A) \to \pi_1(X) \to \pi_1(A) \to \pi_!(X)$$

Since the map $r \circ i : A \to A$ is the identity, the map $r_\star \circ i_\star : \pi_1(A) \to \pi_1(A)$ is the identity; from this it immediately follows that $i_\star$ is one-to-one. Since the map $i \circ r : X \to X$ is homotopic to the identity, the map $i_\star \circ r_\star : \pi_1(X) \to \pi_1(X)$ is the identity by exercise 15.11d. From this it immediately follows that $i_\star$ is onto.

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, hook] & \pi_1(X) \arrow[r, "r_\star"] & \pi_1(A) \arrow[r] \arrow[l, bend right, "i_\star"'] & 0 \end{tikzcd}$$

15.11f: Let H be the kernal of the map $r_\star : \pi_1(X) \to \pi_1(A)$. We have the following diagram:

Let K be the image of $i_\star$. We claim that every element of $\pi_1(X)$ can be written uniquely as hk where $h \in H$ and $k \in K$. Indeed if $g \in \pi_1(X)$, let $k = i_\star(r_\star(g))$ and write $g = (gk^{-1})k$. Notice that gk^{-1} is in H because $r_\star$maps it to the identity, since g and k both map to $r_\star(g)$ by $r_\star$.

This representation is unique, for if $g = h_1k_1 = h_2k_2$ where $k_1 = i_\star(a_1)$ and $k_2 = i_\star(a_2)$, then $r_\star(g) = a_1$ and $r_\star(g) = a_2$, so $a_1 = a_2$, so $k_1 = k_2$ and then $h_1 = h_2$.

Consider the produce $(h_1k_1)(h_2k_2)$. Notice that H is normal because kernals of maps are normal subgroups. We are assuming that K is normal (this is not automatically true).

We can write the above product in two different ways:

$$(h_1k_1)(h_2k_2) = (h_1k_1h_2k_1^{-1})(k_1k_2)$$

$$(h_1k_1)(h_2k_2) = (h_1h_2)(h_2^{-1}k_1h_2k_2)$$

Since H is normal, the first expression is the product of something in H and something in K. Since K is normal, the second expression is the product of something in H and something in K. Since such product expressions are unique, the first term in H is h_1h_2 and the second term in K is k_1k_2. So

$$(h_1k_1)(h_2k_2) = (h_1h_2)(k_1k_2)$$

and thus as a group we have $\pi_1(X) = H \times K$.

15.18b: The map $(if) \star (jg)$ is defined by

$$(if) \star (jg)(t) = \begin{cases} f(2t) \times y_0 & \text{if } 0 \leq t \leq \frac{1}{2} \\ \\ x_0 \times g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

This map is $(f \star \epsilon_{x_0}) \times (\epsilon_{y_0} \star g)$. But $f \star \epsilon_{x_0} : I \to X$ is homotopic to $f : I \to X$ and $\epsilon_{y_0} \star g : I \to Y$ is homotopic to $g : I \to Y$ by the proof that $\epsilon \in \pi$ is an identity element.

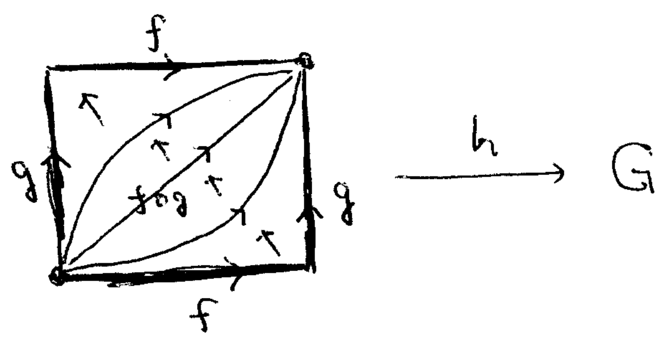
Call these homotopies h and k; their product is a homotopy $h \times k : I \times I \to X \times Y$ from $(f \star \epsilon_{x_0}) \times (\epsilon_{y_0} \star g)$ to $f \times g$.

Similarly the map $(jg) \star (if)$ is $(\epsilon_{x_0} \star f) \times (g \star \epsilon_{y_0})$ and this is also homotopic to $f \times g$. So $(if) \star (jg)$ is homotopic to $(jg) \star (if)$.

15.18c: We will prove that $\pi_1(X) \times \pi_1(Y) \to \pi_1(X \times Y)$ is one-to-one and onto. It is onto, for let $\gamma : I \to X \times Y$ be a closed path in $X \times Y$. Then $\gamma(t) = f(t) \times g(t)$ for each t and so $f : I \to X$ and $g : I \to Y$ are closed paths in X and Y. By the proof of the previous exercise, this map γ is homotopic to $(if) \star (jg)$ and thus in the image of our map.

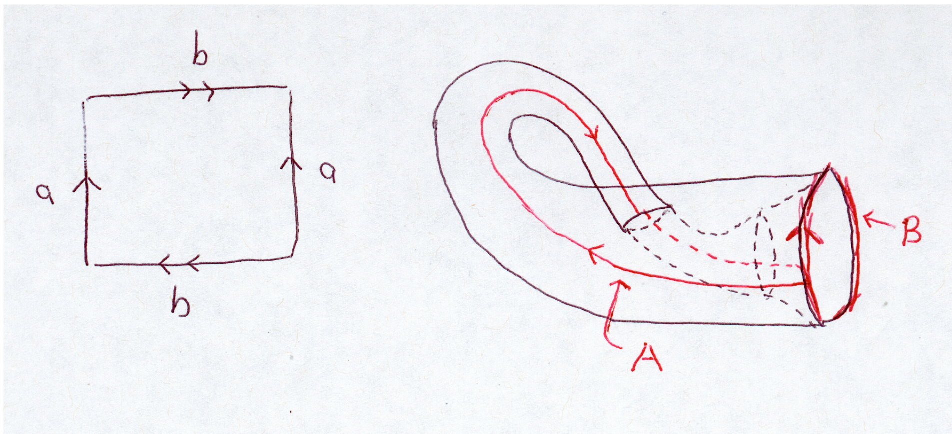
Next we prove that our map is one-to-one. Suppose $f : I \to X$ induces an element of $\pi_1(X)$ and $g : I \to Y$ induces an element of $\pi_1(Y)$. Suppose the sum of these elements is the identity in $\pi_1(X \times Y)$. The projection maps $p : X \times Y \to X$ and $q : X \times Y \to Y$ then map this identity to the identities in $\pi_1(X)$ and $\pi_1(Y)$. But the sum maps to $(if) \star (jg)$ in $\pi_1(X \times Y)$ and thus these projected elements are $f \star \epsilon_{x_0}$ and $\epsilon_{y_0} \star g$ in $\pi_1(X)$ and $\pi_1(Y)$. These paths are homotopic to f and g, so the original elements f and g in $\pi_1(X)$ and $\pi_1(Y)$ are both zero.

15.18d: Consider the map $h : I \times I \to G$ defined by $(t_1, t_2) \to f(t_1) \circ g(t_2)$ where $\circ$ is the group operation in G. The picture below shows a homotopy from the curve $f \star g$ to the curve $f(t) \circ g(t)$ and on to the curve $g \star f$.



Extra Problem 1: Let I^2 be the square pictured on the left below, and let the Klein bottle $\mathcal{K}$ be the quotient space $I^2/\sim$ induced by the indicated equivalences on the boundary of the square. Consider the path $\gamma : I \to I^2$ obtained by mapping $[0, 1/4]$ to the left side of the square traveling upward, mapping $[1/4, 1/2]$ horizontally along the top of the square traveling right, mapping $[1/2, 3/4]$ vertically downward along the right side of the square, and mapping $[3/4, 1]$ horizontally toward the left along the bottom of the square. Composing this map with the quotient map $\pi : I^2 \to \mathcal{K}$ gives a closed path in the Klein bottle which is clearly the map $aba^{-1}b$ in $\mathcal{K}$, up to parameterization which doesn't change the homotopy class.

The path γ is homotopic in I^2 through paths with fixed based point 0×0 to a constant path, by pulling the path downward and to the left. Let $h : I \times I \to I^2$ be this homotopy. Composing this map with the identification map $I^2 \to \mathcal{K}$ gives a homotopy $I \times I \to \mathcal{K}$ from $aba^{-1}b$ to the identity map. So in $\pi_1(\mathcal{K}$ we have $aba^{-1}b = e$ and so $aba^{-1} = b^{-1}$.



Extra Problem 2: Use the picture on the left, and let $r(x, y) = (0, y)$. This map squashes the square horizonally to its left side. Notice that this map respects the equivalence relation and so induces a map $\mathcal{K} \to \mathcal{K}$. Indeed $(0, y) \sim (1, y)$, but $r(0, y) = r(1, y) = (0, y)$. Also $(x, 0) \sim (1 - x, 1)$, but $r(x, 0) = (0, 0)$ and $r(1 - x, 1) = (0, 1)$ and these points are "equal" because they are equivalent. Clearly points in A remain fixed because $r(0, y) = (0, y)$.

Extra Problem 3: The map $A \to \mathcal{K} \to A$ is the identity, so it induces the identity map $\pi_1(A) \to \pi_1(\mathcal{K}) \to \pi_1(A)$, i.e.

$$Z \to \pi_1(\mathcal{K}) \to Z$$

The element $1 \in Z$ induces the circle $a \in \pi_1(\mathcal{K})$. This element has infinite order because it maps back to $1 \in \pi_1(A)$, that is, $1 \in Z$, which has infinite order.

Extra Problem 4: Suppose there were a retraction $r : \mathcal{K} \to B$. Then since

$$Z = \pi_1(B) \to \pi_1(\mathcal{K}) \to \pi_1(B) = Z$$

is the identity map, $r_\star$ would map $b \in \pi_1(\mathcal{K})$ to the identity $1 \in Z$. We don't know what the retraction map looks like off B, so we don't know what $r_\star(a)$ equals. It is some element of $\pi_1(B) = Z$, say an integer m.

Since $aba^{-1} = b^{-1}$, $r_\star(aba^{-1}) = r_\star(b^{-1})$, so $m + 1 - m = -1$ in Z, and thus $1 = -1$ in Z. But this is false, so r cannot exist.