Assignment 2; Due Friday, January 20

13.10o: Define $h : X \times I \to Y$ by

$$h(x,t) = (1-t)f(x) + tg(x)$$

This makes sense because $f(x) \in R^n$ and $g(x) \in R^n$, so these values can be multiplied by scalars and added. Notice that as t increases from 0 to 1, $h(x,t)$ moves along the straight line from $f(x)$ to $g(x)$. Since this line segment is in Y, h is a homotopy in Y from f to g.

13.10p: Define a homotopy $h : X \times I \to S^n$ by

$$h(x,t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

This homotopy moves in R^{n+1} along the straight line from $f(x)$ to $g(x)$ and then points on this line are pushed out to the sphere S^n. Notice that the denominator is never zero, for if $(1-t)f(x) + tg(x) = 0$, then $(1-t)f(x) = -tg(x)$ and so $||(1-t)f(x)|| = ||-tg(x)||$. But then $1 - t = t$ and so $t = \frac{1}{2}$. In particular $\frac{1}{2}f(x) = -\frac{1}{2}g(x)$ and thus $f(x) = -g(x)$, contradicting our assumption.

In particular, suppose $f : X \to S^n$ is not onto; say $p \in S^n$ is not in the image of f. Let $g : X \to S^n$ be identically $-p$. Then $f(x)$ is never $-g(x)$ because $f(x)$ is never p. So f and g are homotopic and in particular f is homotopic to a constant map.

14.6a: Suppose that f, g, and h are constantly x_0. Then $(f \star g) \star h$ and $f \star (g \star h)$ are constantly x_0 and so equal.

Let $f : [0,1] \to R^2$ be $f(t) = (t, 0)$ and let $g(t)$ and $h(t)$ be identically $(1,0)$. Then $(f \star g) \star h$ traverses the x axis from $(0,0)$ to $(1,0)$ at constant speed from time 0 to time $\frac{1}{4}$ and then is identically $(1,0)$ at all later times. But $f \star (g \star h)$ traverses the x axis from $(0,0)$ to $(1,0)$ at constant speed from time 0 to time $\frac{1}{2}$ and then is identically $(1,0)$ at all later times. In particular, $((f \star g) \star h)(1/4) = (1,0)$ and $(f \star (g \star h))(1/4) = (1/2, 0)$.

14.6c: First notice that fh is not the same thing as $f \star h$. Indeed, $f \star h$ makes no sense in the context of this problem. Instead, fh is a reparameterization of the path f and according to the exercise, such reparameterizations do not change the homotopy class of the path.

To avoid confusion, think of f and h as functions of $u \in [0,1]$, reserving t for the homotopy variable. Define a homotopy from f to fh by

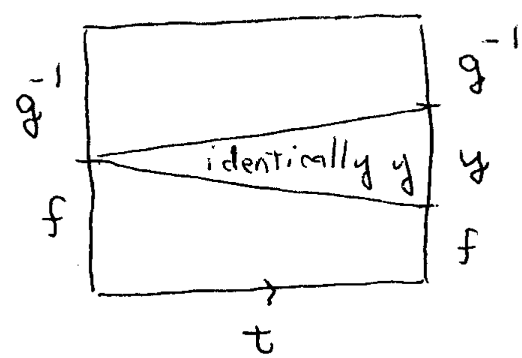
$$H(u,t) = f((1-t)u + th(u))$$

This formula makes sense because $(1-t)u+th(u)$ is a line in $[0,1]$ from u to $h(u)$ and thus is always in $[0,1]$. At $t=0$ we have $f(u)$ and at $t=1$ we have $f(h(u))$.

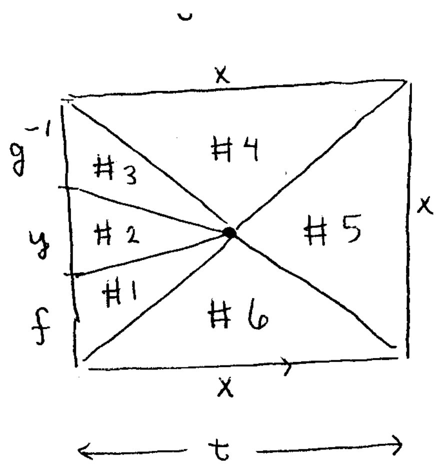
14.6e: First let's clarify the problem. Assume that $f \sim g$ means a homotopy relative to $\{0,1\}$, so all intermediate paths go from x to y. Assume that $f \star g^{-1} \sim \epsilon$ means a homotopy relative to $\{0,1\}$, so all intermediate paths go from x to x. (If we don't put some conditions on endpoints, the problem becomes trivial since f and g would be homotopic to constant paths.)

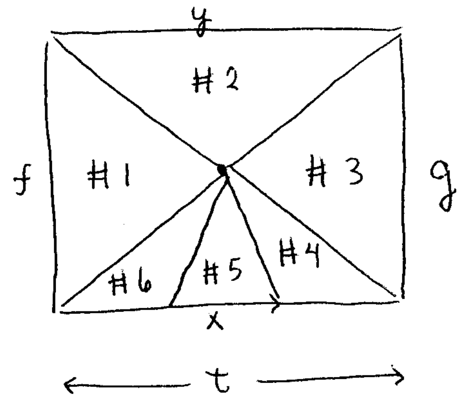
Rather than writing equations for the required homotopies, we'll draw pictures; as you'll see, it would be easy to convert our pictures to equations.

First notice that $f \star g^{-1}$ is homotopic to $f \star \epsilon_y \star g^{-1}$, the path which moves along f from x to y, then remains fixed at y for a time, and then retreats backward along g^{-1} to x. The picture below shows this homotopy.



Next notice that this homotopy can be cut into pieces and reassembled to show a homotopy between f and g fixing endpoints. The picture on the next page shows this disassembly and reassembly.





These pictures are reversible.

For example, suppose $f \star g^{-1}$ is homotopic to ϵ fixing endpoints. Since $f \star g^{-1}$ is homotopic to $f \star \epsilon_y \star g$, there would be a homotopy from $f \star \epsilon_y \star g$ to ϵ_x, as on the left at the top of this page. By cutting this picture apart and gluing it together as on the right, we obtain a homotopy from f to g.

We know what happens on the boundary of the squares as indicated on the pictures, but we don't know what happens inside. For example, in region two there is no assumption that the homotopy is identically y on this region.

Notice that there is a unique linear map from each region on the left to each region on the right, since such a map is completely determined by its value on three points. In particular, the map is linear on the lines between two regions, so the maps agree on the boundary lines. Thus these linear maps glue to form a continuous map from the square on the left to the square on the right.

14.6i: We proved exercise 7.13g on Lebesgue numbers. Notice that $f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{V})$ form an open cover of $[0,1]$. By the exercise, there is a number $\delta > 0$ such that any subset $J \subseteq [0,1]$ of diameter smaller than δ is contained in one of these two sets. Subdivide the interval I via $0 = u_0 < u_1 < \ldots < u_n = 1$ so that for each i we have $|u_i - u_{i-1}| < \delta$. Then for each i, we have $f([u_{i-1}, u_i]) \subseteq \mathcal{U}$ or $f([u_{i-1}, u_i]) \subseteq \mathcal{V}$.

Let $f_i(u) = f((1-u)u_{i-1} + (u)u_i)$ where $0 \le u \le 1$. Notice that f_i is defined on $[0,1]$, but is just f restricted to the interval $[u_{i-1}, u_i]$ and then reparameterized so it starts at the left when $u = 0$ and gets to the right when $u = 1$.

We must now prove that $[f] = [f_1] \star [f_2] \star \ldots \star [f_n]$. Notice that we never defined $f_1 \star f_2 \star \ldots \star f_n$ for for more than two paths. The reason the author is allowed to write a product with n elements is that he is dealing with homotopy classes and the product has been proved associative on homotopy classes. Thus the homotopy class of $(f \star g) \star h$ and the homotopy class of $f \star (g \star h)$ are equal, so we can remove parentheses and talk about the class of $f \star g \star h$. This is easily extended to the product of n elements.

To complete the argument, we will prove that the original f is homotopic to

$$g = f1 \star (f_2 \star (f_3 \star \ldots \star (f_{n-1} \star f_n) \ldots))$$

Notice that f_1 is just the original f on $[u_0, u_1]$ except that the parameter has been linearly changed to go from $u = 0$ to $u = 1$. Notice that g on $[0, \frac{1}{2}]$ is just f_1 on $[0, 1]$ except that the parameter has been linearly changed to go from $u = 0$ to $u = \frac{1}{2}$. Putting these two together, g on $[0, \frac{1}{2}]$ is just f on $[u_0, u_1]$ except that the parameter has been linearly changed to go from 0 to $\frac{1}{2}$.

Similarly f_2 is just the original f on $[u_1, u_2]$ except that the parameter has been linearly changed to go from $u = 0$ to $u = 1$. Notice that g on $[\frac{1}{2}, \frac{3}{4}]$ is just f_1 on $[0, 1]$ except that the parameter has been linearly changed to go from $u = \frac{1}{2}$ to $u = \frac{3}{4}$. Putting these together, g on $[\frac{1}{2}, \frac{3}{4}]$ is just f on $[u_1, u_2]$ except that the parameter has been linearly changed to go from $\frac{1}{2}$ to $\frac{3}{4}$. Etc. Finally, g on $[\frac{2^n-1}{2^n}, \frac{2^n}{2^n}]$ is just f on $[u_{n-1}, u_n]$ except that the parameter has been linearly changed.

15.3a: We can only define $f \star g$ if the path f ends at the spot that the path g begins. If we omit basepoint, then elements of $\pi_1(X)$ would be paths which start and end at the same point, but the paths f and g might start and end at different points, making it impossible to define $f \star g$.

To finish the proof, we need only apply exercise 14.6c. On one side we have the original f, and on the other side we have fh where $h : [0, 1] \to [0, 1]$ is the homeomorphism which maps $[u_0, u_1]$ to $[0, \frac{1}{2}]$ linearly, maps $[u_1, u_2]$ to $[\frac{1}{2}, \frac{3}{4}]$ linearly, maps $[u_2, u_3]$ to $[\frac{3}{4}, \frac{7}{8}]$ linearly, etc., and finally maps $[u_{n-1}, u_n]$ to $[\frac{2^n-1}{2^n}, \frac{2^n}{2^n}]$ linearly.

15.3c: I claim that all continuous paths $f : [0, 1] \to Q$ are constant. If so, the only continuous path $f : [0, 1] \to Q$ mapping 0 and 1 to 0 is the path which is identically 0, so $\pi_1(Q)$ has a single element.

Notice carefully that Q does not have the discrete topology. For instance, $\mathcal{U} = \{0\}$ is not an open set. However we can prove that f is constant because $[0, 1]$ is connected and Q is highly disconnected. Suppose that $a, b \in [0, 1]$ and $f(a) \neq f(b)$. Find an irrational x between $f(a)$ and $f(b)$. Let $\mathcal{V} = (-\infty, x)$ and $\mathcal{W} = (x, \infty)$. Then $[0, 1] = f^{-1}(\mathcal{V}) \cup f^{-1}(\mathcal{W})$ and these open sets are disjoint and nonempty, a contradiction.