Assignment 1; Due Friday, January 13

Extra Problem 1: Notice that 1 generates Z. If $Z \times Z$ were isomorphic to Z, then some $(a,b) \in Z \times Z$ would generate $Z \times Z$. In particular, $(1,0) = n_1(a,b)$ and $(0,1) = n_2(a,b)$. The first of these equations implies that $1 = n_1 a$ and $0 = n_1 b$. But $1 = n_1 a$ implies that $n_1 \neq 0$ and so $0 = n_1 b$ implies that $b = 0$. In exactly the same way, $a = 0$. But $(a,b) = (0,0)$ certainly does not generate $Z \times Z$.

Extra Problem 2: Notice that $(1,1)$ generates $Z_3 \times Z_5$ since the powers of this element are $(1,1), (2,2), (0,3), (1,4), (2,0), (0,1), (1,2), (2,3), (0,4), (1,0), (2,1), (0,2), (1,3), (2,4)$, and $(0,0)$. The powers of $1 \in Z_{15}$ are $1,2,3,4,5,6,7,8,9,10,11,12,13,14,0$. Let a be the element $(1,1) \in Z_3 \times Z_5$ and let b be the element $1 \in Z_{15}$. Writing the groups multiplicately, the isomorphism is given by mapping a^k to b^k.

Extra Problem 3: Every element in $Z_2 \times Z_2 \times Z_3$ has order less than or equal to 6, since $6(a,b,c) = (6a, 6b, 6c) = (0,0,0)$. But $1 \in Z_{12}$ has order 12.

13.4a I'll give a slightly different solution than the book recommends. The idea of this solution is that the cylinder $S^1 \times I$ is homeomorphic to the annulus of all points in the disk with norm between $\frac{1}{2}$ and 1.

Suppose $f : S^1 \to X$ is homotopic to a constant, and let $h : S^1 \times I \to X$ be the homotopy, satisfying $h(x,0) = f(x)$ and $h(x,1) = x_0$. Define $g : D^2 \to X$ by

$$g(p) = \begin{cases} h\left(\frac{p}{||p||}, 2 - 2||p||\right) & \text{if } \frac{1}{2} \leq ||p|| \leq 1 \\ x_0 & \text{if } 0 \leq ||p|| \leq \frac{1}{2} \end{cases}$$

This is clearly continuous and equals $h(p,0) = f(p)$ on the boundary of the disk.

Conversely, suppose $g : D \to X$ exists and equals f on the boundary of the disk. Define homotopy $h : S^1 \times I \to X$ between f and the constant map by

$$h(p,t) = g(p(1-t))$$

When $t = 0$, this is $g(p) = f(p)$. When $t = 1$, this is $g(0)$, a constant.

13.4b If x and y cannot be connected by an arc, then $P(x,y) = \emptyset$ and there is nothing to prove.

Otherwise we must show that $P(x,y)$ and $P(x,x)$ are in one-to-one correspondence. I'll merely sketch the argument, although it would be easy to write down explicit homotopy maps for each assertion.

Once and for all choose a path $\sigma : [0,1] \to X$ from x to y; denote by σ^{-1} this same path traced in the reverse direction from y to x.

If $\gamma : [0,1] \to X$ is a path from x to x, let $\gamma \star \sigma$ be the path which traces γ from x to x and then traces σ from x to y. I claim this induces a map $P(x,x) \to P(x,y)$. Indeed if γ_1 and γ_2 are homotopic with fixed endpoints, it is easy to see that $\gamma_1 \star \sigma$ and $\gamma_2 \star \sigma$ are homotopic.

If $\gamma : [0,1] \to X$ is a path from x to y, let $\gamma \star \sigma^{-1}$ be the path which traces γ from x to y and then σ^{-1} back from y to x. I claim this induces a map $P(x,y) \to P(x,x)$. Indeed if γ_1 and γ_2 are homotopic paths from x to y with fixed endpoints, it is easy to see that $\gamma_1 \star \sigma^{-1}$ and $\gamma_2 \star \sigma^{-1}$ are homotopic.

To finish the argument, it is enough to prove that the maps $P(x,x) \to P(x,y) \to P(x,x)$ and $P(x,y) \to P(x,x) \to P(x,y)$ are both identity maps, for then the individual map $P(x,x) \to P(x,y)$ must be one-to-one and onto.

Consider the first map. It starts with a path γ from x to x, and follows this with the path σ from x to y, and then with the path σ^{-1} from y back to x. This is homotopic to the path which starts with γ from x to x and then follows σ part way from x to y and then follows σ^{-1} part way from this point back to x. At the start of the homotopy, we follow σ all the way from x to y and back. In the middle, we following σ part way and then go back. At the end, we just stay at x and don't move along σ at all. So at the end of the homotopy we just have γ. Hence the map $P(x,x) \to P(x,y) \to P(x,x)$ maps γ to a path homotopic to γ and thus equivalent to γ in the set of equivalence classes. Said another way, this map is the identity map.

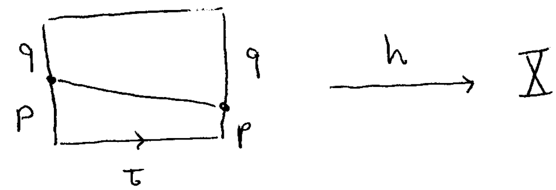
Similarly $P(x,y) \to P(x,x) \to P(x,y)$ is the identity map.

13.4c This wasn't quite fair, since we skipped the earlier section where $p \star q$ was defined. By definition, $p \star q$ is the path which first traces p and then traces q. In detail

$$(p \star q)(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ q(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

In the exercise we must assume $0 < s < 1$. So in words the exercise says that we get the same curve up to homotopy by tracing p very fast and q very slow, or p and q at the same speed, or p very slow and q very fast. We prove this by writing down a homotopy. First we draw a schematic picture of the idea.

Here are the mathematical details. Since t is being used for the homotopy, let us use u for the parameter of the paths. In the picture, we divide between the two paths at s on the right. Let us divide between the two paths at $s(t)$ in the middle. On the left we want $s(0) = \frac{1}{2}$ and we want $s(1) = s$. By linearity, $s(t) = \frac{1}{2}(1-t) + st$. So the homotopy at t



should just be the formula on the right, but replacing s by $s(t)$. Thus

$$h(u,t) = \begin{cases} p\left(\frac{t}{s(t)}\right) & \text{if } 0 \le t \le s(t) \\ p\left(\frac{t-s(t)}{1-s(t)}\right) & \text{if } s(t) \le t \le 1 \end{cases}$$

13.4e Suppose $h : X \times I \to Y$ is a homotopy from f_0 to f_1 relative to A. Then $g \circ h : X \times I \to Y \to Z$ is a homotopy from $g \circ f_0$ to $g \circ f_1$ relative to A.

13.4f As suggested in the hint, $g_0 \circ f_0$ is homotopic to $g_0 \circ f_1$ by the previous exercise. We now prove $g_0 \circ f_1$ is homotopic to $g_1 \circ f_1$. Since homotopy is an equivalence relation, the result will follow. Let $k : Y \times I \to Z$ be a homotopy from g_0 to g_1. Then $k \circ (f_1 \times id) : X \times I \to Y \times I \to Z$ is a homotopy from $g_0 \circ f_1$ to $g_1 \circ f_1$.

13.10a Think of the Mobius Band M as the set $\{(x, y) \mid 0 \le x \le 10, -1 \le y \le 1\}$ with identification of the left and right edges via $(0, y) \sim (10, -y)$. This band contains a circle $\{(x, 0) \mid 0 \le x \le 10\}$. Define a restract from M to the circle by $f(x, y) = (x, 0)$. Notice that this map preserves the glueing.

Define a homotopy $h(x, y, t)$ from $M \times I$ to M by $h(x, y, t) = (x, (1-t)y)$. Notice that $(0, y)$ is mapped to $(0, (1-t)y)$ and $(10, -y)$ is mapped to $(10, -(1-t)y)$ and so this homotopy respect the glueing. At time $t = 0$ this map is $h(x, y, 0) = (x, y)$, the identity mapping. At $t = 1$ the map is $h(x, y, 1) = (x, 0)$, sending M to the circle. Finally, points on the circle are mapped to themselves since $h(x, 0, t) = (x, 0)$. So our retract is a strong deformation retract.

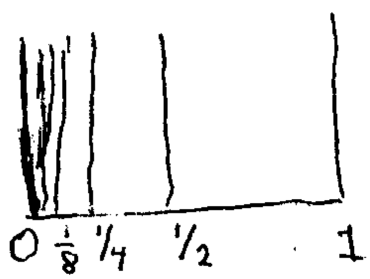
13.10b By definition, X is contractible if there is a space consisting of a single point p, and two maps $f : X \to \{p\}$ and $g : \{p\} \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps on $\{p\}$ and X respectively. The map $f \circ g : \{p\} \to \{p\}$ is the identity map and the homotopy condition is automatic. The map $g \circ f : X \to X$ sends every point in X to $g(p)$ and so is a constant, so the condition that X is contractible boils down to the assertion that this constant map is homotopic to the identity.

13.10l I inadvertently gave this problem away. Think of the torus as a square with the standard identification of boundaries, and think of the omitted point as the center of the square. Given a point p not at the center, draw a line from the center through p and on to the boundary, and map p to this boundary point. This map is homotopy to the identity map, since we can push p out gradually to the boundary. But the boundary of the square is a figure eight, since the two vertical sides map to one circle in the figure eight and the two horizontal sides map to the other circle in the figure eight.

13.10e Let D be the closed unit disk in the plane, and let $\mathcal{U} \subset D$ be the open unit disk. Then $\mathcal{U}$ is not a retract of D because a continuous map $D \to D$ which is the identity on $\mathcal{U}$ must be the identity on all of D and thus cannot map D to $\mathcal{U}$.

But $\mathcal{U}$ is a weak retract; we can map D to the closed disk of radius $\frac{1}{2}$ inside $\mathcal{U}$ by shrinking by a factor of $\frac{1}{2}$. When we restrict this map to $\mathcal{U}$ we obtain the map which shrinks the open disk by $\frac{1}{2}$ and this map is clearly homotopy to the identity map $\mathcal{U} \to \mathcal{U}$.

13.10f We'll give a tricky example. Let X be the comb space illustrated below, and let $A \subset X$ be the vertical line at the left side. Map X to A by mapping (x, y) to $(0, y)$, squashing the space X horizonally to the line segment on the left. This is a retract because it leaves points in A fixed.



I claim this map is a deformation retract, that is, the map $X \to A \to X$ is homotopic to the identity map. Indeed this map pushes X to the vertical line A, and it is clearly homotopic to the constant map which sends all of X to $(0, 0)$ because we can just push the vertical line on the left gradually down to $(0, 0)$. But the identity map from $X \to X$ is also homotopic to this constant map, because we can push the comb gradually down until all of its points are along the x-axis, and then we can gradually push this x-axis over to $(0, 0)$. Since homotopy is an equivalent relation on maps from X to X, and since the identity map $X \to X$ and the retract map $X \to A \to X$ are both homotopic to the constant map sending everything to $(0, 0)$, they are homotopic to each other.

But the retraction is *not* a strong deformation retract, because we cannot keep A fixed as we deform the identity map to the map $X \to A$. Indeed, a homotopy must gradually deform the point (x, y) to the final point $(0, y)$, and the only way to get there is to push the point down to the x-axis, and then over to the origin, and then up to $(0, y)$. But if A is fixed, lines very close to A must be almost fixed, and certainly cannot make this long journey down and around.

We can make this final argument rigorous as follows. Suppose $h : X \times I \to X$ is a homotopy which fixes points in A. Let $\mathcal{U}$ be the open subset of X consisting of points (x, y) with $\frac{1}{2} < y$. Let p be the point $(0, 1) \in X$. Since $h(p, t) = p$, there is an open neighborhood $\mathcal{V}_t$ of p and an open interval I_t about $t \in I$ such that h maps $\mathcal{V}_t \times I_t$ into $\mathcal{U}$.

The intervals I_t form an open cover of I, which is compact. Form a finite subcover, and let $\mathcal{V}$ be the intersection of the corresponding I_t. Then h maps $\mathcal{V} \times I$ into $\mathcal{U}$. But $\mathcal{V}$ contains points in adjacent spikes of the comb, and so these points are not mapped down to the x-axis by h and so cannot be sent to the vertical interval A on the left.