Review 1: Mathematics 432/532

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1 Homotopy

Definition 1 *Let $f, g : X \to Y$ be continuous maps. We call these maps* homotopic *if f can be gradually deformed to g through intermediate maps. More precisely, where should be a continuous map $h(x,t) : X \times I \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$.*

Remark: In practice, we may put additional restrictions on the homotopy:

- If $x_0 \in X$ and $y_0 \in Y$ are base points and f and g map x_0 to y_0, we may require that all intermediate maps preserve the base point, so $h(x_0, t) = y_0$ for all t.
- If $\gamma, \tau : I \to X$ are closed paths which start and end at a base point x_0, so $\gamma(0) = \gamma(1)$ and $\tau(0) = \tau(1)$, we may require that intermediate paths also start and end at x_0, so $h(0,t) = x_0$ and $h(1,t) = x_0$ for all t.
- If $\gamma, \tau : I \to X$ are closed paths, so $\gamma(0) = \gamma(1)$ and $\tau(0) = \tau(1)$, we may require that intermediate paths are also closed, so $h(u,0) = h(u,1)$ for all t. In this case there is no base point, and the entire curve can move provided it remains closed.
- If $\gamma, \tau : I \to X$ are paths which begin at the same point x_0 and end at the same point x_1, we may require that all intermediate paths begin at x_0 and end at x_1, so $h(0,t) = x_0$ and $h(1,t) = x_1$ for all t.

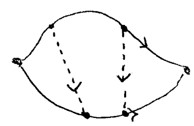
Usually the context will make any additional restriction clear.

Remark: Sometimes I will ask you to show that two maps are homotopic. In these cases, I'd appreciate a picture or a written explanation of the basic idea of the homotopy, followed with an explicit h. Sometimes it is messy to write h down. In the examination, I'll make it clear whether I want a formula or whether an explanation will do. It is really hard to decipher just a formula for h, so please let me know the idea first.

Example: If Y is a convex subset of R^n, for example R^n or D^n, then any two maps $f, g : X \to Y$ are homotopic. Indeed, we can push $f(x)$ to $g(x)$ along the straight line joining these points. Symbolically

$$h(x,t) = (1-t)f(x) + tg(x)$$

Notice that if f and g are curves which begin at y_0 and end at y_1, this homotopy will be through curves which also begin at y_0 and end at y_1.



Example: If $f, g : X \to S^n$ are maps and for each fixed x, $f(x) \neq -g(x)$, then f and g are homotopic. Indeed we can again push along a straight line, but we must project this line outward to the sphere:

$$h(x,t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

Exercise 1 *Prove that homotopy is an equivalence relation on maps $f : X \to Y$. Do this first in words, and then symbolically by writing down formulas for h. Concentrate on the proof that $f_1 \sim f_2$ and $f_2 \sim f_3$ implies $f_1 \sim f_3$.*

2 The Fundamental Group

Definition 2 *Suppose X is a space with base point $x_0 \in X$. The set $\pi_1(X, x_0)$ consists of equivalence classes of closed curves $\gamma : I \to X$ which begin and end at x_0, that is, $\gamma(0) = \gamma(1) = x_0$. Two such curves are equivalent if they are homotopic through intermediate curves which also begin and end at x_0.*

Remark: We always parameterize these curves with u, reserving t for later homotopies.

Definition 3 *Suppose $\gamma, \tau : I \to X$ are curves and the end of γ is the beginning of τ. Then $\gamma \star \tau : I \to X$ is a new curve formed by tracing γ and then tracing τ. Symbolically*

$$\gamma \star \tau(u) = \begin{cases} \gamma(2u) & 0 \le u \le \frac{1}{2} \\ \tau(2u-1) & \frac{1}{2} \le u \le 1 \end{cases}$$

Theorem 1 *This operation induces an operation on $\pi_1(X, x_0)$. Under this operation, $\pi_1(X, x_0)$ becomes a group.*

Remark: The group $\pi_1(X, x_0)$ is called the *fundamental group* of X. As we discovered in exercises, there are also higher homotopy groups $\pi_n(X, x_0)$. Since these groups are not discussed in this course, I will omit the one and just write $\pi(X, x_0)$. For convenience I'll sometimes write $\pi(X)$, but the base point is important; otherwise we cannot define a group operation.

You should be able to prove this theorem, first intuitively by just saying a few sentences, and then precisely by writing down formulas for homotopies. Here are two examples:

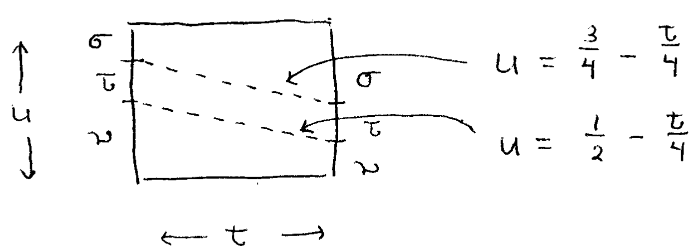
We must check that the operation is well-defined. Suppose γ_1 and γ_2 are homotopic, and τ_1 and τ_2 are homotopic. We must prove that $\gamma_1 \star \tau_1$ is homotopic to $\gamma_2 \star \tau_2$. The intuitive idea is that for the first half of the path we will push γ_1 to γ_2 using the homotopy h for the γ's, and the second half of the path we push τ_1 to τ_2 using the homotopy k for the τ's. In formulas define a homotopy $H : I \times I \to X$ by

$$H(u,t) = \begin{cases} h(2u,t) & 0 \le u \le \frac{1}{2} \\ k(2u-1,t) & \frac{1}{2} \le u \le 1 \end{cases}$$

We must check that the operation is associative. If $\gamma, \tau, \sigma : I \to X$, then $\gamma \star (\tau \star \sigma)$ spends half the time on γ and a quarter of the time on τ and σ, while $(\gamma \star \tau) \star \sigma$ spends a quarter of the time on γ and τ and half the time on σ. These are homotopic by slowing down γ and speeding up τ. Symbolically

$$h(u,t) = \begin{cases} \gamma\left(\frac{u}{\frac{1}{2}-\frac{t}{4}}\right) & 0 \le u \le \frac{1}{2} - \frac{t}{4} \\ \gamma\left(\frac{u-\left(\frac{1}{2}-\frac{t}{4}\right)}{\frac{1}{4}}\right) & \frac{1}{2} - \frac{t}{4} \le u \le \frac{3}{4} - \frac{t}{4} \\ \gamma\left(\frac{u-\left(\frac{3}{4}-\frac{t}{4}\right)}{\frac{1}{4}+\frac{t}{4}}\right) & \frac{3}{4} - \frac{t}{4} \le u \le 1 \end{cases}$$

In this last case, you should be able to write down the formulas for h by thinking through what it is trying to do.



Calculations: Our course is about calculating $\pi(X)$ in difficult cases. At the moment we are mainly able to show that it is the trivial group in some cases. For example $\pi(R^n) = \{1\}$ and $\pi(D^n) = \{1\}$ (why?). You should be able to prove that $\pi(S^n) = \{1\}$ for $n \geq 2$ using the idea of exercise 15.16c.

3 Functorial Properties

Suppose $f : X \to Y$ is a continuous map and f preserves base points, that is, $f(x_0) = y_0$. To indicate this, we often write $f : (X, x_0) \to (Y, y_0)$. Then f induces a map

$$f_\star : \pi(X, x_0) \to \pi(Y, y_0)$$

induced by sending a path $\gamma(u) : I \to X$ to the path $f(\gamma(u)) : I \to Y$. The reader can easily prove that this map is well-defined and is a group homomorphism. Moreover, if $id : X \to X$ is the map $id(x) = x$, then $id_\star$ is the identity map $id \star (g) = g$. If $f : X \to Y$ and $g : Y \to Z$, then $(g \circ f)_\star = g_\star \circ f_\star$.

Exercise 2 *Let $f = z^k : S^1 \to S^1$ be the map which wraps the circle around itself k times. If you prefer, write this map using angles as $\theta \to k\theta$. Jumping ahead, $\pi(S^1) = Z$. Show that $f_\star : Z \to Z$ is the map $f_\star(a) = ka$.*

Remark: The following theorem is crucial; you should know the easy proof:

Theorem 2 *Let $f, g : (X, x_0) \to (Y, y_0)$ be homotopic by a homotopy always mapping x_0 to y_0. Then*

$$f_\star = g_\star : \pi(X, x_0) \to \pi(Y, y_0)$$

Remark: Some of the first definitions in the course look toward the introduction of this induced map:

Definition 4 *Let $A \subseteq X$. We say A is a retract of X if there is a map $r : X \to A$ such that $A \xrightarrow{i} X \xrightarrow{r} A$ is the identity map.*

Definition 5 *We say A is a strong deformation retract if the map $X \xrightarrow{r} A \xrightarrow{i} X$ is homotopic to the identity map by a homotopy which fixes each point of A.*

Theorem 3 *The following results follow immediately from results above:*

- *If X and Y are homeomorphic via a homeomorphism mapping x_0 to y_0, then $\pi(X, x_0)$ and $\pi(Y, y_0)$ are isomorphic*
- *If $r : X \to A$ is a retract, then $i_\star : \pi(A) \to \pi(X)$ is one-to-one, so $\pi(A)$ sits inside $\pi(X)$*
- *If $r : X \to A$ is a deformation retract, then $i_\star : \pi(A) \to \pi(X)$ is an isomorphism*

Examples: The map $r : R^n \to \{0\}$ is a deformation retract, so we obtain another proof that $\pi(R^n) = \pi(\{0\}) = \{1\}$.

The circle is a deformation retract of $R^2 - \{0\}$, so $\pi(R^2 - \{0\}) = \pi(S^1) = Z$.

The set A below is a retract of the Klein bottle, so $Z = \pi(A)$ is a subgroup of $\pi(\mathcal{K})$ and in particular the fundamental group of the Klein bottle is not trivial.

There cannot be a retraction $r : D^2 \to S^1$ because $Z = \pi(S^1)$ is not a subgroup of $\{1\} = \pi(D^2)$.

Every continuous map $f : D^2 \to D^2$ has a fixed point, because otherwise we could define a retraction from D^2 to S^1 as follows: given $x \in D^2$, draw a line segment from $f(x)$ to x and continue this segment until it hits the boundary at $f(x)$. (This is the Brouwer fixed point theorem for $n = 2$.)

Theorem 4 $\pi(X \times Y, x_0 \times y_0) = \pi(X, x_0) \times \pi(Y, y_0)$

Remark: Please consult the handout sheet for an easy proof. Don't be misled by the rather complicated way we proved this result in exercises.

4 Intermission: Dependence of $\pi(X, x_0)$ on the Base Point

In the calculation of $\pi(X, x_0)$, the only points of X which matter are points which can be connected to x_0 by a path. So we often assume that X is pathwise connected, since otherwise we can replace X by the path component of x_0.

Suppose x_0 and x_1 are base points. Choose a path τ from x_0 to x_1. Define a map

$$\pi(X, x_0) \to \pi(X, x_1$$

as follows: if γ represents an element of $\pi(X, x_0$, then find a corresponding loop at x_1 by following τ backward from x_1 to x_0 and then following γ around the loop, and then following τ from x_0 to x_1. Symbolically:

$$\gamma \to \tau^{-1} \star \gamma \star \tau$$

It is easy to prove that this map is well-defined and that it is a group isomorphism. So in a sense the fundamental group is independent of base point.

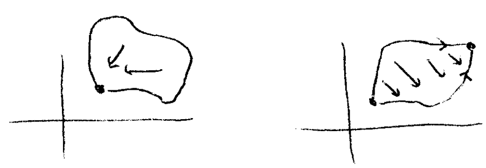
This can be somewhat tricky because the isomorphism depends on the choice of τ. Exercises show that it does not depend on τ when $\pi(X)$ is abelian, but does otherwise.

5 Simply Connected Spaces; Locally Pathwise Connected Spaces

Definition 6 *A topological space is simply connected if it is pathwise connected and if* $\pi(X, x_0) = \{1\}$ *for some (and hence all)* x_0*.*

Remark: Hence D^n and R^n are simply connected. Also S^n is simply connected if $n \geq 2$. Poincare conjectured that S^3 is the only simply connected 3-manifold. A proof is worth \$1,000,000. The proof may have already have been given by Grigori Perelman; the proof is being checked right now.

Remark: It is easy to prove that a pathwise connected space is simply connected if and only if whenever two paths start at a point p and end at a point q, they are homotopic via a homotopy through paths starting at p and ending at q.



Definition 7 *A space X is locally pathwise connected if whenever $p \in \mathcal{U} \subseteq X$, there is a pathwise connected open $\mathcal{W}$ such that $p \in \mathcal{W} \subseteq \mathcal{U}$.*

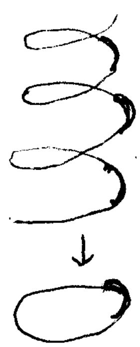
Remark: A pathwise connected space need not be locally pathwise connected; you should know a counterexample.

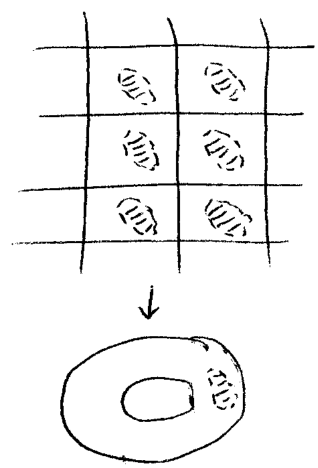
6 Covering Spaces

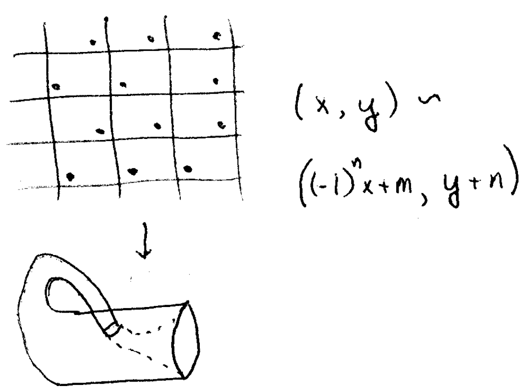
The next section leads to our first serious attempt to calculate fundamental groups. But the section can also be read backwards: a knowledge of the fundamental group of X gives a classification of covering spaces of X. We'll discuss that point of view after the midterm.

Definition 8 *Let X be a connected space. We will always assume X is Hausdorff. A covering space of X is a connected topological space $\tilde{X}$ and continuous onto map $\pi : \tilde{X} \to X$ such that each point $p \in X$ has an open neighborhood which is evenly covered. That is, there is an open $\mathcal{U}$ such that $p \in \mathcal{U} \subseteq X$ and $\pi^{-1}(\mathcal{U})$ can be written as a disjoint union $\cup\, \mathcal{U}_\alpha$ and for each α, $\pi : \mathcal{U}_\alpha \to \mathcal{U}$ is a homeomorphism.*

Remark: Using techniques developed after the midterm, we will prove that each subgroup of $\pi(X, x_0)$ corresponds to a unique covering space of X. Thus a knowledge of $\pi(X, x_0)$ leads to a classification of covering spaces. But at the moment we are interested in the reverse direction: if we can construct a covering space, we can deduce properties of the fundamental group.



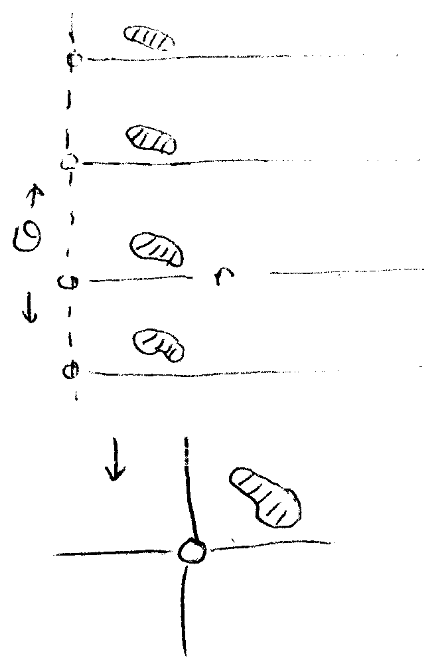




7 Fundamental Lifting Theorems

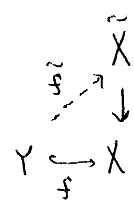
The central idea of the rest of the course is best understood from a concrete example. I recommend that you study this example carefully and refer back to it whenever our remaining theorems seem unnecessarily abstract.

Let X be the space $R^2 - \{0\}$. The points in X are conveniently described in polar coordinates. Let $\tilde{X}$ be the space of possible coordinates: $\tilde{X} = \{ \ (r, \theta) \mid 0 < r < \infty \ \}$. The map $\pi : \tilde{X} \to X$ is the standard polar coordinate map $\pi(r, \theta) = (r \cos\theta, r \sin\theta)$. Notice that the angle variable θ has infinitely many values separated by multiples of 2π; the covering space contains all of these possible choices.

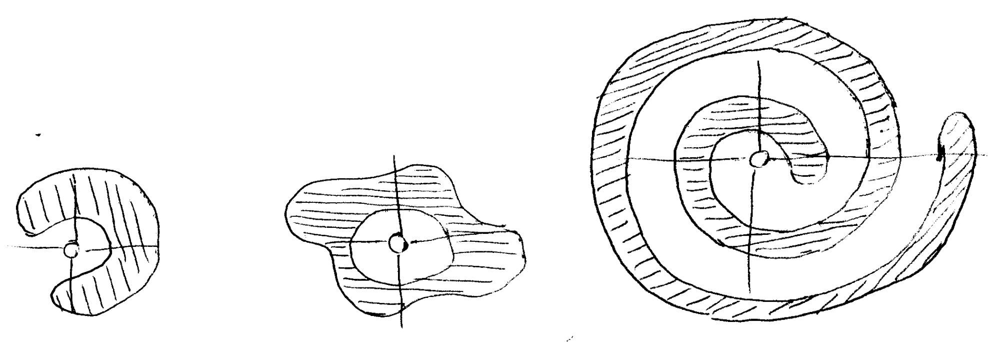


Now suppose Y is a subset of $R^2 - \{0\}$. Then we have an inclusion map $f : Y \to R^2 - \{0\}$. The general theory works for an arbitrary f and not just an inclusion, but it is convenient to consider the case of a subset.

Suppose we want to assign polar coordinates to each point y of Y so these coordinates vary continuous as y varies. In particular we want to assign a unique angle θ to each such point. This assignment will be given by a map $\tilde{f} \to \tilde{X}$ because $\tilde{X}$ is the space of all polar coordinates. We want $\tilde{f}(y)$ be polar coordinates for $f(y) = y$, so the diagram below should commute.

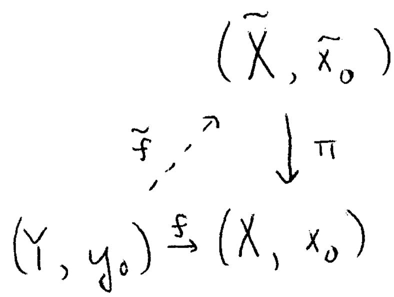


Sometimes it is possible to make this assignment continuously, and sometimes it isn't. In the picture on the left, we get a continuous assignment by choosing $-\pi < \theta < \pi$. In the example on the center, no assignment is possible because as we circle the center, the angle increases by 2π. In the example on the right, a continuous assignment is possible. We can choose θ to be approximately 0 at points inside the spiral on the right side. As the spiral circles once, points on the right further out are assigned $\theta \sim 2\pi$ and as the spiral circles again, outside points on the right are assigned $\theta \sim 4\pi$.



We are going to prove that an assignment is possible whenever Y is simply connected. Here is the general theorem:

Theorem 5 *Let Y be an arbitrary connected topological space, $\tilde{X} \to X$ an arbitrary covering space, and $f : Y \to X$ an arbitrary continuous map. If the diagram below can be completed, the lifting map $\tilde{f}$ is unique.*



Remark: You should know the proof.

Theorem 6 *A lifting exists if Y is simply connected and locally pathwise connected.*

Remark: This theorem is proved by first proving two special cases of great importance:

Theorem 7 *A lift exists if $Y = I$ or $Y = I \times I$*

Remark: It is very important to know the proof of the last theorem. If you have time, study the proof of the more general theorem.

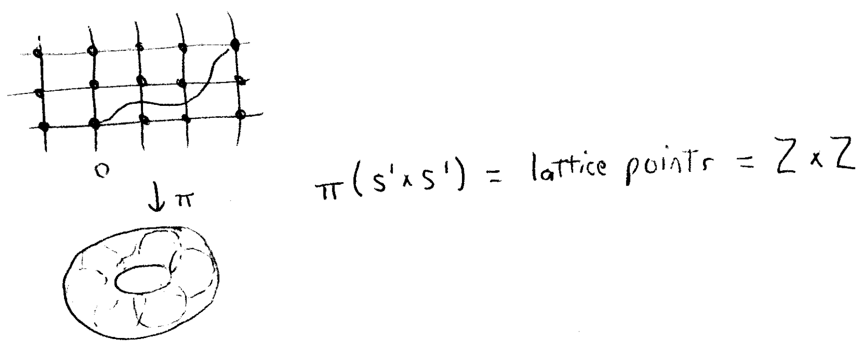
8 Applications of Lifting Theorems

Theorem 8 *Suppose $(\tilde{X}, \tilde{x}_0)$ and $(\tilde{\tilde{X}}, \tilde{\tilde{x}}_0)$ are simply connected covering spaces of a connected and locally pathwise connected space X. Then $\tilde{X}$ and $\tilde{\tilde{X}}$ are homeomorphic by a homeomorphism commuting with π. So there is essentially one simply connected covering space of X. This space is called the* universal covering space *of X.*

Remark: We will prove that this space exists after the midterm. None of the remaining results in the review sheet depend on this existence theorem because in our applications we explicitly construct the universal cover.

Important remark: Here is the central construction of the course. Let $\gamma : (I, 0) \to (X, x_0)$ represent an element of the fundamental group. Suppose $\tilde{X} \to X$ is any covering space. Lift the path γ to a path $\tilde{\gamma} : (I, 0) \to (\tilde{X}, \tilde{x}_0)$. Since γ is a closed path, $\tilde{\gamma}(1)$ lies over x_0, so $\tilde{\gamma}(1) \in \pi^{-1}(x_0)$. In this way we obtain a map

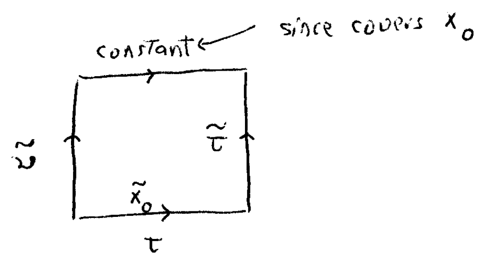
$$\pi(X, x_0) \to \pi^{-1}(x_0) \subseteq \tilde{X}$$



Theorem 9 *The map $\pi(X, x_0) \to \pi^{-1}(x_0)$ is*

- *well-defined;*
- *onto;*
- *one-to-one in case $\tilde{X}$ is simply connected*

Proof: The proof is very important. To get you started, I'll prove that the map is well-defined. Suppose γ and τ are homotopic and thus represent the same element of $\pi(X)$. Find such a homotopy $h(u, t) : I \times I \to X$ from γ to τ. Lift the homotopy to a map $\tilde{h}(u, t) : I \times I \to \tilde{X}$. The lift is a homotopy from $\tilde{\gamma}$ to $\tilde{\tau}$. Since $h(1, t)$ is identically x_0, $\tilde{h}(1, t)$ is constant. In particular, $\tilde{\gamma}(1) = \tilde{\tau}(1)$.

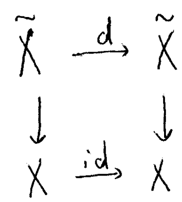


Important Example In the special case $R \to S^1$ given by $\pi(x) = e^{2\pi i x}$, the inverse image of $1 \in S^1$ is $Z \subseteq R$. So each element $\gamma \in \pi(S^1)$ is mapped to an integer called *the degree* *of* γ. Two closed paths are homotopic if and only if they have the same degree. A brief calculation shows that the degree of the path z^k which maps the circle around itself k times is k.

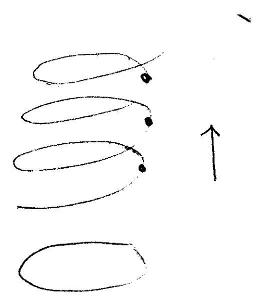
You may wish to show that the group law on $\pi(S^1)$ is the standard addition of integers in Z. Later we prove this using deck transformations.

9 Deck Transformations and All That

Definition 9 *Let* $\tilde{X} \to X$ *be a covering space. A deck transformation is a homeomorphism* $d : \tilde{X} \to X$ *such that the diagram below commutes. The set of all deck transformations is denoted by* Γ. *This set is obviously a group under composition.*



Remark: This diagram just states that $\tilde{x}$ and $d(\tilde{x})$ lie over the same point of X. So a deck transformation just "shuffles the decks" as illustrated by the example below.



Theorem 10 *Let* $\tilde{x} \in \tilde{X}$. *A deck transformation* d *is completely determined by* $d(\tilde{x})$. *That is, if* d_1 *and* d_2 *are deck transformations and* $d_1(\tilde{x}) = d_2(\tilde{x})$, *then* $d_1 = d2$.

Remark: This theorem is a special case of theorem 5.

This theorem allows us to easily find all deck transformations. Suppose $x_0 \in X$ and suppose $\tilde{x}_0 \in \pi^{-1}(X)$. We know that deck transformations are determined by the values of $d(\tilde{x}_0)$ and we know that these values also lie in $\pi^{-1}(x_0)$. If we can directly construct deck transformations which take $\tilde{x}_0$ to each of these possible images, then we must know all possible deck transformations.

Example: Consider $\pi : R \to S^1$. It is easy to see that $x \to x + n$ is a deck transformation of R whenever $n \in Z$. This transformation takes $x = 0$ to $x = n$. Since the integers are exactly $\pi^{-1}(1)$ for $1 \in S^1$, these are all possible deck transformations.

Theorem 11 *Suppose X is connected and locally pathwise connected, and let $\tilde{X} \to X$ be a universal covering space. Then whenever $x_0 \in X$, the deck transformation group Γ acts transitively on $\pi^{-1}(x_0)$. That is, if we fix $\tilde{x}_0$ in this set, then for each $\tilde{x}_1 \in \pi^{-1}(x_0)$ there is a unique deck transformation mapping $\tilde{x}_0$ to $\tilde{x}_1$.*

Remark: This is an immediate consequence of theorem 6.

10 $\pi(X)$ and Γ Are Isomorphic

We now come to our climactic theorem. Let X be connected and locally pathwise connected. Let $\tilde{X}$ be a universal cover of X — that is, a covering space which is simply connected.

Fix $x_0 \in X$ and fix $\tilde{x}_0 \in \pi^{-1}(x_0)$. According to theorem 9, there is a canonical one-to-one onto map $\pi(X, x_0) \to \pi^{-1}(x_0)$. According to theorem 11, there is a canonical one-to-one onto map $\Gamma \to \pi^{-1}(x_0)$. Combining these maps, we discover a canonical one-to-one onto map $\pi(X, x_0) \to \Gamma$.

Theorem 12 *This map is a group isomorphism.*

Proof: Let γ and τ represent elements of $\pi(X, x_0)$. Suppose γ maps to the deck transformation d_γ and τ maps to the deck transformation d_τ. Working our way through the maps, we conclude that if $\tilde{\gamma}$ is a lift of γ to a path in $\tilde{X}$ starting at $\tilde{x}_0$, then $d_\gamma(\tilde{x}_0) = \tilde{\gamma}(1)$. Similarly $d_\tau(\tilde{x}_0) = \tilde{\tau}(1)$

We need to lift $\gamma \star \tau$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$. The lift of γ should be $\tilde{\gamma}$. This path will end at $d_\gamma(\tilde{x}_0)$. We want the lift of τ to start at this point so we can glue the maps together, To achieve this, we should lift via $d_\gamma(\tilde{\tau})$. Hence the proper lift is

$$\begin{cases} \tilde{\gamma}(2u) & 0 \le u \le \frac{1}{2} \\ d_\gamma(\tilde{\tau}(2u-1)) & \frac{1}{2} \le u \le 1 \end{cases}$$

This lifts ends at

$$d_\gamma(\tilde{\tau}(1)) = d\gamma(d\tau(\tilde{x}_0))$$

Therefore $\gamma \star \tau$ maps to the deck transformation $d_\gamma \circ d_\tau$.

Example 1: The group of deck transformations of the universal cover R of S^1 is $x \to x+n$. Clearly translations compose by adding the transformation amount. So $\pi(S^1) \cong \Gamma \cong Z$.

Example 2: The group of deck transformations of the universal cover $R \times R$ of $S^1 \times S^1$ is $(x, y) \to (x + m, y + n)$ for integers m and n. Since translations compose by adding the translation amount, $\pi(S^1 \times S^1) \cong \Gamma \cong Z \times Z$.

Example 3: Recall that RP^n is constructed from S^n by identifying antipodal points. Each point on RP^n is covered by exactly two points, so the deck transformation group has two maps, the identity, and the map J which sends each point to its antipodal point. Thus $\pi(RP^n) \cong \Gamma \cong Z_2$.

Example 4: Consider the Klein bottle $\mathcal{K}$. The universal cover is R^2 with equivalence relation $(x, y) \sim (-1)^n x + m, y + n)$. It immediately follows that the following maps are deck transformations:

$$(x, y) \to ((-1)^n x + m, y + n)$$

Notice that this map sends $(0, 0)$ to (m, n). Since the set of all (m, n) is the full set $\pi^{-1}(x_0)$ for the point $x_0 \in \mathcal{K}$ represented by $(0, 0)$, it follows that the above transformations form the complete deck transformation group Γ.

Notice that

$$(x, y) \overset{(m_2,n_2)}{\to} ((-1)^{n_2} x + m_2, y + n_2) \overset{(m_1,n_1)}{\to} ((-1)^{n_1}((-1)^{n_2} x + m_2) + m_1, y + n_2 + n_1)$$

and this last element is

$$((-1)^{n_1+n_2} x + m_1 + (-1)^{n_1} m_2, n_1 + n_2)$$

we conclude that $\Gamma = Z \rtimes Z$, the semidirect product with group law

$$(m_1, n_1) \circ (m_2, n_2) = (m_1 + (-1)^{n_1} m_2, n_1 + n_2).$$

11 Sample Exercises

The following exercises are interesting and could be profitably studied for the midterm.

- First assignment: 13.4c (pay attention to the idea before worrying about the formula), 13.10a, 13.10l
- Second assignment: 13.10o, 13.10p, 14.6c (be sure to apply the homotopy to the parameter rather than to the maps)

- Third assignment: 15.18b (I'm interested in the fact that (f, g) is homotopic to first doing f while fixing the second component, and then doing g while fixing the first component), 15.18d, and all four extra problems
- Fourth assignment: 15.16c, 16.11g parts ii and iii
- Fifth assignment: 17.9b, 17.9j, 10.7a