Classifying Covering Spaces

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1 Previous Work

We have been using covering spaces to compute fundamental groups. To find $\pi(X)$, we must *guess* a universal covering space $\pi : \tilde{X} \to X$. The theory does not help us guess. But if we succeed

1. $\pi(X, x_0)$ is in one-to-one correspondence with the set $\pi^{-1}(x_0)$ of points in $\tilde{X}$ over x_0
2. $\pi(X, x_0)$ is isomorphic as a group to the group Γ of deck transformations of $\tilde{X}$
3. if $\tilde{x}_0$ if a point over x_0, then for each $\tilde{x}_1$ over x_0 there is a unique deck transformation mapping $\tilde{x}_0$ to $\tilde{x}_1$; hence if we are able to guess these particular deck transformations, we know the complete group Γ and so the group $\pi(X, x_0)$

Example 1: Let $\pi(x) = e^{2\pi i x} : R \to S^1$. Under this map R is the universal cover of S^1. So $\pi(S^1, 1)$ is set theoretically the collection of all points over 1, which is Z.

The map $x \to x + n$ is clearly a deck transformation. This transformation takes 0 to n. Since these maps carry 0 to every possible image, they form the complete set of *all* deck transformations. Notice that $R \stackrel{x+m}{\to} R \stackrel{x+n}{\to} R$ sends x to $(x + m) + n = x + (m + n)$, so composition in Γ corresponds to addition in Z. Thus $\pi(S^1, 1) = Z$ as a group.

Example 2: Write down the corresponding calculations for the Klein bottle $\mathcal{K}$.

2 Going Backward

We are going to turn this theory on its head. Suppose that we have managed to compute $\pi(X)$ some other way. We will then use this group to classify all possible covering spaces of X.

From now on we always assume that X is connected and locally pathwise connected. It follows easily that X is pathwise connected.

In the lectures, covering spaces are assumed connected. It follows from our assumption on X that any covering space is locally pathwise connected and so pathwise connected.

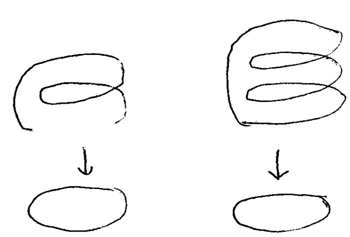
Definition 1 *Suppose $\tilde{X}$ and $\tilde{\tilde{X}}$ are covering spaces of X. We say these covering spaces are* isomorphic *if there is a homeomorphism $\psi : \tilde{X} \to \tilde{\tilde{X}}$ making the following diagram commute:*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\psi"] \arrow[d] & \tilde{\tilde{X}} \arrow[d] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd}$$

This is a fancy way of saying that if $x \in \tilde{X}$ is a point over $p \in X$, then $\psi(x) \in \tilde{\tilde{X}}$ must also be a point over x.

Example: Consider the spaces below. Both covering spaces are homeomorphic to S^1, but the covering spaces are not isomorphic. Indeed the covering space on the left covers each point in X twice, while the space on the right covers each point three times. A necessary condition that two covering spaces be isomorphic is that they be homeomorphic and cover X the same number of times. These conditions are not sufficient; In the exercises you will discover two covering spaces of the torus $S^1 \times S^1$ which both cover twice and yet are not isomorphic.

$$\begin{tikzcd} S^1 \arrow[d, "z^2"'] \\ S^1 \end{tikzcd} \qquad\qquad \begin{tikzcd} S^1 \arrow[d, "z^3"'] \\ S^1 \end{tikzcd}$$



3 The Lifting Theorem To End All Lifting Theorems

Our classification theorem requires just one technical theorem, which generalizes all of our previous lifting theorems:

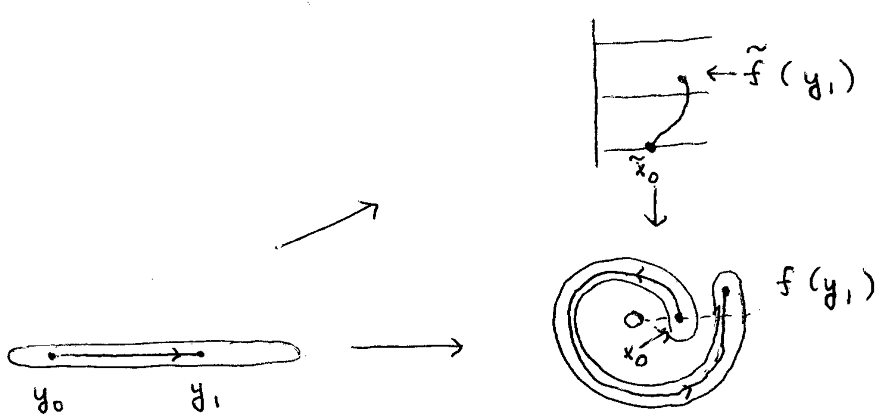
Theorem 1 *Let Y be a connected, locally pathwise connected space. Suppose $\tilde{X}$ is a covering space of X and $f : Y \to X$ is a continuous map. This map lifts to $\tilde{f}$ making the following diagram commute, if and only if* $f_\star\Big(\pi(Y, y_0)\Big) \subseteq \pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big)$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "\pi"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Remark: If Y is simply connected, then $\pi(Y)$ is trivial and the last condition is automatically true. Thus our previous theorem follows from this one.

I'll let you provide the (easy) proof that this condition is necessary.

Proof: We'll follow the pattern of the previous proof. Suppose $y_1 \in Y$. We want to define $\tilde{f}(y_1)$. Find a path $\gamma : I \to Y$ in Y from y_0 to y_1; this is possible because Y is path connected. Then $f \circ \gamma$ is a path in X from x_0 to $f(y_1)$. Lift this path to a path $\widetilde{f \circ \gamma}$ in $\tilde{X}$ starting at $\tilde{x}_0$ and define $\tilde{f}(y_1)$ to be the end of this lifted path.

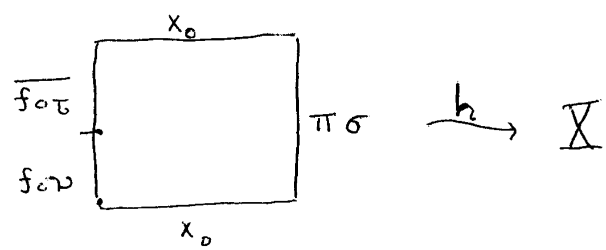


When Y is simply connected, our previous proof now separated into two pieces. First we proved that $\tilde{f}$ is well-defined; that step used the assumption that Y is simply connected.

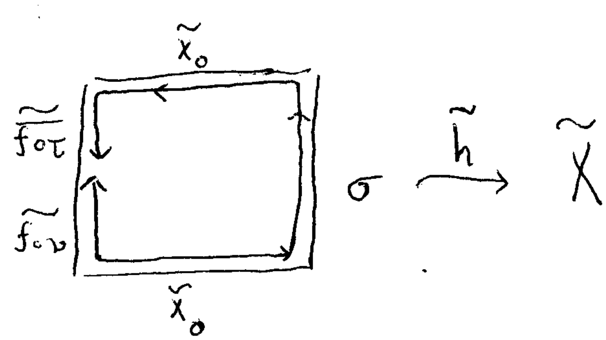
Then we proved that $\tilde{f}$ is continuous; that step used the assumption that Y is locally path-connected.

The second argument applies unchanged to the current situation. So we need only prove that $\tilde{f}$ is well-defined. Suppose, then, that γ and τ are two paths in Y starting at y_0 and ending at y_1. Then $f \circ \gamma$ and $g \circ \tau$ are two paths in X starting at x_0 and ending at $f(y_1)$. We are going to use the book's notation that $\overline{\tau}$ represents τ traced backward. Notice that $\gamma \star \overline{\tau}$ represents an element of $\pi(Y)$ and so $(f \circ \gamma) \star \overline{(f \circ \tau)}$ represents an element of $f_\star\Big(\pi(Y, y_0)\Big)$; by assumption this element is in $\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big)$.

Consequently, this path is homotopic to the image of a closed path σ in $\tilde{X}$ under π. Let $h : I \times I \to X$ be such a homotopy. The picture below shows where various portions of the square go under this homotopy.



By our standard result, this homotopy can be lifted to a map $\tilde{h} : I \times I \to \tilde{X}$ sending 0×0 to $\tilde{x}_0$. By uniqueness of lifts, the images of various portions of the square's boundary under this map are shown below. We work around this square counterclockwise starting with $\widetilde{f \circ \gamma}$.



Looking at the middle of the left side of the square, we see that $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \tau}(1)$, as desired.

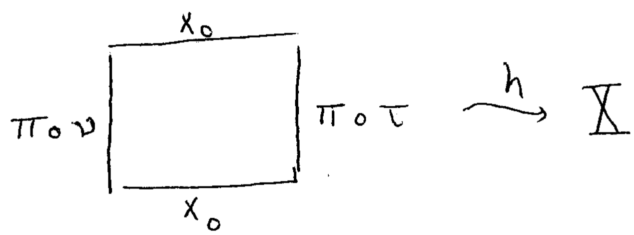
4 Covering Spaces of X and Subgroups of $\pi(X)$

Our next goal is to show that each covering space $\tilde{X}$ of X corresponds to a subgroup of $\pi(X)$ which is unique up to conjugacy.

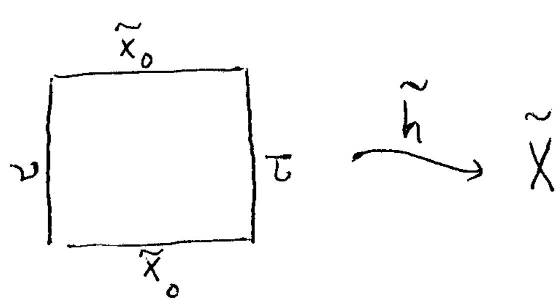
Theorem 2 *If $\tilde{X} \xrightarrow{\pi} X$ is a covering space, then $\pi_\star : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is one-to-one. Hence the fundamental group of $\tilde{X}$ can be considered to be a subgroup of $\pi(X)$.*

Proof: This was an exercise in the midterm. Suppose γ and τ are closed paths in $\tilde{X}$ representing elements of $\pi(\tilde{X})$, and suppose these paths induce the same element of $\pi(X)$. We must prove that γ and τ represent the same element of $\pi(\tilde{X})$.

Since $\pi \circ \gamma$ and $\pi \circ \tau$ represent the same element of $\pi(X)$, they are homotopic in X. Let $h : I \times I \to X$ be such a homotopy.



Lift this homotopy to $\tilde{h} : I \times I \to \tilde{X}$ mapping 0×0 to $\tilde{x}_0$. Then this lifted homotopy is a homotopy from γ to τ through closed paths by uniqueness of lifts, as indicated in the picture below. So γ and τ represent the same element of $\pi(\tilde{X})$.



Theorem 3 *Suppose $\tilde{x}_0$ and $\tilde{x}_1$ are base points in $\tilde{X}$ projecting to the same point $x_0 \in X$. Then $\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big)$ and $\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_1)\Big)$ are conjugate subgroups of $\pi(X)$.*

Proof: Let $\sigma : I \to \tilde{X}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. By previous results, $\pi(\tilde{X}, \tilde{x}_0)$ is isomorphic to $\pi(\tilde{X}, \tilde{x}_1)$ by the map sending γ to $\overline{\sigma} \star \gamma \star \sigma$. Project this result to X. The path σ now becomes a closed curve in X because $\tilde{x}_0$ and $\tilde{x}_1$ project to the same point, so $\pi \circ \sigma$ induces an element $g \in \pi(X)$. Clearly the isomorphism between fundamental groups of $\tilde{X}$ then projects to an isomorphism $\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big) \to \pi_\star\Big(\pi(\tilde{X}, \tilde{x}_1)\Big)$ given by $\gamma \to g^{-1}\gamma g$, which is a conjugation.

Theorem 4 *Conversely suppose $g \in \pi(X)$. If by choosing the base point $\tilde{x}_0 \in \tilde{X}$ we obtain the subgroup $\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big)$ of $\pi(X)$, then we can find a new base point $\tilde{x}_1$ in $\tilde{X}$ inducing the conjugate subgroup $g^{-1}\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big)g$.*

Proof: The element g is represented by a path $\sigma : I \to X$ starting and ending at x_0. Lift this path to a path $\tilde{\sigma}$ starting at $\tilde{x}_0$. This path ends at another point in $\tilde{X}$ over x_0; call this point $\tilde{x}_1$. Now read the proof of theorem 3 to check that this $\tilde{x}_1$ is the desired point.

5 The Big One

Here then is the first of our two big theorems classifying covering spaces:

Theorem 5 *Let $\tilde{X}$ and $\tilde{\tilde{X}}$ be two covering spaces of X. If the subgroups of $\pi(X, x_0)$ corresponding to these spaces are conjugate, then the spaces are isomorphic as covering spaces.*

Proof: By the previous theorem, we can choose base points $\tilde{x}_0 \in \tilde{X}$ and $\tilde{\tilde{x}}_0 \in \tilde{\tilde{X}}$ so the subgroups of $\pi(X)$ corresponding to $\tilde{X}$ and $\tilde{\tilde{X}}$ are not just conjugate, but instead actually equal.

Consider now the following diagram.

$$
\begin{tikzcd}
 & (\tilde{\tilde{X}}, \tilde{\tilde{x}}_0) \arrow[dl, dashed, bend left, "\Phi"] \arrow[d, "\pi"] \\
(\tilde{X}, x_0) \arrow[ur, dashed, bend left, "\Psi"] \arrow[r, "\pi"'] & (X, x_0)
\end{tikzcd}
$$

Since

$$\pi_\star\Big(\pi(\tilde{X}, \tilde{x}_0)\Big) \subseteq \pi_\star\Big(\pi(\tilde{\tilde{X}}, \tilde{\tilde{x}}_0)\Big)$$

(and actually these groups are equal), theorem one gives a map $\psi : \tilde{X} \rightarrow \tilde{\tilde{X}}$ lifting the identity map. Looking at the diagram backward, we similarly discover a map $\phi : \tilde{\tilde{X}} \rightarrow \tilde{X}$ lifting the identity map.

Consequently $\phi \circ \psi : \tilde{X} \rightarrow \tilde{X}$ is a lift of the identity map. But lifts are unique, and the identity is one possible lift. So $\phi \circ \psi$ is the identity from $\tilde{X}$ to $\tilde{X}$. Similarly $\psi \circ \phi$ is the identity from $\tilde{\tilde{X}}$ to $\tilde{\tilde{X}}$. It follows that ψ and ϕ are both homeomorphisms, and each is a covering space isomorphism as required.

6 Constructing Covering Spaces

Our next goal is to show that each subgroup of $\pi(X)$ actually comes from a covering space. This is easy *provided we know that* X *has a universal cover.* Indeed, we'll give a remarkably concrete method of constructing the covering space $\tilde{X}$ corresponding to the subgroup. So we have our second fundamental theorem:

Theorem 6 *Suppose* X *is connected and locally pathwise connected, and suppose* X *has a universal covering space* $\mathcal{C}$*. Then every subgroup* $G \subseteq \pi(X, x_0)$ *comes from a covering space* $\tilde{X}$ *of* X*.*

Corollary 7 *Suppose* X *is connected and locally pathwise connected, and suppose* X *has a universal covering space. Then there is a one-to-one correspondence between conjugacy classes of subgroups of* $\pi(X)$ *and covering spaces of* X*.*

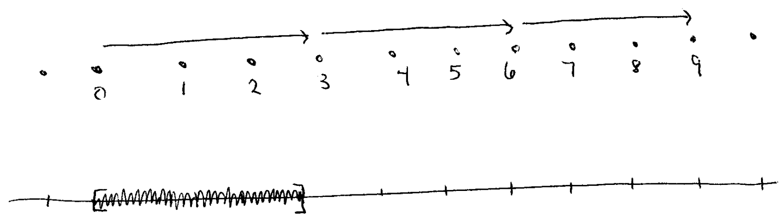
Proof: For convenience, let $\mathcal{C}$ be the universal cover. Let Γ be the group of deck transformations of this universal cover. Then we have an isomorphism from Γ to $\pi(X, x_0)$. Suppose we have a subgroup $G \subseteq \pi(X, x_0)$. Let Γ_G be the corresponding subgroup of the deck transformation group.

Define an equivalence relation on the universal cover by calling two points equivalent if one can be mapped to the other by an element of Γ_G. Let $\tilde{X}$ be the quotient space $\mathcal{C}/\Gamma_G$. The natural projection $\pi : \mathcal{C} \rightarrow X$ maps points equivalent under Γ to the same point, so it certainly maps points equivalent under Γ_G to the same point. Thus π induces a map $\tilde{X} = \mathcal{C}/\Gamma_G \rightarrow X$. We claim this is a covering map, and the fundamental group of this covering space is G. This will complete the argument.

Eventually we will give the general argument, but it is much more illuminating to examine some examples.

Example 1: Let $X = S^1$ so $\pi(S^1) = Z$. Let G be the subgroup $3Z$ of all multiples of 3. The full deck transformation group contains all maps $x \rightarrow x + n$; clearly the subgroup Γ_G contains all deck transformations of the form $x \rightarrow x + 3n$.

It is clear that every point in R is equivalent to an element in $[0, 3]$ under Γ_G. So the set of equivalence classes $\mathcal{C}/\Gamma_G$ is equal to the set $[0, 3] \subseteq R$ except that 0 and 3 are equivalent.

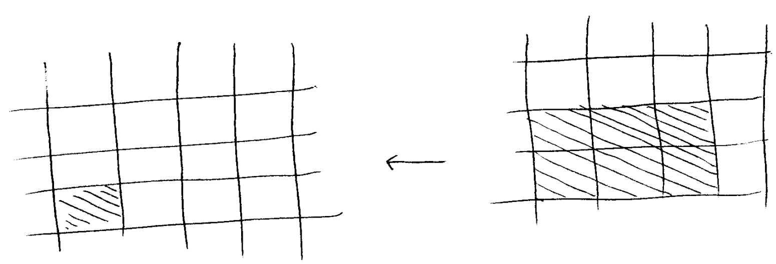


If we make exactly the same construction using the full group Γ, we get the original X. Indeed the set of equivalence classes under all of Γ is $[0, 1]$ except that 0 and 1 are equivalent. Of course this is equivalent to S^1.

Notice that the map $\mathcal{C}/\Gamma_G \to \mathcal{C}/\Gamma$, which is $\tilde{X} \to X$, is in this case $[0, 3] \to [0, 1]$ corresponding to mapping the circle around itself three times. Theorem 6 says that this sort of construction takes place in general.

Example 2: Let the base space $X = S^1 \times S^1$. Its universal covering space is $R \times R$ and its fundamental group is $Z \times Z$. The group of deck transformations is the set of transformations $(x, y) \to (x + m, y + n)$.

Now consider the subgroup $3Z \times 2Z$ of all (m, n) such that m is even and n is a multiple of 3. The corresponding group of deck transformations Γ_G is the set of maps $(x, y) \to (x + 2k, y + 3l)$.



Every point of $R \times R$ is equivalent under the group Γ_G to a point in the rectangle $0 \le x \le 3$, $0 \le y \le 2$. We must glue the left and right edges, and the top and bottom edges, getting a torus. This gives $\tilde{X}$. The covering map π maps each of the six squares inside this rectangle onto the original torus, so it is a six-fold cover.

Example 3: Here is a more complicated example. Let us find one of the covering spaces of the Klein bottle. The universal cover of this space is $R \times R$ and its fundamental group is $Z \rtimes Z$. Recall that $Z \rtimes Z$ is the set of all (m, n) with group law $(m_1, n_1) \circ (m_2, n_2) = (m_1 + (-1)^{n_1}, n_1 + n_2)$. There is an exact sequence

$$0 \to Z \xrightarrow{i} Z \rtimes Z \xrightarrow{r} Z \to 0$$

where the first map sends m to (m, n) and the second map sends (m, n) to n.

Suppose that we have a subgroup $G \subseteq Z \rtimes Z$. The image of this group under r must be a subgroup of the Z on the right; let us suppose that this subgroup is $3Z$. The kernel of r must be a subgroup of the left Z; suppose this subgroup is $2Z$.

The generator of $3Z$ comes from an element of G, but we cannot assume that this element has the form $(0, 3)$. We *can* assume that it is either $(0, 3)$ or $(1, 3)$ since we can modify it by elements coming from $2Z$. For fun, suppose that $(1, 3) \in G$.

Then $(1, 3) \circ (1, 3) = (1 + (-1)^3 1, 3 + 3) = (0, 6) \in G$, etc. In general $(0, 3n) \in G$ for even n and $(1, 3n) \in G$ for n odd. From exactness we conclude that G is exactly

$$\{(2m, 3n) \mid n \text{ is even}\} \ \cup \ \{(2m + 1, 3n) \mid n \text{ is odd}\}$$

Notice that the group is generated by the two elements $(2, 0)$ and $(1, 3)$.

The element (m, n) in the full group corresponds to the transformation

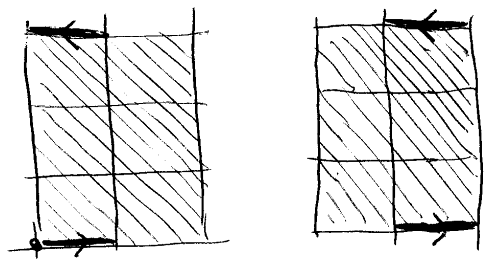
$$(x, y) \to ((-1)^n x + m, y + n).$$

So the generators correspond to the transformations $(x, y) \to (x + 2, y)$ and $(x, y) \to (-x + 1, y + 3)$.

Multiples of the map $(x, y) \to (-x + 1, y + 3)$ raise or lower points by multiples of three. By applying an appropriate such power, we can find an equivalent point (x, y) with $0 \leq y \leq 3$.

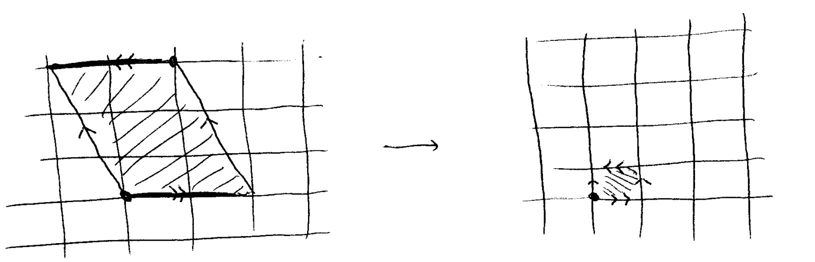
Multiples of the map $(x, y) \to (x + 2, y)$ slide points horizontally by multiples of 2. By applying an appropriate such power, we can find an equivalent point (x, y) with $0 \leq x \leq 2$.

Consequently every point of the plane is equivalent under Γ_G to a point in the shaded rectangle below. However, it will soon turn out that this is not the best choice of representatives for $\tilde{\mathcal{K}}$.



We must glue the left and right sides of this rectangle together, since these points are equivalent under $(x, y) \to (x + 2, y)$. We must also glue the heavily shaded lines on the top and bottom together as indicated since these points are equivalent under $(x, y) \to (-x{+}1, y{+}3)$. Finally, we must glue the remaining points on the top and bottom together as indicated in the second picture because they are equivalent under $(x, y) \to (-x{+}1, y{+}3) \to (-x + 1 + 2, y + 3) = (-x + 3, y + 3)$.

It is a little difficult to understand the resulting space. But notice that we can find another shaded region by sliding horizontal segments toward the left like a series of dinner plates as indicated below. This time, the entire bottom is glued to the top by $(x, y) \to (-x{+}1, y{+}3)$. It is clear that this space is another Klein bottle, covering the original $\mathcal{K}$ six times.



The Klein bottle can also be covered by tori and cylinders. Explain how.

7 The General Theory

We now give the general proof of theorem six. We earlier proved that the full deck transformation group Γ of the universal cover $\mathcal{C} \to X$ is transitive on $\pi^{-1}(x)$ whenever $x \in X$.

So when we glue equivalent points under Γ together, we identify all the points over x to a single point. It immediately follows that $\mathcal{C}/\Gamma = X$.

Define $\tilde{X} = \mathcal{C}/\Gamma_G$ with the quotient topology. If two points are equivalent under Γ_G, they certainly are equivalent under the full Γ, so there is a natural mapping $\mathcal{C}/\Gamma_G \to \mathcal{C}/\Gamma$. Said another way, there is a natural mapping $\tilde{X} \to X$.

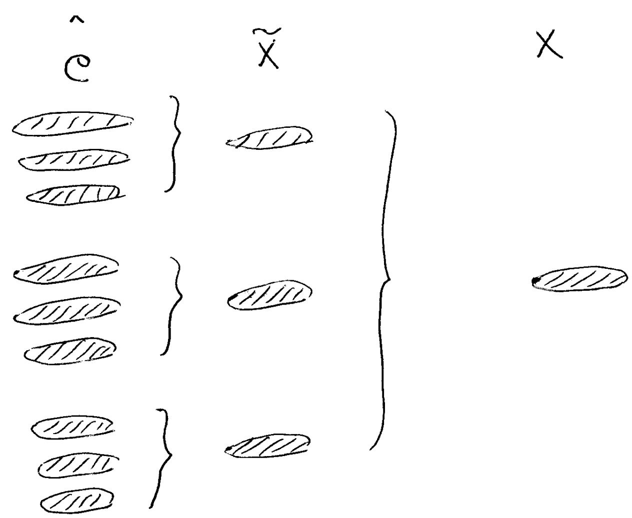
It remains to show every point of X has an evenly covered open neighborhood, and that the fundamental group of $\pi(\tilde{X})$ is G.

Let $x \in X$. Choose an open neighborhood $\mathcal{U}$ of x which is evenly covered in $\mathcal{C}$. Since X is locally pathwise connected, we can suppose that $\mathcal{U}$ is pathwise connected. Write $\pi^{-1}(\mathcal{U}) = \cup\, \mathcal{U}_\alpha$ in $\mathcal{C}$. Pick $\tilde{x}_1 \in \mathcal{C}$ over x; this point is in one of the $\mathcal{U}_\alpha$, say $\mathcal{U}_1$. The deck transformation group Γ acts simply transitively on the $\{\mathcal{U}_\alpha\}$. This is a fancy way of saying that if $\tilde{x}_\alpha$ is the point in $\mathcal{U}_\alpha$ over x, there is a unique deck transformation mapping $\tilde{x}_1$ to $\tilde{x}_\alpha$ and thus mapping $\mathcal{U}_1$ to $\mathcal{U}_\alpha$. Thus we can identify the "plates" over $\mathcal{U}$ in $\mathcal{C}$ with the elements of Γ.

Now write Γ as a union of right Γ_G cosets:

$$\Gamma = \Gamma_G e \cup \Gamma_G g_2 \cup \Gamma_G g_3 \cup \dots$$

This decomposition then induces a similar decomposition of the plates over $\mathcal{U}$ into equivalence classes. Here is the picture:



Plates in a particular coset are equivalent under Γ_G. So when we identify points equivalent under Γ_G, these equivalent plates are glued together. Thus the various cosets correspond to the open sets in $\tilde{X}$ which cover $\mathcal{U}$. This shows that $\mathcal{U}$ is evenly covered in $\tilde{X}$.

Finally, we must find the fundamental group of $\tilde{X}$. By our general theory, this group is isomorphic to the deck transformation group of the covering $\mathcal{C} \to \tilde{X}$. We can see this deck transformation group in the above picture. Indeed, take one of the small stacks of plates which will glue together in $\tilde{X}$ to form a subset $\mathcal{V}$. The deck transforms of $\mathcal{C} \to \tilde{X}$ must permute these stacks. But the small stacks correspond to cosets of Γ_G, and the deck transformations which permute a particular coset are just the deck transformations in Γ_G. So the fundamental group of $\tilde{X}$ is Γ_G as desired.