Assignment 9; Due Friday, March 17

Each of the following exercises asks for a calculation of $\pi(X, x_0)$ for a particular X. I'm giving these exercises so you can try calculations on your own and become confident that you can apply Seifert-Van Kampen. You'll be able to check your answers against mine in the end. If you run out of time but feel confident with the examples you've done, I'll be satisfied.

24.4 b, d, e, k

25.1b, f (part f has a lot of examples; do examples a and h)

Exercise: The final exercise, for graduate students only, is supposed to tie the course together by showing that the lens spaces from the previous exercise can also be obtained from natural ideas in covering space theory. The tricky part is showing that the two descriptions of a lens space are really the same.

Earlier we computed the fundamental group of a lens space using Seifert-Van Kampen; now we compute it using covering space theory.

The three-sphere $S^3 \subseteq R^4$ is the set of all (x, y, z, w) such that $x^2 + y^2 + z^2 + w^2 = 1$. Clearly this is the same as

$$\{(z_1, z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

Let $p \geq 2$ be an integer and let q be an integer relatively prime to p such that $1 \leq q \leq p$. Define an action of Z_p on S^3 by letting the generator act as

$$(z_1, z_2) \xrightarrow{h} \left(e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2\right)$$

Show that h^p is the identity. Show that h^k has no fixed points for $1 \leq k < p$. Explain briefly why the quotient space S^3/Z_p is a compact 3-manifold and

$$S^3 \to S^3/Z_p$$

is a covering map. Using our covering space theory, conclude that the fundamental group of this manifold is Z_p. Show that the special case $p = 2, q = 1$ gives RP^3.

Explain roughly why $L(p, q)$ is homeomorphic to the lens space discussed in the previous exercise.