

Assignment 8; Due Friday, March 10

The previous two exercise sets covered lots of material. We'll end the course with two short assignments. This one asks you to visualize an important family of three dimensional manifolds called *lens spaces*, and compute their fundamental groups.

Several exercises in our book describe lens spaces as quotient manifolds obtained by a group action on S^3 . We'll come to that description in the last exercise set. In these exercises a more concrete description of lens spaces will be used, taken from Thurston's book *Three Dimensional Geometry and Topology* published in 1997. This book describes Thurston's research from the 1980's, which had led to a deeper understanding of compact three-manifolds, particularly if Perelman's proof of the Thurston Geometrization Conjecture pans out.

I cannot resist showing a irrelevant picture from the book. The picture below shows the universal cover of a two-holed doughnut. The central shaded region is an octagon, just as our polygonal model of the two-holed doughnut last term was an octagon with sides identified. Thurston then tiles the *non-Euclidean plane* with copies of this octagon. Eight copies of the next smallest octagon are shown in the picture. There are infinitely many copies altogether, but the rest of them are so close to the boundary that they are invisible in the picture. The entire set of octagons covers the interior of the unit disk, which is the universal cover of the two-holed doughnut.

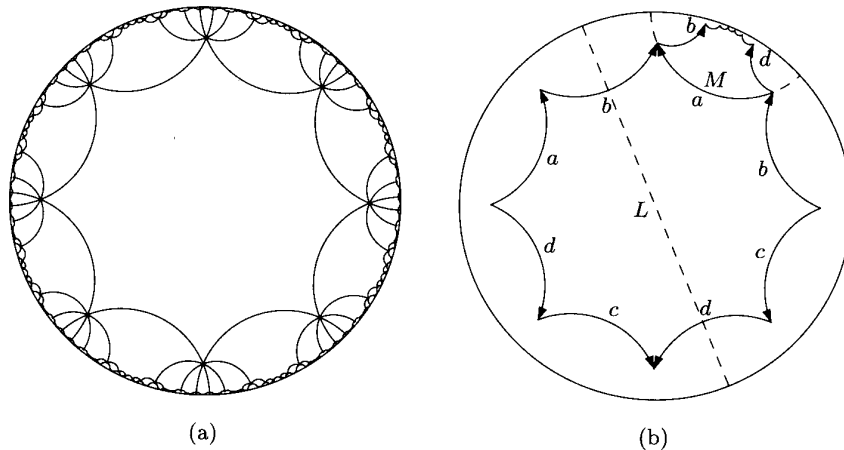
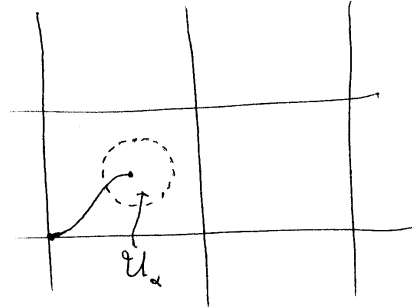
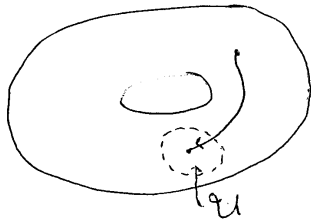


Figure 1.13. A tiling of the hyperbolic plane by regular octagons. (a) A tiling of the hyperbolic plane by identical regular octagons, seen in the Poincaré disk projection. (b) To get the small octagon from the big one, reflect in L , then in M .

Compare this to our picture of the universal cover of a torus, which tiles the *Euclidean plane* with squares.



Enough for irrelevancies. I'm going to let you directly read what Thurston writes. This section is from chapter one, where Thurston is describing several interesting three-manifolds.

Example 1.4.6 (lens spaces). Consider a ball with its surface divided into two hemispheres along the equator. What happens when we glue one hemispherical surface to the other?

If we glue with no twist at all, so that the identification is the identity on the equator, the resulting manifold is S^3 . This is analogous to the way S^2 can be formed by dividing the boundary of a disk into two intervals, and gluing one to the other so as to match each endpoint with itself.

On the other hand, if the hemispheres are glued with a q/p clockwise revolution, where q and p are relatively prime integers, each point along the equator is identified to $p - 1$ other points. A neighborhood of such a point in the resulting identification space is like p wedges of cheese stuck together to form a whole cheese, so the identification space is a manifold, called a *lens space* $L_{p,q}$.

To form a geometric model for a lens space, we need a solid something like a lens, where the angle between the upper surface and the lower surface is $2\pi/p$. This is easy to do within S^3 . Any great circle in S^3 has a whole family of great two-spheres passing through it. From this family it is easy to choose two that meet at the desired angle $2\pi/p$. Now when the two faces are glued together, neighborhoods of p points on the rim of the lens which are identified fit exactly. This corresponds to a tiling of S^3 by p lenses: see Figure 1.23, left.

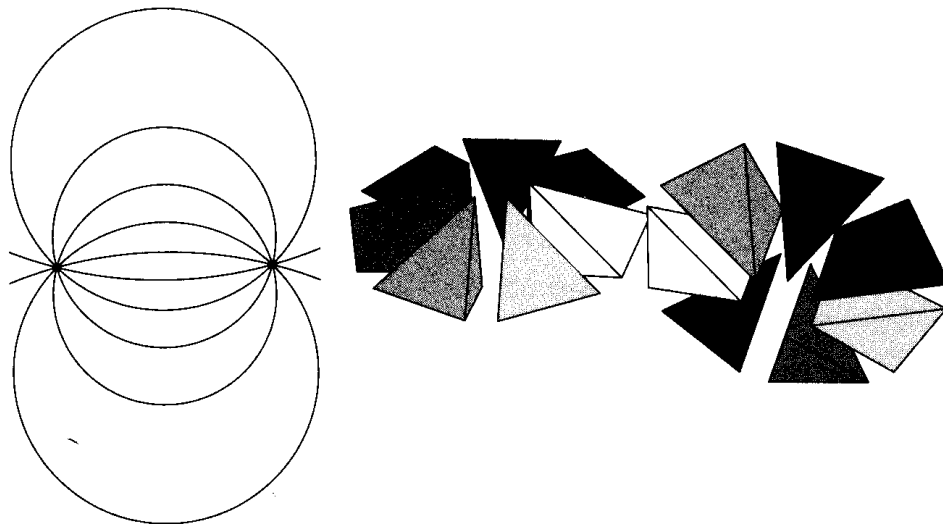


Figure 1.23. Lens spaces. On the left, S^3 is seen in cross section, tiled with twelve copies of the fundamental region for $L_{12,q}$ (we're using stereographic projection: see Exercise 2.2.8). On the right, $L_{7,2}$ is disassembled and reassembled in a different way, showing that it equals $L_{7,3}$.

Problem 1.4.7 (reworking lens spaces). (a) The lens that was glued to form $L_{p,q}$ can be cut up into p tetrahedra, meeting around one edge through its central axis. When this is done, the p tetrahedra can be assembled by gluing first the faces which came from the surface of the lens (Figure 1.23, right). What figure does this form? What identities among lens spaces can you construct?

- (b) Cut out a solid cylinder around the central axis of the lens used to form $L_{p,q}$. Its upper face is glued to its lower face to form a solid torus, under the identifications. What happens to the rest of the lens when the part of its boundary on the surface of the lens is glued together? Sketch a picture for $L_{3,2}$. Describe how lens spaces can be constructed by gluing together two solid tori.
- (c) Show that two lens spaces $L_{p,q}$ and $L_{p',q'}$ are homeomorphic if and only if $p = p'$ and either $q' = \pm q \pmod p$ or $qq' = \pm 1 \pmod p$. (This is much harder; see [Bro60] for a proof.)

Incidentally, look at exercise c) above. Thurston warns that it isn't easy. Lens spaces have been around since about 1900, but this result was proved by Brody in 1960:

E. J. Brody. The topological classification of lens spaces.
Annals of Math. (2), 71:163-184, 1960.

This year, one of our PhD candidates is reproving this theorem using a new machine.

Exercise 1: Consider a closed unit ball B^3 in R^3 . Identify boundary points on this ball as follows: If p is on the upper hemisphere, rotate p by θ about the z -axis, and then push p straight down until you reach the corresponding point q on the lower hemisphere. Glue p to q .

Let L be the quotient space of B^3 with this equivalence relation. For a moment, ignore the equator. Explain why all other points of the quotient space have open neighborhoods homeomorphic to an open set in R^3 . I expect a rough picture or paragraph explaining the essential idea, not a precise argument with formulas. You'll have no trouble with interior points, but points on the boundary require a little thought.

Exercise 2: Now worry about the equator. The equivalence relation on this equator is more complicated: If p is in the equator and we rotate by θ , the new point q is also in the equator and can be rotated by θ to yet another point r . Etc. So we must glue p, q, r, \dots together to form a single point in the quotient space.

If θ is not a rational multiple of 2π , explain why the resulting quotient space couldn't possibly be Hausdorff.

Exercise 3: From now on suppose θ is a rational multiple of 2π , say $\theta = \left(\frac{q}{p}\right) 2\pi$ where p and q are positive integers. We may as well assume that this fraction is in lowest terms, so p and q are relatively prime. Moreover, we may as well assume that $0 < \theta < 2\pi$, so $q < p$. In this case, the quotient space is called a *lens space* and denoted $L(p, q)$.

Explain why the resulting lens space is compact. This is easy.

Explain why it is a three-manifold. This is the hard step, which simply consists of showing that each point on the equator has a neighborhood homeomorphic to an open subset of R^3 .

Hint: The explanation is in Thurston. We are free to deform the original ball B^3 before we glue. Thurston flattens the ball into a lens-shaped structure, hence the name of the space. He cuts this lens into tetrahedra (not necessarily regular) and pulls the tetrahedra apart slightly.

Thurston shows two pictures of this lens, but you can concentrate on the center picture. We are supposed to glue points on the top of the lens to equivalent points on the bottom. Notice that the top of one tetrahedron will be glued to the bottom of a different tetrahedron, determined by the rotation θ .

At the center of the picture you can see that edges of the tetrahedra have been glued to a central axis. This central axis has a Euclidean neighborhood made up of pieces from the various tetrahedra.

Now look at the all important equator. Explain why the image of the equator in $L(p, q)$ is a circle and each point on this circle has p representatives on the equator. To show how the pieces on the equator glue, we must pull the tetrahedra apart and turn each by ninety degrees. Then we must rearrange them in conformance with the gluing of the top and bottom of the original lens, and reglue.

Important Remark: I am anxious that you understand what happened. But after you understand, you may have difficulty writing things down. Don't worry about that; write something and get on with your life. For comparison, you can see what Thurston wrote.

Exercise 4: Show that the fundamental group of $L(p, q)$ is Z_p .

Hint: Use the Seifert-Van Kampen theorem twice. In the first stage, let \mathcal{U} be the open ball without boundary points, and let \mathcal{V} be all points in the lens space except the center of the ball. Explain first why $\mathcal{U} \cap \mathcal{V}$ has a strong deformation retract to S^2 . Proceed. Show that the fundamental group of $L(p, q)$ is equal to the fundamental group of the two-dimensional object X formed by starting just with boundary points of the ball and gluing.

Now use the Seifert-Van Kampen theorem again. Every point in X is equivalent to a point which comes from the upper hemisphere. Let \mathcal{U} be points in X which come from points

strictly above the equator, and let \mathcal{V} be points in X which come from anything except the north pole. Explain first why $\mathcal{U} \cap \mathcal{V}$ has a strong deformation retract to S^1 . Proceed.

A Final Remark

It follows from the exercise that many lens spaces are not homeomorphic to each other. For example, $L(3, 1)$ and $L(5, 2)$ aren't homeomorphic.

On the other hand, many lens spaces aren't obviously homeomorphic and yet have the same fundamental group. For example, $L(7, 1)$ and $L(7, 2)$ both have fundamental group Z_7 . By Thurston's exercise c), which is a hard theorem, these lens spaces are not homeomorphic.

We will shortly prove that two compact two-dimensional surfaces are homeomorphic if and only if they have the same fundamental group. It came as a great surprise to Poincare that this result is false for three-manifolds. Hence additional topological invariants are required if we want to be able to distinguish such spaces.

However, Thurston later shows that the fundamental group often suffices. For example, if his geometrization conjecture is true, then all compact three-manifolds with finite fundamental group are known. These spaces are completely determined up to homeomorphism by their fundamental groups *except in the lens space case*.