Assignment 6; Due Friday, March 3

- Exercises 22.3 a, b, c
- Exercises 23.1 b, c, and (graduate students only) d, f
- (graduate students only) Recall the following useful theorem:

  Theorem 1 *Suppose Y is connected and locally pathwise connected. The diagram below can be completed if and only if*

  $$f_\star(\pi(Y, y_0)) \subseteq \pi_\star(\pi(\tilde{X}, \tilde{x}_0)).$$

  $$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "\pi"] \\ (Y, y_0) \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & (X, x_0) \end{tikzcd}$$

  Using this result, prove that the deck transformation group of a cover $\tilde{X} \to X$ acts transitively on $\pi^{-1}(x_0)$ if and only if the subgroup $\pi(\tilde{X}, \tilde{x}_0)$ of $\pi(X, x_0)$ is normal.

  Hint: Suppose you wish to obtain a deck transformation d of $\tilde{X}$ mapping $\tilde{x}_0$ to $\tilde{x}_1$. Obtain d as follows:

  $$\begin{tikzcd} & (\tilde{X}, \tilde{x}_1) \arrow[d] \\ (\tilde{X}, \tilde{x}_0) \arrow[r] \arrow[ur, dashed, "d"] & (X, x_0) \end{tikzcd}$$

  To show that d is a deck transformation, you must prove that d is a homeomorphism. But you can obtain d^{-1} from the diagram by reversing the roles of the two copies of $\tilde{X}$.

- (graduate students only) In the preceeding case when $\pi(\tilde{X}, \tilde{x}_0) \subseteq \pi(X, x_0)$ is normal, prove that the deck transformation group of $\tilde{X} \to X$ is isomorphic to the quotient group
  $$\pi(X, x_0)/\pi(\tilde{X}, \tilde{x}_0)$$
- (graduate students only) In an earlier exercise, you proved that if G is a topological group which is connected, locally pathwise connected, and semi-locally simply connected, then the universal covering space of G can be made into a topological group

so that $\pi : \tilde{G} \to G$ is a group homomorphism. Show that this result actually holds for *any* covering space of G.

Actually, you only need to define the map $\tilde{G} \times \tilde{G} \to \tilde{G}$ by lifting another map, as in your earlier proof. The remaining steps repeat what you did earlier and need not be repeated. Thus the key point is to prove that the condition for a lift is satisfied in the topological group case.

- (graduate students only) Consider the special case $SO(3) \times SO(3)$. A very interesting thing happens when you construct all covering groups of this group.

  Explain why the fundamental group of $SO(3) \times SO(3)$. is $Z_2 \times Z_2$. Show that this fundamental group has exactly five subgroups. Consequently, there are five covering groups of $SO(3) \times SO(3)$.

  Recall that $Sp(1) = S^3$ is a two-fold covering group of $SO(3)$. Using this $Sp(1)$, identify four of the five covering groups of $SO(3) \times SO(3)$ which exist by the previous exercise.

- But the remaining case is the most interesting. It yields a covering group which corresponds to the subgroup of $Z_2 \times Z_2$ containing 0×0 and 1×1. Astonishingly, this group is $SO(4)$. That is, $SO(4)$ is *almost* $SO(3) \times SO(3)$. This amazing coincidence happens only in dimension four. In all other dimensions, $SO(n)$ cannot be broken apart (it is a simple group in the technical Lie sense, which is slightly different than the finite group sense).

  Use the following hints to construct this way of looking at $SO(4)$. The group $Sp(1)$ consists of all quaternions of norm one. Let V be the set of all quaternions of the form $a_1 i + a_2 j + a_3 k$. If $q \in Sp(1)$ then q acts on V by

  $$v \to qvq^{-1}.$$

  In this way, $Sp(1)$ induces a rotation in three space.

  But $Sp(1)$ also acts on the full set of quaternions $H = \{a_0 + a_1 i + a_2 j + a_3 k\}$ by $h \to qh$. Indeed, we can let $Sp(1) \times Sp(1)$ act on this set by letting one factor act from the left and one factor act from the right:

  $$h \to q_1 h q_2^{-1}$$

  Since q_1 and q_2 have norm one, this map preserves the norm of h and so acts on H via rotations in $SO(4)$.

  Prove that (q_1, q_2) and (q_1', q_2') define the same map if and only if $(q_1', q_2') = \pm(q_1, q_2)$. This plus or minus corresponds to the group $\{0 \times 0, 1 \times 1\}$ which gave rise to this covering space.

Prove that every rotation in $SO(4)$ is one of the maps from $Sp(1) \times Sp(1)$. You can be fairly cavalier about how rotations work. The key observations are

- Show that if $v \in H$ is a unit vector, there is an element of $Sp(1) \times Sp(1)$ which rotates v to 1.
- Suppose that $R \in SO(4)$ is a rotation which fixes 1. Then R preserves the three dimensional subspace perpendicular to 1 and thus is a rotation of this subspace. Show that all such rotations come from $Sp(1) \times Sp(1)$.
- Mumble one or two sentences from group theory to conclude the argument.