Assignment 6; Due Friday, February 24

You have a week and a half to work on this assignment. I may assign a few more problems on Friday, but I won't assign more than three or four problems then.

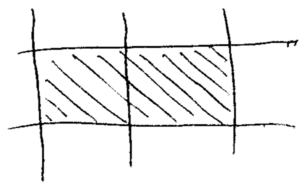
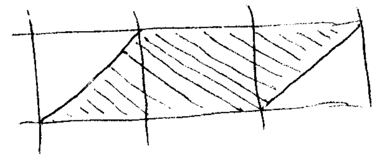
It is important that you completely understand our calculation of $\pi(S^1)$. So I'm going to ask you to write down a clean proof as follows:

- Let $\gamma : I \to S^1$ be a path starting at 1. Prove carefully that this path can be lifted to a path $\tilde{\gamma} : I \to R$ starting at 0.
- Let $h : I \times I \to S^1$ be a map such that $h(0,0) = 1$. Prove that this map can be lifted to a map $\tilde{h} : I \times I \to R$ such that $\tilde{h}(0,0) = 0$.
- Let $\gamma : I \to S^1$ be a path starting and ending at 1. Explain carefully why $\tilde{\gamma}(1)$ is an integer.
- Let γ and τ be paths in S^1 starting and ending at 1, and suppose these paths induce the same element of $\pi(S^1, 1)$. Explain carefully why $\tilde{\gamma}(1) = \tilde{\tau}(1)$. Give all necessary details.
- By the previous result, there is a well-defined map $\pi(S^1, 1) \to Z$. Prove that this map is one-to-one.
- Prove that the previous map is onto.

Remark: Suppose X is a connected, locally pathwise connected, semilocally simply connected space . In class we completely classified all covering spaces of X. We proved that classifying spaces of X are in one-to-one correspondence with subgroups of $\pi(X)$ up to conjugacy. If $\tilde{X}$ is a covering space, then $\pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is one-to-one, so $\pi(\tilde{X})$ is a subgroup of $\pi(X)$. Changing the base point $\tilde{x}_0$ changes this subgroup to a conjugate subgroup. If two covering spaces induce the same subgroup up to conjugacy, they are isomorphic as covering spaces. Finally, every subgroup comes from a covering space.

- Explain why RP^n has exactly two covering spaces. What are they?
- Find all covering spaces of a Mobius band by explicitly describing each covering space. Show that the covering spaces of a Mobius band are homeomorphic to a Mobius band, a cylinder, or an open disk.
- Below are two pictures of different covering spaces of a torus. We are thinking of the torus as a unit square with opposite sides identified in the standard way. We are

thinking of these covering spaces as parallelograms in the plane with opposite sides identified, so the covering spaces are also tori. Finally, the covering projection is the map induced by the equivalence relation $(x, y) \sim (x + m, y + n)$.

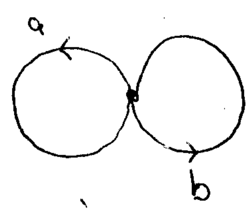



Explain briefly why each space is a covering space. Explain why each point in X comes from two points in $\tilde{X}$ in each of the two examples. Finally prove that these covering spaces are *not* isomorphic as covering spaces, even though both coverings are two to one and both covering spaces are homeomorphic. (Hint: Find a closed path in X which lifts to a closed path in one of the covering spaces, but not in the other.)

- Since $\pi(S^1 \times S^1) = Z \times Z$ is abelian, covering spaces of $S^1 \times S^1$ are in one-to-one correspondence with subgroups of $Z \times Z$. Find the subgroups corresponding to the two covering spaces of the previous exercise.

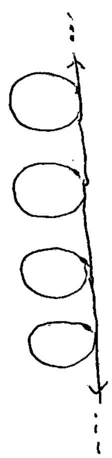
- (Graduate Students) Find all subgroups of $Z \times Z$. Indeed, show that these groups are 0×0 or else all multiples of (a, b) for a nonzero element of $Z \times Z$ or else the set of all $k(M, 0) + l(A, N)$ for integers k and l, where M and N are positive integers and $0 \le A < M$.

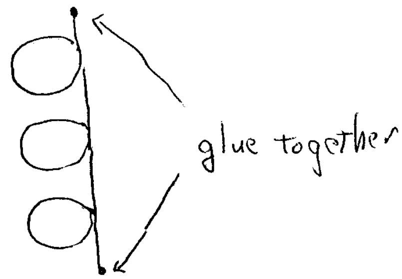
- Let X be a bouquet of two circles.

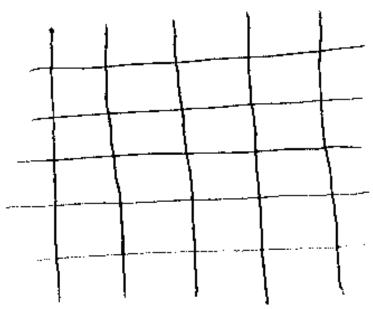


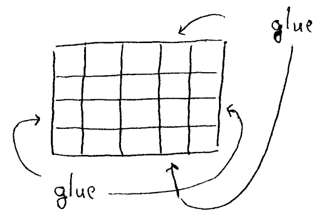
We will later prove that the fundamental group of this space is $F(a, b)$, the free product on two generators a and b. This means that each element of the group can be written as a finite product of powers of a and b. For instance, $a^3b^2a^{-3}ba^{-2}$ and b^2abab^2 are two elements of the group. Here a is a loop around one of the circles and b is a loop around the other circle.

Explain how each picture below is a natural covering space of X, and find the subgroup of $F(a,b)$ corresponding to each picture. (Undergraduates need only study the first two pictures.)







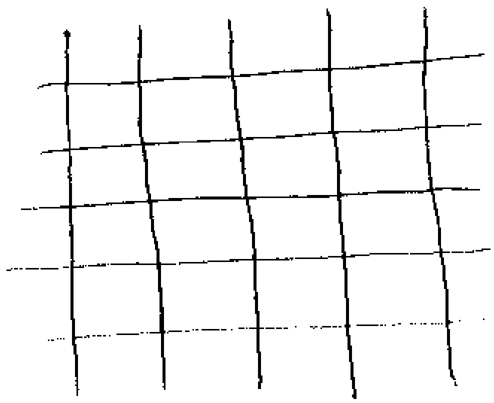


- (Graduate Students) Suppose G is an arbitrary group. If $g, h \in G$, their commutator is the element $ghg^{-1}h^{-1} \in G$. Notice that this commutator is zero if G is abelian. By definition, the commutator subgroup of G is the normal subgroup generated by all commutators; this means that it is the subgroup consisting of products of the form

$$\left(g_1 c_1 g_1^{-1}\right) \left(g_2 c_1 g_2^{-1}\right) \, \dots \left(g_n c_n g_n^{-1}\right)$$

where the g_i are arbitrary elements of G and the c_i are arbitrary commutators.

Notice that G modulo this commutator subgroup is abelian. It is called the *abelianization of G*. Prove that the fundamental group of the covering space below is the commutator subgroup of the free group on two generators.



The final exercise is for graduate students.

There are many applications of covering spaces to other areas of mathematics. I want to tell you about some of them, but am a little frustrated because these applications require results which aren't prerequisites for this course. However, one very important application can be introduced fairly easily.

Let G be a topological group. Thus G is simultaneously a group and a topological space, and the group operations are continuous, so the maps $G \times G \stackrel{g_1 \circ g_2}{\rightarrow} G$ and $G \stackrel{g^{-1}}{\rightarrow} G$ are continuous.

Assume also that G is a nice topological space: it is connected, locally pathwise connected, and semilocally simply connected. These hypotheses guarantee that G has a universal covering space $\tilde{G}$.

First Exercise: Fix $\tilde{e} \in \tilde{G}$ projecting to the identity $e \in G$. Prove that there is a unique topological group structure on $\tilde{G}$ making $\pi : \tilde{G} \rightarrow G$ a group homomorphism and $\tilde{e}$ the identity element.

Hint: Consider the map $\tilde{G} \times \tilde{G} \rightarrow G \times G \rightarrow G$ where the first map is $\pi \times \pi$ and the second map is the group product in G. Explain why this map lifts to a map from $\tilde{G} \times \tilde{G} \rightarrow \tilde{G}$. The group axioms for $\tilde{G}$ can then be proved by using uniqueness of lifts. Details please!

Remark: Many interesting groups are constructed this way. Let me give some examples.

The only one-dimensional connected topological groups are R and S^1 and we know all about them. Notice that our covering map $R \rightarrow S^1$ is indeed a group homomorphism.

The only two-dimensional connected topological topological groups are $R \times R$, $R \times S^1$, $S^1 \times S^1$, and the $ax + b$ group. This last group is the set of all maps $x \rightarrow ax + b$ where $a, b \in R$ and $a > 0$. Notice that it is simply connected.

But in three dimensions, there are many connected topological groups (a complete classification is known). By far the most interesting are $SO(3)$, the group of rotations of three space, and $SL(2, R)$, the set of 2×2 matrices of determinant one. It is just a coincidence that the set of rotations of R^3 has dimension three.

Exercise 2: Recall the quaternions $H = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in R\}$. Define a multiplication by $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k, jk = -kj = i$, and $ki = -ik = j$. If $q = a_0 + ia_1 + ja_2 + ka_3$, define $\overline{q} = a_0 - a_1 i - a_2 j - a_3 k$ and define $q\overline{q} = ||q||^2$. Prove that $q\overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2$ and prove that $||q_1 q_2|| = ||q_1|| \, ||q_2||$.

Exercise 2 Continued: Prove that the set $Sp(1)$ of unit quaternions is a topological group homeomorphic to S^3.

Exercise 2 Continued: Let V be the set of quaterions such that $\overline{q} = -q$. Show that V is isomorphic to R^3. If q is a unit quaternion, show that the map $V \to V$ by $v \to qvq^{-1}$ is well-defined. Show that this map preserves the norm of elements in V and so is a rotation.

Exercise 2 Continued: Show that q_1 and q_2 induce the same rotation if ans only if $q_1 = \pm q_2$.

Exercise 2 Continued: Consider the specific rotation generated by $q = \cos\theta + i\sin\theta$. Show that it becomes rotation about the x axis by 2θ. Conclude that the image of the map $Sp(1) \to SO(3)$ contains all rotations about the x-axis. Explain why it must then contain all rotations about the x, y, or z axis. Explain finally why the image must thus contain all rotations.

Exercise 2 Concluded Conclude that $Sp(1) \to SO(3)$ is a 2-fold covering space. It is not surprising that the universal covering group of $SO(3)$ has this form since $SO(3)$ is homeomorphic to RP^2 and the fundamental group of this space is Z_2.

Remark: I wish I could ask questions about $SL(2, R)$, but maybe not. The group $SL(2, R)$ is not compact. Notice that $S^1 = SO(2, R) \subseteq SL(2, R)$. It turns out that $SO(2, R)$ is a strong deformation retract of $SL(2, R)$. Consequently the fundamental group of $SL(2, R)$ is Z, and thus the universal covering group of $SL(2, R)$ is an infinite-sheeted cover. This covering group is interesting because it is *not* a matrix group (this requires proof, of course). So it is not often written down explicitly.

Remark: It can be shown that the fundamental group of $SO(n)$ is Z_2 for all $n \geq 3$. Thus each of these groups has a 2-fold universal cover, called $Spin(n)$. These groups are compact and simply connected, but they are not homeomorphic to spheres. It can be proved that S^1 and S^3 are the only spheres which can be made into topological groups.

According to the Poincare conjecture, S^3 is the only compact simply-connected 3 manifolds, just as S^2 is the only compact simply-connected 2 manifold. The groups $Spin(n)$ show that in higher dimensions, the spheres are not the only compact simply-connected manifolds.

Remark: Finally, a vague remark about where all of this leads. Finite dimensional topological groups are called *Lie groups.* These groups can all be given differentiable structures and studied using calculus. Near $e \in G$, these groups look like Euclidean space, and so tangent vectors starting at e look like a vector space L of dimension equal the dimension of the group. This vector space is known as the *Lie algebra* of G.

This Lie algebra has an additional structure called the Lie bracket; if X and Y are tangent vectors, the Lie bracket $[X, Y]$ is another vector in L which measures the non-commutativity of group elements in the direction X and group elements in the direction Y. Since G and

$\tilde{G}$ are locally homeomorphic, they have the same tangent space at e and the same Lie algebra.

It can be proved that two simply connected Lie groups are isomorphic if and only if their Lie algebras are isomorphic. Consequently, Lie theory breaks into two disjoint pieces. The first is a purely algebraic study of Lie algebras; it is equivalent to studying simply connected Lie groups. The second piece is essentially our covering space theory and leads to a classification of all Lie groups with a given Lie algebra.