Assignment 3; Due Friday, January 27 (Revised List)

Read sections 15 and 16 on the fundamental group, and on $\pi_1(S^1)$. Do the following exercises.

- 15.11ab (the first should be very easy, and the second a little harder)
- read 15.11d, e, and g, noticing that these are essentially results from class; you need not do these exercises
- (graduate students only) 15.11f
- 15.18bc (this is an important result, namely $\pi_1(X \times Y, x_0 \times y_0) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$. For example, it follows that the fundamental group of a torus is $Z \times Z$. This exercise will require you to keep track of the meaning of the notation, a useful skill.)
- 15.18d Prove only the first part of this exercise, up to the moment when you show that π_1 of a topological group is abelian. This exercise may be slightly tricky, but it has a beautiful solution. Notice that S^1 is the set of all complex numbers of absolute value one; this is a group under multiplication. It follows that $S_1 \times S_1$ is also a topological group. The fundamental groups of these objects are Z and $Z \times Z$, both abelian. The exercise shows that this is no accident.

  We will later show that the only compact connected surfaces with abelian fundamental group are S^2, P^2, and T^2. This exercise then shows that it is impossible to define a topological group structure on any of the remaining compact connected surfaces.

  Actually S^2 and P^2 cannot be given a topological group structure either, but this is for a different reason. The only spheres which can be made into topological groups are S^1 and S^3.

Extra Problem: The previous exercises did not use our calculation of $\pi_1(S^1) = Z$. But in this exercise, you definitely need this result.

I'd now like to convince you that purely topological results can be proved using our algebraic machinery. Consider the Klein bottle $\mathcal{K}$. The pictures below show two circles A and B embedded in this bottle. Notice that these circles intersect at a point p. Take this point to be the base point for homotopy. Then the circles induce elements a and b of $\pi_1(\mathcal{K})$. Of course a and b might be zero in the group, or they might be equal in the group, or they might be related in another unusual way.

- Prove that $aba^{-1} = b^{-1}$ in $\pi_1(\mathcal{K})$.
- Show that A is a retract of $\mathcal{K}$. I'm not asking you to show that it is a strong deformation retract, but only a retract. Find an explicit retraction.
- Conclude that a is a nonzero element of $\pi_1(\mathcal{K})$ with infinite order.
- Finally, show that B is not a retract of $\mathcal{K}$. This is the moment when the algebra is used to prove a topological result. As a hint, assume that there is a retract $r : \mathcal{K} \to B$. Then there is an induced map $r_\star : \pi_1(\mathcal{K}) \to \pi_1(B)$. Explain why the various properties of this supposed map are inconsistent.