Mathematics 431/531 Midterm

October 31, 2005

Name ____

1. Define:

   (a) continuous map between topological spaces

   Answer: A map $f : X \to Y$ is continuous if whenever $\mathcal{U} \subseteq Y$ is open, $f^{-1}(\mathcal{U}) \subseteq X$ is open.

   (b) product topology on $X \times Y$, where X and Y are topological spaces

   Answer: We give $X \times Y$ a topology by calling $\mathcal{W} \subseteq X \times Y$ open if there exist open $\mathcal{U}_\alpha \subseteq X$ and $\mathcal{V}_\alpha \subseteq Y$ such that $\mathcal{W} = \cup\, \mathcal{U}_\alpha \times \mathcal{V}_\alpha$.

   (c) induced topology on $A \subseteq X$, where A is a subset of a topological space X

   Answer: We give A a topology by calling $\mathcal{U} \subseteq A$ open if there is an open $\mathcal{V} \subseteq X$ such that $\mathcal{U} = \mathcal{V} \cap A$.

2. Prove that a composition of two continuous maps $f : X \to Y$ and $g : Y \to Z$ is continuous.

   Answer: Suppose $\mathcal{U} \subseteq Z$ is open. Then $(g \circ f)^{-1}(\mathcal{U}) = f^{-1}(g^{-1}\mathcal{U})$. Since g is continuous, $g^{-1}(\mathcal{U})$ is open. Since f is continuous, $f^{-1}(g^{-1}\mathcal{U})$ is open.

3. If a statement below is true, write "yes." If it is false, give a counterexample.

   (a) Every metric space is Hausdorff.

   Answer: This is true.

   (b) If two subsets of R^n are homeomorphic and one is bounded, the other is bounded.

   Answer: This is false because $(-1, 1) \subseteq R$ and $R \subseteq R$ are homeomorphic.

   (c) If $f : X \to Y$ is continuous, one-to-one, and onto, then f is a homeomorphism.

   Answer: This is false. Let $X = R$ with the discrete topology and let $Y = R$ with the usual topology. Then the identity map $id : X \to Y$ is continuous, one-to-one, and onto, but not a homeomorphism.

(d) If X is a topological space and $A \subseteq X$ is compact, then A is closed.

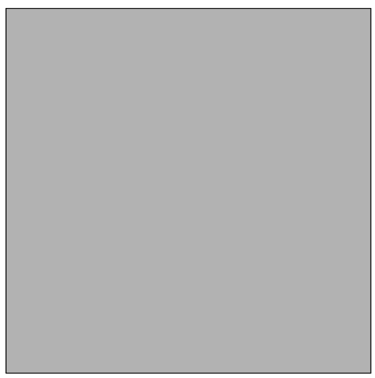
Answer: This is false. Let X be R with the concrete topology. Then every subset is compact because the only sets in a cover will be the empty set and R and just one of these already covers. But if A is nonempty and not everything, it is not closed.

4. Prove that $[a, b]$ is compact.

Answer: Let $\{\mathcal{U}_\alpha\}$ be an open cover. Define $s = \sup\{c \mid [a, c] \text{ can be finitely covered}\}$. The point s is in some $\mathcal{U}_{\alpha_1}$, so there is an $\epsilon > 0$ such that $(s - \epsilon, s] \subseteq \mathcal{U}_{\alpha_1}$. By definition of s, there must be a $c_1 > s - \epsilon$ so that $[a, c_1]$ is finitely covered. But then $[a, c_1] \cup (s - \epsilon, s] = [a, s]$ is finitely covered. If $s = b$, we are done.

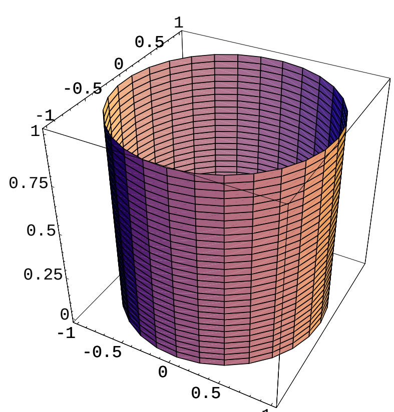
Otherwise we could have chosen ϵ so $(s-\epsilon, s+\epsilon) \subseteq \mathcal{U}_{\alpha_1}$. But then $[a, c_1] \cup (s-\epsilon, s+\epsilon) = [a, s+\epsilon)$ could be finitely covered, so $[a, s + \epsilon/2]$ could be finitely covered, contradicting the definition of s.

5. Let S be the standard square $S = \{\ (x, y) \mid 0 \le x \le 1, 0 \le y \le 1\ \}$.



(a) Define an equivalence relation on S by declaring that for each y we have $(0, y) \sim (1, y)$. Draw a picture of the resulting object $S/\!\sim$.

Answer:



(b) Expand the equivalence relation by also calling all $(x, 0)$ equivalent. Draw a picture of the resulting $S/\sim$.

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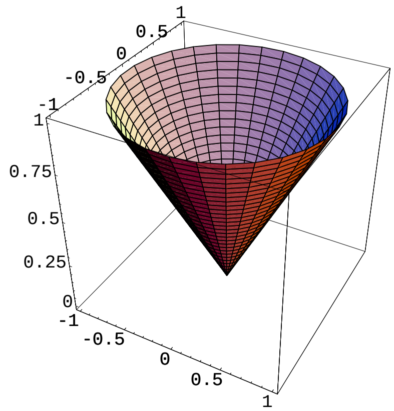
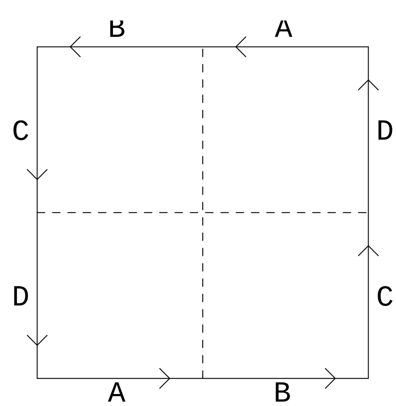


(c) Let D be the unit disk in the plane. Define $f : S \to D$ by $f(x, y) = (y \cos 2\pi x, y \sin 2\pi x)$. Explain why f induces a map $\tilde{f} : S/\sim \to D$. Explain why this map is continuous. Explain why it is a homeomorphism (you may assume in your explanation that it is one-to-one and onto).

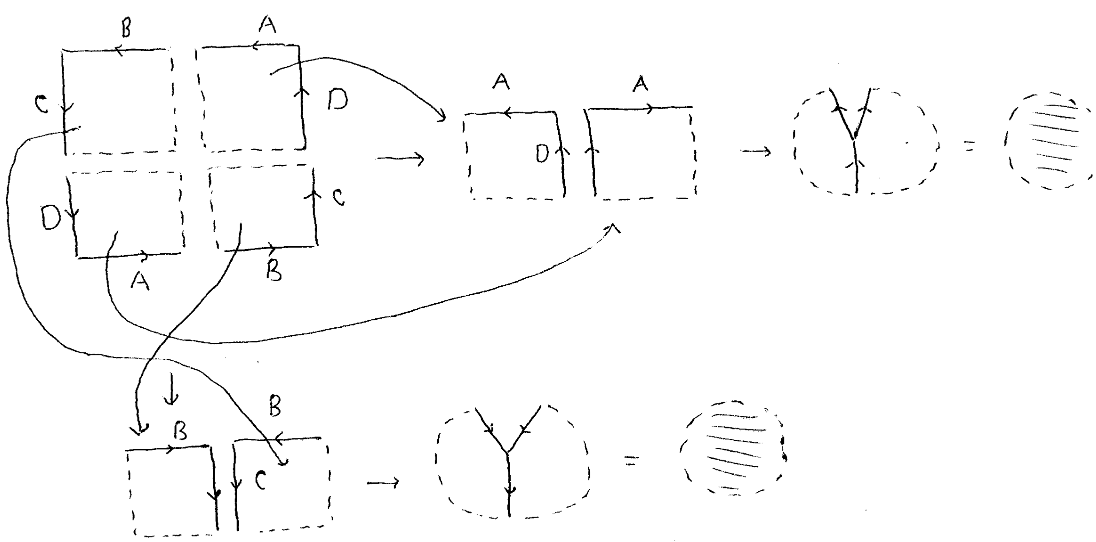
Answer: We must check that $\tilde{f}$ is well defined. The points $(0, y)$ and $(1, y)$ are equivalent and map to $(y \cos 0, y \sin 0) = (y, 0)$ and $(y \cos 2\pi, y \sin 2\pi) = (y, 0)$ respectively, and so to the same point. The points $(x_1, 0)$ and $(x_2, 0)$ are equivalent and they both map to $(0, 0)$.

It follows that f induces $\tilde{f}$ and this map is automatically continuous. It is easy to see that $\tilde{f}$ is one-to-one and onto. It is a homeomorphism because S is compact and D is Hausdorff.

6. The object below, with opposite points of the boundary identified, is RP^2. Suppose this object is cut along the dotted lines and these dotted lines are removed. Show pictorially that the resulting object is homeomorphic to two disjoint open disks in the plane.



Answer: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