Mathematics 431/531 Final Examination

December 7, 2005

Name ______________________________

1. (a) Give the definition of a *connected* topological space.

   Answer: A topological space X is connected if it is impossible to write $X = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are nonempty, disjoint, open subsets of X.

   (b) If X is connected and $f : X \to Y$ is continuous, prove that $f(X)$ is connected.

   Answer: Suppose $f(X) = \mathcal{U} \cup \mathcal{V}$ as above. Then $f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{V})$ are open in X and $X = f^{-1}(\mathcal{U}) \cup f^{-1}(\mathcal{V})$. Moreover, these open sets are disjoint, for $x \in f^{-1}(\mathcal{U}) \cap f^{-1}(\mathcal{V})$ implies $f(x) \in \mathcal{U} \cap \mathcal{V}$ which is impossible because this set is empty. Since X is connected, one of $f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{V})$ is empty, so one of $\mathcal{U}$ and $\mathcal{V}$ is empty, a contradiction.

2. (a) Give an example of a Hausdorff space X and onto map $\pi : X \to Y$ such that the quotient topology on Y is not Hausdorff.

   Answer: Let $X = R^2$ and let $\sim$ be the equivalence relation $x \sim y$ if $\exists r \neq 0$ such that $x = ry$. Then $Y = S^1 \cup \{0\}$ because each nonzero element x is equivalent to $\frac{x}{||x||} \in S^1$. If $\mathcal{U}$ is an open neighborhood of 0, then $\pi^{-1}(\mathcal{U})$ is open in R^2 and thus contains a point equivalent to every element of R^2, so $\mathcal{U} = Y$. Thus 0 cannot be separated from any other element of Y.

   (b) In class we proved that in the previous situation if X is compact and π is a closed map, then Y is Hausdorff. Using this result, prove that RP^n is Hausdorff. Recall that RP^n is the quotient space obtained by identifying antipodal points of S^n.

   Answer: It is enough to prove that π is a closed map. Let $\phi : S^n \to S^n$ be the antipodal map. Suppose $A \subseteq X$ is closed. We want to prove that $\pi(A) \subseteq RP^n$ is closed, which will be true if $\pi^{-1}(\pi(A)) \subseteq S^n$ is closed. But this set is $A \cup \phi(A)$, which is a union of two closed sets and thus closed.

3. (a) Give the definition of a *compact* topological space.

   Answer: A topological space X is compact if whenever $\{\mathcal{U}_\alpha\}$ is an open cover of X, i.e., $\cup\, \mathcal{U}_\alpha = X$, then there is a finite subcover $\mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_k}$, so $\mathcal{U}_{\alpha_1} \cup \ldots \cup \mathcal{U}_{\alpha_k} = X$.

   (b) If $f : X \to Y$ is continuous and X is compact, prove that $f(X)$ is compact.

   Answer: Let $f(X) \subseteq \cup\, \mathcal{U}_\alpha$. Then $X \subseteq \cup f^{-1}(\mathcal{U}_\alpha)$. Since X is compact, there is a finite collection of indices $\alpha_1, \ldots, \alpha_k$ such that $X \subseteq f^{-1}(\mathcal{U}_{\alpha_1}) \cup \ldots \cup f^{-1}(\mathcal{U}_{\alpha_k})$. So $f(X) \subseteq \mathcal{U}_{\alpha_1} \cup \ldots \cup \mathcal{U}_{\alpha_k}$.

(c) In class we proved that a subset $A \subseteq R^n$ is compact if and only if it is closed and bounded. Show that this result is false if we replace the space R^n with an infinite discrete metric space M.

Proof: Let A be an infinite subset of M. Then A is closed because every subset of a discrete space is closed. Also A is bounded because the distance between any two points of M is either 0 or 1. However, A is not compact because A is covered by the open subsets $\{a\}$ for $a \in A$, and no finite number of these covers A.

4. Are the surfaces with surface symbols $aba^{-1}b$ and $abab$ homeomorphic? Why or why not?

Answer: Using the rule $xPxQ \sim x_1x_1P^{-1}Q$, we have $aba^{-1}b = ab_1b_1a = aab_1b_1$, so this space is homeomorphic to $RP^2 \# RP^2$. Using the same rule, $abab \sim a_1a_1b^{-1}b$. But $b^{-1}b$ can be glued together and eliminated, so the second symbol is equivalent to a_1a_1 and the second space is homeomorphic to RP^2. These two spaces are not homeomorphic.

5. Prove that $[0,1]$ is connected OR prove that $[0,1]$ is compact.

Answer: The first holds, for suppose that $[0,1] \subseteq \mathcal{U} \cup \mathcal{V}$ with the usual properties. Without loss of generality, say $0 \in \mathcal{U}$. Let $s = \sup\{t \mid [0,t] \subseteq \mathcal{U}\}$. If $s \in \mathcal{U}$, then there is an $\epsilon > 0$ such that the set of all $x \in [0,1]$ with $|x - s| < \epsilon$ is contained in $\mathcal{U}$. This contradicts the definition of s if $s < 1$ because then we can find s_1 greater than s with $[0, s_1] \subseteq \mathcal{U}$. However if $s = 1$, then $[0,1] \subseteq \mathcal{U}$, making $\mathcal{V} = \emptyset$ contrary to our choice of $\mathcal{U}$ and $\mathcal{V}$.

So $s \in \mathcal{V}$. But then a similar ϵ can be chosen so $x \in [0,1]$ and $|x - s| < \epsilon$ implies $x \in \mathcal{V}$. By definition of s, we can find s_1 with $[0, s_1] \subseteq \mathcal{U}$ and $|s_1 - s| < \epsilon$, a contradiction. This contradiction shows that $\mathcal{U}$ and $\mathcal{V}$ cannot exist.

The proof that $[0,1]$ is compact is similar. Suppose $\{\mathcal{U}_\alpha\}$is an open cover of $[0,1]$. Let

$$s = \sup\{t \mid [0,t] \text{can be covered by a finite subcover}\}$$

Find β so $s \in \mathcal{U}_\beta$ and find $\epsilon > 0$ such that whenever $x \in [0,1]$ and $|x - s| < \epsilon$ then $x \in \mathcal{U}_\beta$. By definition of sup, we can find s_1 so $[0, s_1]$ can be covered by a finite subcover and $|s_1 - s| < \epsilon$. But then whenever $s < s_2$ such that $s_2 \in [0,1]$ and $|s_2 - s| < \epsilon$ we have $[0, s_2]$ covered by the finite subcover for $[0, s_1]$ together with $\mathcal{U}_\beta$. By definition of s we conclude that $s = 1$ and $[0,1]$ can be finitely covered.

6. As a set, the long ray is $J \times [0,1)$ where J is a specific index set. List the key properties of this index set. Carefully define the topology on this set and explain why the resulting space is not second countable.

Answer: The index set should be totally ordered and well-ordered. This last condition means that every nonempty subset of J should have a smallest element. Moreover, J should be uncountable. But whenever $j \in J$, the set of all elements less than j should be countable.

Order $J \times [0,1)$ by letting $(j_1, t_1) < (j_2, t_2)$ if $j_1 < j_2$ or if $j_1 = j_2$ and $t_1 < t_2$. Let an open interval be a set of the form $(\alpha, \beta) = \{p \in J \times [0,1) \mid \alpha < p < \beta\}$ or else $[\tilde{0}, \beta) = \{p \in J \times [0,1) \mid p < \beta\}$. Call a subset of $J \times [0,1)$ open if it is a union of open intervals.

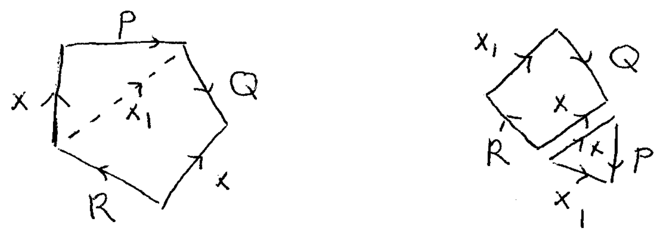
Fix an index $j \in J$. The set $\mathcal{U}_j$ of all (j,t) with $t \in (0,1)$ is open in the long ray. As j varies, these open sets run through an uncountable collection of disjoint open sets. The long ray cannot then have a countable collection of open sets such that every open set is a union of these, because then each of the $\mathcal{U}_j$ would be a union of certain of these sets, and the sets whose union gives $\mathcal{U}_j$ would be disjoint from the sets which give another $\mathcal{U}_k$, so there would have to be uncountably many of these sets.

7. Show that $R^3 - R$ is homeomorphic to $S^1 \times R^2$.

   Answer: We can suppose R is the z-axis in R^3. A point in $R^3 - R$ is thus a nonzero point in the plane, together with a height h in the z direction. Write this point in the plane in polar coordinates (r, θ). The result is cylindrical coordinates (r, θ, h) for points in $R^3 - R$. Map $R^3 - R$ to $S^1 \times R^2$ by $(r, \theta, h) \to (\theta, \ln r, h)$ where $\theta \in S^1$ and $(\ln r, h) \in R^2$. The inverse map is $(\theta, x, y) \to (e^x, \theta, y)$ and both of these maps are clearly continuous.

8. (a) Show that the surface symbol $aPQa^{-1}R$ can be replaced with an equivalent surface symbol $a_1QPa_1^{-1}R$.

   Answer:



   (b) Using this result, show that the surface symbol $aPbQa^{-1}Rb^{-1}S$ can be replaced with an equivalent surface symbol with $a_1b_1a_1^{-1}b_1^{-1}T$ where T is an appropriate concatenation of P, Q, R, S in some order.

   Answer:

$$a(P)(bQ)a^{-1}Rb^{-1}S \to a_1(bQ)(P)a_1^{-1}Rb^{-1}S =$$
$$b(QP)(a_1^{-1}R)b^{-1}Sa_1 \to b_1(a_1^{-1}R)(QP)b_1^{-1}Sa_1 =$$
$$a_1^{-1}(RQP)(b_1^{-1}S)a_1b_1 \to a_2^{-1}(b_1^{-1}S)(RQP)a_2b_1 =$$
$$a_2b_1a_2^{-1}b_1^{-1}SRQP$$

9. Suppose X and Y are connected topological spaces and $A \subset X$ and $B \subset Y$ are subspaces such that $A \neq X$ and $B \neq Y$. Prove that $X \times Y - A \times B$ is connected.

   Answer: Let $x_0 \notin A$ and let $y_0 \notin Y$. The union of all connected subsets of $X \times Y - A \times B$ containing $x_0 \times y_0$ is connected; call this connected set Z.

   Notice that $X \times \{y_0\}$ and $\{x_0\} \times Y$ are connected subsets which intersect at $x_0 \times y_0$, so their union W is connected and $W \subseteq Z$.

   Let $(x, y) \in X \times Y - A \times B$. Then $x \notin X$ or $y \notin B$. Without loss of generality, say $x \notin A$. Then $x \times Y$ is a connected set which intersects W, so their union is connected and contains $x_0 \times y_0$ and thus is a subset of Z. In particular, $x \times y \in Z$. This argument works whenever $x \times y \in X \times Y - A \times B$, so Z is everything in $X \times Y - A \times B$.

10. (a) Prove that $[-1,1]/[-1,0]$ is homeomorphic to $[-1,1]$.

Answer: Map $[-1,1]$ to $[-1,1]$ by mapping $[-1,0]$ to -1 and mapping $[0,1] \to [-1,1]$ via $f(t) = 2t - 1$. This map is continuous and it induces a mapping $[-1,1]/[-1,0] \to [-1,1]$. Clearly this map is one-to-one.

Since $[-1,1]$ is compact and $[0,1]$ is Hausdorff, this map is a homeomorphism.

(b) Prove that $[-1,1]/[-1,0)$ is not homeomorphic to any closed interval in R.

Answer: This will follow if we can show that $[-1,1]/[-1,0)$ is not Hausdorff. Notice that $[-1,0)$ maps to a single point p in the quotient space, and notice that $0 \in [-1,1]$ maps to a different point q in the quotient space. If $\mathcal{U}$ is an open neighborhood of q in the quotient space, then $\pi^{-1}(\mathcal{U})$ must be an open neighborhood of 0 and thus must include points equivalent to p. So q cannot be separated from p in the quotient space.