

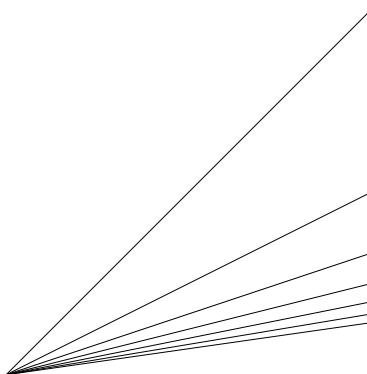
Assignment 9; Due Wednesday, November 30

12.10a Every map $\gamma : [0, 1] \rightarrow X$ is continuous. So we can always find a path from p to q by letting $\gamma(t) = p$ for $t < 1$ and $\gamma(1) = q$.

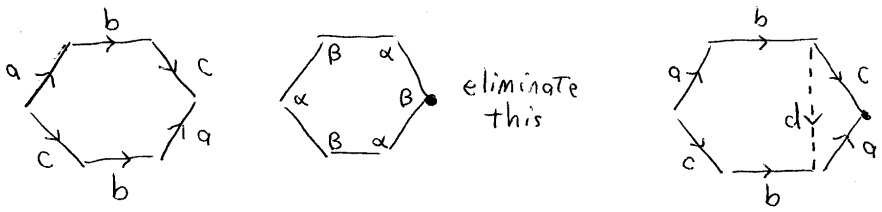
12:10f The set looks like the picture below. The set A is the series of infinitely many rays starting at the origin, and the set B is the interval $[1/2, 1]$ on the x -axis. Both of these sets are arcwise connected, hence connected.

The set $A \cup B$ is connected, for suppose $A \cup B \subseteq \mathcal{U} \cup \mathcal{V}$ where \mathcal{U} and \mathcal{V} are disjoint open sets in $A \cup B$. Since A and B are connected, we must have $A = \mathcal{U}$ and $B = \mathcal{V}$. But this is impossible because if $p \in B$ then there is an $\epsilon > 0$ such that $B_\epsilon(p) \subseteq \mathcal{V}$, and clearly $B_\epsilon(p)$ intersects A and thus intersects \mathcal{U} .

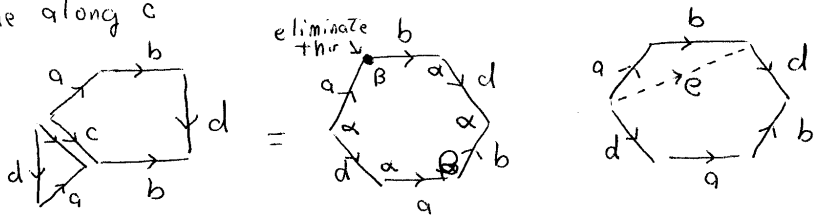
We still must prove that $A \cup B$ is not arcwise connected. Suppose $\gamma(t)$ is a path starting at the origin and ending at $(1/2, 0)$. The set $\gamma^{-1}(B)$ is closed, so it contains its infimum. Thus there is a first time t_0 such that $\gamma(t_0)$ is in this set. By continuity, there is a delta such that $|t - t_0| < \delta$ implies that the x -component of γ is larger than $1/4$. But $A \cap \{(x, y) \mid x > 1/4\}$ is a union of line segments, each open and connected in this set. Since the image of γ on $(t_0 - \delta, t_0)$ is in this set and connected, it must be in one of the line segments. But this line segment is bounded away from B , so $\gamma(t_0) \in B$ is impossible.



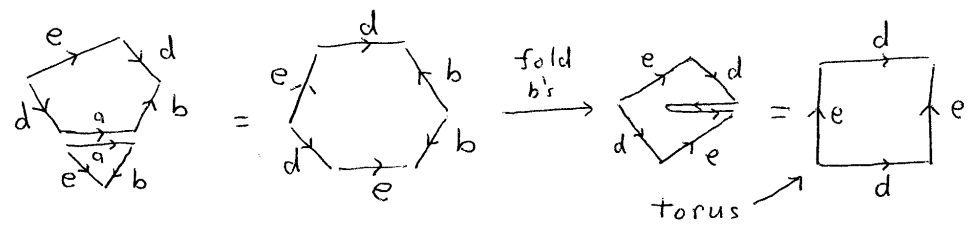
Extra Problem 1 A brief analysis shows that the vertices of the figure correspond to two points in the identification space; we have labeled the vertices α and β accordingly. The first step of the reduction to canonical form eliminates one of these vertices. We will eliminate β . This is shown on the next page. At the end, we find that we have the symbol of a torus, so this manifold is T^2 .



According to the algorithm, we pick an edge from this β which goes to a different kind of vertex, and then glue back along the other edge. In this case both edges go to different kinds of vertices, so either can be used for gluing. We'll glue along c



We'll glue back along a ; either choice works



Extra Problem 2 This time all vertices correspond to the same point. So we reduce to canonical form using the formula $xPQx^{-1}R = x_1QPx_1^{-1}R$. This rule allows us to permute pieces of the symbol in certain circumstances.

Notice that we can apply this formula even if there are extra entries at the front because $NxPQx^{-1}R = xPQx^{-1}RN \rightarrow x_1QPx_1^{-1}RN = Nx_1QPx_1^{-1}R$. We'll surround P and Q with brackets to indicate how the rule is being applied.

Sometimes we have to cyclically permute the symbols; that will always be done by writing the original form of a symbol at the end of a line and its permuted form at the beginning of the next line.

$$\begin{aligned}
ab(cd)(a^{-1})b^{-1}c^{-1}d^{-1} &\rightarrow ab_1(a^{-1})(cd)b_1^{-1}c^{-1}d^{-1} \\
b_1a^{-1}(cd)(b_1^{-1}c^{-1}d^{-1})a &\rightarrow b_1a_1^{-1}(b_1^{-1}c^{-1}d^{-1})(cd)a_1 \\
& a_1b_1a_1^{-1}b_1^{-1}c^{-1}d^{-1}cd
\end{aligned}$$

This is the canonical form of $T^2\#T^2$.

Extra Problem 3 Again we find that all vertices correspond to the same point. We will apply the rule $xxPQ \rightarrow x_1P^{-1}x_1Q$.

This rule can be applied even if there are extra entries at the front because

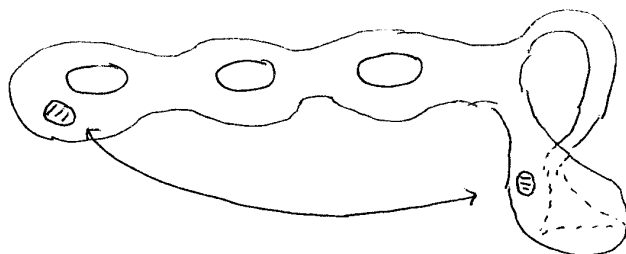
$$RxxPQ = xxPQR = x_1P^{-1}x_1QR = Rx_1P^{-1}x_1Q$$

To indicate the rule, we surround P in parentheses. As usual, cyclic permutation is indicated by writing the original form at the end of a line and the permuted form at the beginning of the next line.

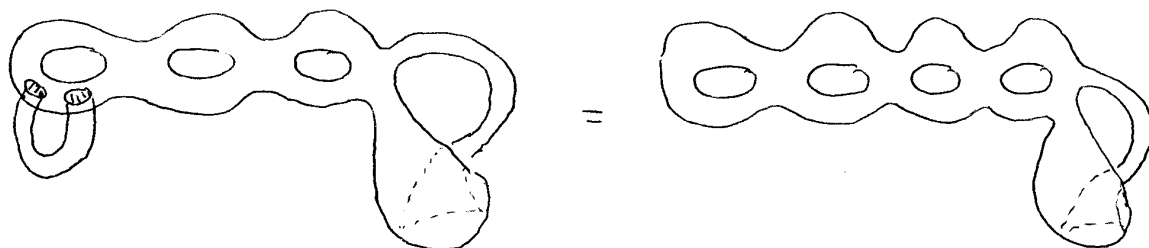
$$\begin{aligned}
a(bc)adb^{-1}d^{-1}c^{-1} &\rightarrow a_1a_1(bc)^{-1}db^{-1}d^{-1}c^{-1} = a_1a_1c^{-1}b^{-1}db^{-1}d^{-1}c^{-1} \\
a_1a_1c^{-1}(b^{-1}db^{-1}d^{-1})c^{-1} &\rightarrow a_1a_1c_1^{-1}c_1^{-1}(b^{-1}db^{-1}d^{-1})^{-1} = a_1a_1c_1^{-1}c_1^{-1}dbd^{-1}b \\
& a_1a_1c_1^{-1}c_1^{-1}db(d^{-1})b \rightarrow a_1a_1c_1^{-1}c_1^{-1}db_1b_1d \\
& a_1a_1c_1^{-1}c_1^{-1}d(b_1b_1)d \rightarrow a_1a_1c_1^{-1}c_1^{-1}d_1d_1b_1^{-1}b_1^{-1}
\end{aligned}$$

This final symbol is the symbol of $RP^2\#RP^2\#RP^2\#RP^2 = T^2\#RP^2\#RP^2 = T^2\#K$.

Extra Problem 4 This picture shows $T^2 \# T^2 \# T^2 \# K$ except that there is then an additional connected sum between the first T^2 and the last K . Actually if we form $M = T^2 \# T^2 \# T^2 \# K$ first, this extra connected sum is a connected sum from M to itself. It is formed by removing two disks, and then gluing their boundaries together. The book asserts (without proof) that such connected sums do not depend on the location of the disks being removed, so we can suppose that the disks are close to each other. But then the picture below shows that this last connected sum is like adding an extra handle, i.e., an extra T^2 . So the final object is $T^2 \# T^2 \# T^2 \# T^2 \# K$.



Slide the right disk left until it is next to the original disk



Extra Problem 5 The object on the left is $T^2 \# T^2 \# T^2 \# T^2 \# T^2$ with an extra connected sum from the first T^2 to the last one. As in the previous problem, this is like adding an extra handle, and thus gives six copies of the torus: $T^2 \# T^2 \# T^2 \# T^2 \# T^2 \# T^2$.

The object on the right is a central T^2 connected summed with four more copies. It doesn't matter where each connected sum takes place, so we could string the tori all in a line if we wanted. At any rate, there are five tori: $T^2 \# T^2 \# T^2 \# T^2 \# T^2$.

In particular, these two objects are not homeomorphic.