Assignment 8; Due Friday, November 18

10.7b The first part of this proof is exactly the first part of the proof in the book. The line L_x divides A into two pieces of equal area and divides B into pieces of area $b_1(x)$ and $b_2(x)$, where b_1 is the area of the piece furthest from x and b_2 is the area of the piece closest to x. Consider the continuous function $f(x) = b_1(x) - b_2(x)$. Notice that $f(-x) = -f(x)$ because the piece furthest from x is the piece closest to $-x$ and the piece closest to x is the piece furthest from $-x$. So as we move around the circle from x to $-x$ the function f changes sign, and thus it must be zero at an intermediate point. At this point, the line which divides A into two equal pieces also divides B into two equal pieces.

11.2 a Let $\mathcal{W} \subseteq M$ be open and suppose M is an n-manifold. Let $p \in \mathcal{W}$. Since M is a manifold, there is an open set $p \in \mathcal{U} \subseteq M$ and an open $\mathcal{V} \subseteq R^n$ and a homeomorphism $\varphi : \mathcal{U} \to \mathcal{V}$. Then $\varphi|_{\mathcal{U} \cap \mathcal{W}} : \mathcal{U} \cap \mathcal{W} \to \mathcal{V} \cap \mathcal{W}$ is a homeomorphism, and thus each point in $\mathcal{W}$ has an open neighborhood in $\mathcal{W}$ homeomorphic to an open set in R^n.

11.2 b Suppose $(z_1, z_2, \ldots, z_{n+1})$ represents a point in CP^n. Then

$$|z_1|^2 + |z_2|^2 + \ldots + |z_{n+1}|^2 = 1$$

and therefore some $z_i \neq 0$. Suppose, for example, that $z_{n+1} \neq 0$. In that case, z_{n+1} can be written in "polar coordinates" as $e^{i\theta} r$ where r is real and positive. Multiplying all the coordinates by $e^{-i\theta}$ gives another representative of the same point; the new representative has the form $(w_1, w_2, \ldots, w_n, r)$ for complex numbers $w_1, w_2, \ldots, w_n$. This new representative still lives in S^{2n+1}, so

$$|w_1|^2 + |w_2|^2 + \ldots + |w_n|^2 + r^2 = 1$$

and therefore

$$r = \sqrt{1 - |w_1|^2 - |w_2|^2 - \ldots - |w_n|^2}$$

The conclusion is that points in CP^n with $z_{n+1} \neq 0$ have unique representatives of the form

$$\left(w_1, w_2, \ldots, w_n, \sqrt{1 - |w_1|^2 - |w_2|^2 - \ldots - |w_n|^2}\right)$$

where $|w_1|^2 + |w_2|^2 + \ldots + |w_n|^2 < 1$ and thus

$$(w_1, w_2, \ldots, w_n) \in D^{2n}$$

where D^{2n} is the open disk of radius one.

Now I'd like to show formally that these provide local coordinates, i.e., that each point in CP^n has an open neighborhood homeomorphic to an open set in R^{2n}. I'll assume that my point has $z_{n+1} \neq 0$; analogous arguments work if some other $z_i \neq 0$.

Let $\mathcal{U}$ be the set of all points in CP^n whose representatives satisfy $z_{n+1} \neq 0$ and let $\mathcal{V}$ be the open unit disk D^{2n}. Map $\mathcal{V} \rightarrow \mathcal{U}$ by

$$(w_1, w_2, \ldots, w_n) \rightarrow \left(w_1, w_2, \ldots, w_n, \sqrt{1 - |w_1|^2 - |w_2|^2 - \ldots - |w_n|^2}\right)$$

Map $\mathcal{U} \rightarrow \mathcal{V}$ by

$$(z_1, z_2, \ldots, z_n, z_{n+1}) \rightarrow \left(\frac{z_1}{z_{n+1}}|z_{n+1}|, \frac{z_2}{z_{n+1}}|z_{n+1}|, \ldots, \frac{z_n}{z_{n+1}}|z_{n+1}|\right)$$

These maps are inverse to each other, so each map is one-to-one and onto. We are done if $\mathcal{U}$ is open and if both of these maps are continous. But $\mathcal{U}$ is open because its inverse image in S^{2n+1} is open, being $\{(z_1, z_2, \ldots, z_{n+1}) \in S^{2n+1} \mid z_{n+1} \neq 0\}$. The first map is a map $D^{2n} \rightarrow CP^n$ induced by a continuous map $D^{2n} \rightarrow S^{2n+1}$ and so continuous. The bottom map is a map $\mathcal{U} \subseteq CP^n \rightarrow D^{2n}$ induced from a continuous map from a subset of S^{2n+1} to D^{2n} and thus continuous.

To complete the argument, we need only show that CP^n is Hausdorff. Since S^{2n+1} is compact Hausdorff, we can use theorem 8.11, so it suffices to show that $\pi : S^{2n+1} \rightarrow CP^n$ is a closed map. Thus we want to show that if $A \subseteq S^{2n+1}$ is closed, then $\pi^{-1}\pi(A)$ is closed. This set is the set of all points in S^{2n+1} which are equivalent to points in A. Equivalently, it is the image of $S^1 \times A$ under the map

$$S^1 \times A \rightarrow S^1 \times S^{2n+1} \rightarrow S^{2n+1}$$

where this last map is

$$\lambda \times (z_1, z_2, \ldots, z_{n+1}) \rightarrow (\lambda z_1, \lambda z_2, \ldots, \lambda z_{n+1})$$

But $A \subseteq S^{2n+1}$ is closed, so compact. Thus $S^1 \times A$ is compact, and so its image in S^{2n+1} is compact, and so closed.

11.2 f Each point in M has an open neighborhood $\mathcal{U}$ homeomorphic to an open $\mathcal{V} \subseteq R^n$. Shrinking $\mathcal{V}$ if necessary, we can suppose that $\mathcal{V}$ is a disk. Magnifying, we can suppose that $\mathcal{V}$ is the open disk of radius 1, and thus that $\mathcal{U}$ is homeomorphic to such a disk.

M is covered by the union of these $\mathcal{U}$ and so by finitely many of them, $\mathcal{U}_1, \ldots \mathcal{U}_k$. By 8.14j, $X/(X - \mathcal{U}_i)$ is homeomorphic to U_i^∞, and thus homeomorphic to $(\mathring{D})^\infty$, which is homeomorphic to S^n.

The map $M \rightarrow M/(M - \mathcal{U}_i)$ is continuous, so $M \rightarrow M/(M - \mathcal{U}_i) \rightarrow S^n$ is continuous. Putting these maps together for all i gives a map

$$M \rightarrow M/(M - \mathcal{U}_1) \times \ldots \times M/(M - \mathcal{U}_k) \rightarrow S^n \times \ldots \times S^n \subseteq R^{n+1} \times \ldots \times R^{n+1} = R^{k(n+1)}.$$

We now claim this map is one-to-one. If so, we are done, because M is compact Hausdorff, so a continuous one-to-one map onto its image is automatically a homeomorphism.

Suppose x and $y \in M$ and $x \neq y$. If $x \in \mathcal{U}_i$ and $y \in \mathcal{U}_i$, then $x \notin M - \mathcal{U}_i$ and $y \notin M - \mathcal{U}_i$, so x and y represent different elements in $M/(M - \mathcal{U}_i)$ and thus map to different points in $R^{k(n+1)}$. If $x \in \mathcal{U}_i$ and $y \notin \mathcal{U}_i$, then $x \notin X - \mathcal{U}_i$ and $y \in M - \mathcal{U}_i$ and so x and y represent different points in $M/(M - \mathcal{U}_i)$. But x is certainly in *some* $\mathcal{U}_i$.

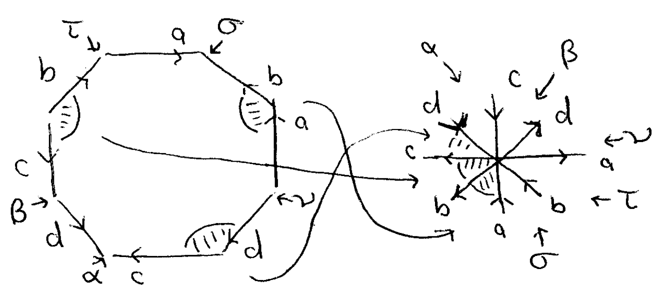
Extra Problem 1

Notice that the two top b arrows are glued together, so their ends become the same point. But these ends are the two ends of a. so the ends of b and both ends of a are the same point.

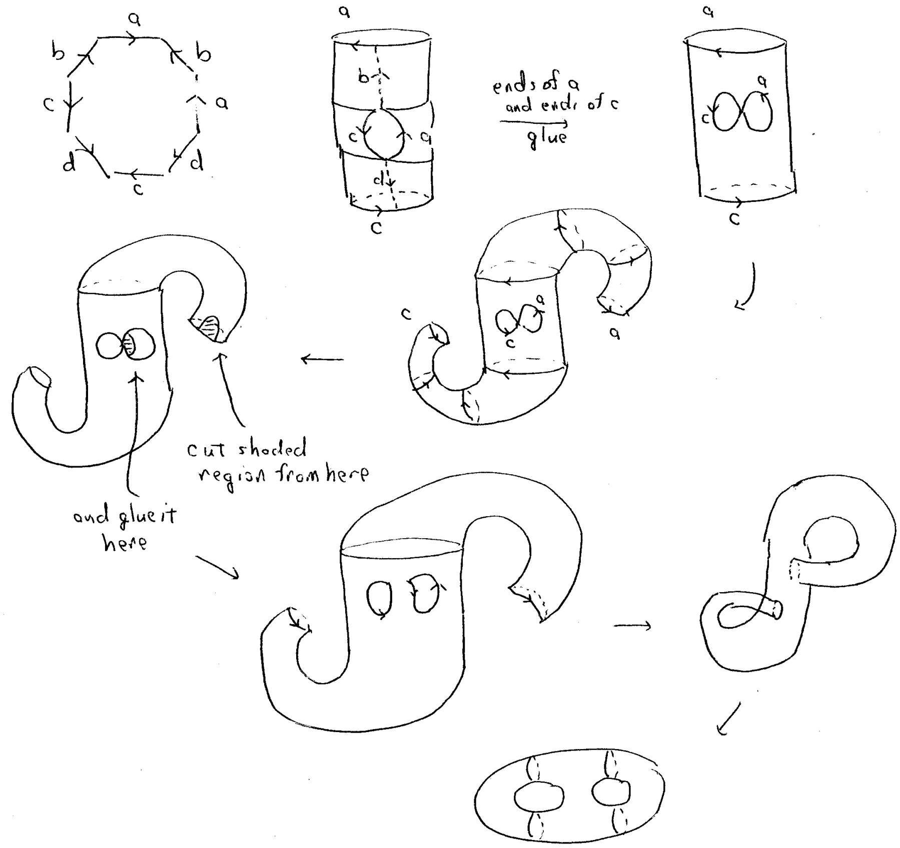
On the right side, we see that the end of a is the start of b, so both ends of b and both ends of a all glue to the same point.

The same reasoning applied to the bottom of the diagram shows that both ends of d and both ends of c glue to the same point.

But the a on the right goes from one end of d to an end of b, so all ends of a and b are glued to all ends of c and d.

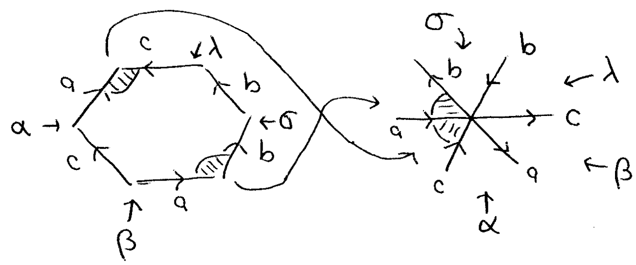


Extra Problem 2



Extra Problem 3

Reading from bottom to top counterclockwise, the end of a is glued to the start of b, which is glued to the end of b, which is glued to the start of c, Moreover, from the top left, the end of c is glued to the end of a. So the start and end of c are glued to the start and end of b which are glued to the end of a. But the start of c is glued to the start of a. So all vertices are glued together.



Extra Problem 4