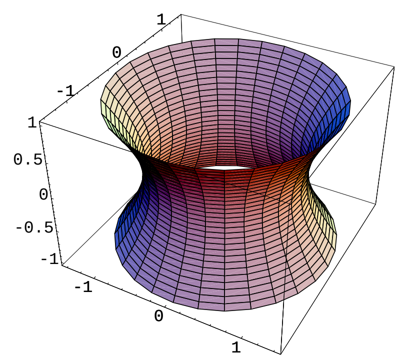
Assignment 7; Due Friday, November 11

9.8 a The set Q is not connected because we can write it as a union of two nonempty disjoint open sets, for instance $\mathcal{U} = (-\infty, \sqrt{2})$ and $\mathcal{V} = (\sqrt{2}, \infty)$. The connected subsets are just points, for if a connected subset C contained a and b with $a < b$, then choose an irrational number ξ between a and b and notice that $C = ((-\infty, \xi) \cap A) \cup ((\xi, \infty) \cap A)$.

9.8 b The exercise uses a fancy definition of an interval because it doesn't want to list them: $(-\infty, \infty), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (a, b), (a, b], [a, b)$, and $[a, b]$. These sets are connected by 9.6. Conversely, suppose $C \subseteq R$ is connected and let $c \in C$. Let $s = \sup\{b \geq c \mid [c, b] \subseteq C\}$. Note that s could equal infinity. If s is finite and $s \notin C$, then C contains $[c, s)$ but nothing larger, for otherwise $C = ((-\infty, s) \cap C) \cup ((s, \infty) \cap C)$ would decompose C into two nonempty disjoint open sets. If $s \in C$, then C contains $[c, s]$ but nothing larger, for if $d > s$ is in C, then since $[c, d]$ is not in C there must be a ξ with $s < \xi < d$ and $\xi \notin C$, and then $((-\infty, \xi) \cap C) \cup ((\xi, \infty) \cap C)$ decomposes C.

A similar argument at the lower end of C completes the proof.

9.8 d The sets $\{ x; ||x|| < 1 \}$ and $\{ x; ||x|| > 1\}$ are connected, but $\{ x; ||x|| \neq 1 \}$ is not because it is a union of the two disjoint open sets listed first. Also $\{ x; x_1^2 + x_2^2 - x_3^2 = 1 \}$ is connected because it looks like this:



and $\{ x; x_1^2 + x_2^2 + x_3^2 = -1 \}$ is the empty set and thus connected, and $\{ x; x_1 \neq 1 \}$ is not connected because it is the union of two open sets, one on one side of the plane $x_1 = 1$ and one on the other side.

9.8 e We will prove that X is not connected if and only if there is a continuous nonconstant $f : X \to Y$ whenever Y is discrete with at least two points.

Suppose $f : X \to Y$ is not constant,. Let $y \in Y$ be one of the values of Y and notice that $\{y\}$ and $Y - \{y\}$ are open sets. Since f is continuous, $f^{-1}(y)$ and $f^{-1}(Y - \{y\})$ are disjoint nonempty open sets whose union is X, so X is not connected.

Suppose X is not connected and write $X = \mathcal{U} \cup \mathcal{V}$ for disjoint nonempty open $\mathcal{U}$ and $\mathcal{V}$. If Y has at least two points, and thus has $p \neq q$, define $f : X \to Y$ by sending $\mathcal{U}$ to p and $\mathcal{V}$ to q. This f is not constant, and it is continuous because the inverse image of any set is either $\emptyset, \mathcal{U}, \mathcal{V}$ or X.

9.8 f If Y is not connected, we can find write $Y = \mathcal{U}_1 \cup \mathcal{U}_2$ where the $\mathcal{U}_i$ are disjoint, nonempty open subsets of Y. According to the induced topology, there are open sets $\mathcal{V}_1$ and $\mathcal{V}_2$ in X such that $Y \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$ and $Y \cap \mathcal{V}_1$ and $Y \cap \mathcal{V}_2$ are disjoint and nonempty. Notice that $A \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$ and $A \cap \mathcal{V}_1$ and $A \cap \mathcal{V}_2$ are disjoint. Since A is connected, one of $A \cap \mathcal{V}_1$ and $A \cap \mathcal{V}_2$ must be empty. Say for instance that $A \cap \mathcal{V}_1$ is empty. Since $Y \cap \mathcal{V}_1$ is not empty, let $y \in Y \cap \mathcal{V}_1$. But then $\mathcal{V}_1$ is an open neighborhood of y which does not intersect A, so by a theorem much earlier in the course, $y \notin \overline{A}$, contradicting the assumtpion $Y \subseteq \overline{A}$.

9.8 h Fix i. First we prove that $\{x \in R^{n+1} - \{0\} \mid x_i > 0\}$ is connected. Indeed fix a point p in this set. Whenever q is in the set, the straight line joining p and q is in the set. Parameterizing this line, we can find a continuous map

$$f : [0, 1] \to \{x \in R^{n+1} - \{0\} \mid x_i > 0\}$$

such that $f(0) = p$ and $f(1) = q$. Call the image of this line Y_q and notice that Y_q is connected by 9.4. The intersection of all Y_q contains p and thus is nonempty, so the union of these Y_q is connected. Clearly this union is all of $\{x \in R^{n+1} - \{0\} \mid x_i > 0\}$.

Notice that $(1, 1, \ldots, 1)$ is in the set defined by $x_i > 0$ for each i. Since these sets are all connected and have nonempty intersection, their union is connected. So the following set is connected:

$$\{x \in R^{n+1} - \{0\} \mid x_i > 0 \text{ for some } i\}$$

Similarly the following set is connected:

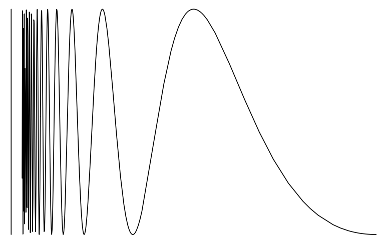
$$\{x \in R^{n+1} - \{0\} \mid x_i < 0 \text{ for some } i\}$$

The intersection of the two displayed sets is nonempty, so their union is connected. But this union is all of $R^{n+1} - \{0\}$.

Let $f : R^{n+1} - \{0\} \to S^n$ be the map $f(x) = \frac{x}{||x||}$. This map is continuous. Since the image of a connected set under a continuous map is connected, S^n is connected.

Recall that RP^n is S^n with opposite points identified, and the quotient topology from $\pi : S^n \to RP^n$. Since π is continuous and S^n is connected, RP^n is connected.

9.8 i Below is a picture of this set. The set is a famous example in topology: it is a connected set which is not arcwise connected. That is, it is impossible to draw a continuous path connecting a point in the right side to a point on the vertical line, because such a path would have to go up and down infinitely often as it approached the left, which is impossible in a continuous manner.



Let A be the vertical line on the left and let B be the cosine curve on the right. Both A and B are connected because they are continuous images of connected sets under obvious continuous maps. Suppose that $A \cup B$ is not connected, and write it as a disjoint union of nonempty open sets $\mathcal{U}$ and $\mathcal{V}$. The $A = (A \cap \mathcal{U}) \ \cup \ (A \cap \mathcal{V})$, so one of these sets must be empty. The same thing is true for B. We conclude that $\mathcal{U}$ and $\mathcal{V}$ must be A and B.

However, A is not open in $A \cup B$ because any open ball about a point on the vertical A intersects B.

10.7 a If both have the same center, any cut through the center will do. Otherwise cut on the straight line joining their centers.

Extra Problem 1 This exercise follows immediately from theorem 9.6 in the text.

Extra Problem 2 Let C_p and C_q be the connected components of p and q. If these sets are not disjoint, then theorem 9.6 states that $C_p \cup C_q$ is connected. Since this connected set contains p, it must equal C_p. Similarly it equals C_q.

Extra Problem 3 There are two connected components of the set of points in the plane with $y \neq 0$: the open upper half plane and the open lower half plane.

The connected components of Q are the $\{q\}$ for $q \in Q$.

Extra Problem 4 Let C_q be the connected component of q. By exercise 9.8f, $\overline{C_q}$ is connected; since C_q is the largest connected set containing q, $\overline{C_q} = C_q$. So connected components are closed.

Note that the connected components of Q are not closed in Q.

Extra Problem 5 By bilinearity we have $||v + w||^2 =< v + w, v + w > =< v, v > + < v, w > + < w, v > + < w, w > = ||v||^2 + 2 < v, w > + ||w||^2$, and the formula in the problem immediately follows. So $||Av|| = ||v||$ for all v implies $< Av, Aw > = < v, w >$ for all v and w. Conversely, if this result holds, then $< Av, Av > = < v, v >$, i.e., $||Av||^2 = ||v||^2$.

Write $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. Then

$$< Av, Aw > = \sum_i \left(\sum_j A_{ij} v_j \right) \left(\sum_k A_{ik} w_k \right) = \sum_{jk} (A^T A)_{kj} w_k v_j$$

and this expression equals $\sum_k w_k v_k$ for all v and w. Setting $w_k = \delta_{ks}$ and $v_k = \delta_{kt}$ we conclude that $(A^T A)_{st} = \delta_{st}$ and consequently $A^T A = I$. Conversely, if $A^T A = I$, this equation is clearly true for all v and w.

We will prove that $O(n)$ is compact by showing that it is closed and bounded. Clearly it is closed, for if $A_n \to A$ and $A_n^T A_n = I$ for all n, then $A_n^T A_n \to A^T A$ and so $A^T A = I$.

Notice that

$$1 = (A^T A)_{ii} = \sum_k A_{ik}^T A_{ki} = \sum_k A_{ki}^2$$

and so $|A_{ki}| \le 1$. Thus $O(n)$ is bounded.

Finally we compute the connected components of A. Notice that $1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2$, so $\det(A) = \pm 1$. I claim there are two components, the set $\mathcal{U}$ of elements with determinant 1 and the set $\mathcal{V}$ of elements with determinant -1. These sets are certainly disjoint, and they are open because $\det : O(n) \to R$ is continuous.

The matrix

$$B = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

is in $O(n)$ and has determinant minus one. We have a map $\mathcal{U} \to \mathcal{V}$ by sending A to AB. This map is clearly a homeomorphism. So to finish the argument we need only show that $\mathcal{U}$ is connected.

We prove this by showing that every element $A \in \mathcal{U}$ can be connected to the identity matrix by a continuous path in $\mathcal{U}$; this is enough by theorems 9.4 and 9.6 in the book. Write $A = BCB^{-1}$ as in the problem set where C has 1×1 and 2×2 blocks down the diagonal. The number of 1×1 blocks with a -1 must be even because A has determinant one and so C has determinant 1. Replace each pair of -1 blocks by the following 2×2 matrix with $\theta = \pi$:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In the end, our $A = BCB^{-1}$ where C has 2×2 blocks down the diagonal corresponding to rotations by $\theta_1, \ldots, \theta_k$, and then 1×1 blocks down the diagonal, each containing 1.

Now a continuous path to the identity is easily constructed. Replace θ_i by $t\theta_i$ where $t \in [0, 1]$. When $t = 1$, we have $BCB^{-1} = A$ and when $t = 0$ we have $BIB^{-1} = I$.