Assignment 6; Due Friday, November 4

8.2a If X is finite, then every set is open and X has the discrete topology. Thus if $x \neq y$, the open sets $\mathcal{U} = \{x\}$ and $\mathcal{V} = \{y\}$ separate x and y.

Conversely, suppose X is Hausdorff and let $x \neq y$. Choose disjoint $\mathcal{U}$ and $\mathcal{V}$ with $x \in \mathcal{U}$ and $y \in \mathcal{V}$. Notice that $\mathcal{U} = X - A$ and $\mathcal{V} = X - B$ where A and B are finite. Then $\mathcal{U} \cap \mathcal{V} = X - (A \cup B) = \emptyset$ and so $X = A \cup B$ is also finite.

8.14a Since $f : X \to Y$ is continuous, $\mathcal{U} \subseteq Y$ open implies $f^{-1}(\mathcal{U})$ open. Conversely, suppose $f^{-1}(\mathcal{U})$ open. Then $X - f^{-1}(\mathcal{U})$ is closed in the compact set X, so compact. By one of our theorems, $f(X - f^{-1}(\mathcal{U})) \subseteq Y$ is compact in the Hausdorff space Y, so closed. But $f(X - f^{-1}(\mathcal{U})) = Y - \mathcal{U}$ since f is onto. So $Y - \mathcal{U}$ is closed and $\mathcal{U}$ is open.

8.14b Suppose Y is Hausdorff. Whenever $x \times y \notin D$ we have $x \neq y$ and we can find open $x \in \mathcal{U}$ and $y \in \mathcal{V}$ with $\mathcal{U} \cap \mathcal{V} = \emptyset$. It follows that $x \times y \in \mathcal{U} \times \mathcal{V} \subseteq Y \times Y$ and $(\mathcal{U} \times \mathcal{V}) \cap D = \emptyset$. The union of all such $\mathcal{U} \times \mathcal{V}$ is an open set which is exactly $Y \times Y - D$, so D is closed.

Conversely, suppose D is closed. Then $\mathcal{W} = Y \times Y - D$ is open. If $x \neq y$, then $x \times y \in \mathcal{W}$, so by definition of open sets in the product topology there is a rectangle $\mathcal{U} \times \mathcal{V}$ with $x \times y \in \mathcal{U} \times \mathcal{V} \subseteq Y \times Y - D$. Since $\mathcal{U} \times \mathcal{V}$ does not intersect D, $\mathcal{U} \cap \mathcal{V} = \emptyset$.

8.14c Consider the map $f \times f : X \times X \to Y \times Y$. Since Y is Hausdorff, the diagonal $D \subseteq Y \times Y$ is closed, so $(f \times f)^{-1}(D)$ is closed. But this is exactly the set of all $x \times y \in X \times X$ such that $f(x) \times f(y) \in D$, that is $f(x) = f(y)$.

8.14e Suppose X is compact, Y is Hausdorff, and $f : X \to Y$ is continuous and onto. If $A \subseteq X$ is closed, then A is compact, so $f(A) \subseteq Y$ is compact in a Hausdorff space and so closed.

Conversely, suppose f takes closed sets to closed sets. Apply theorem 8.11: If Y is the quotient space of the compact Hausdorff space X determined by an onto map $f : X \to Y$, and if f is a closed mapping, then Y is Hausdorff.

Continuation of 8.14e Now we must prove that under the same hypotheses, Y is Hausdorff if and only if $E = \{ x_1 \times x_2 \mid f(x_1) = f(x_2)\}$ is closed. Half of this was proved in 8.14c. We must still prove that if this set is closed, then Y is Hausdorff. This will follow from the first part of the problem if we can prove that E closed implies that f is a closed mapping.

So suppose $A \subseteq X$ is closed. We must prove that $f(A)$ is closed; since Y has the quotient topology, we must prove that $f^{-1}(f(A))$ is closed, and thus that

$$\{x \in X \mid \exists a \in A \text{ with } f(x) = f(a)\}$$

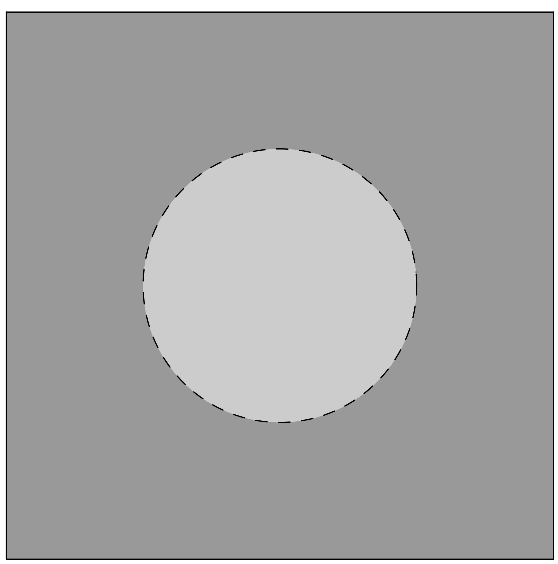
is closed. But A closed in X implies that $X \times A$ is closed in $X \times X$. We are assuming E is closed, so $(X \times A) \cap E$ is closed. This set is $\{x \times a \mid f(x) = f(a)\}$. Since the natural projection $\pi : X \times X \to X$ is continuous and since X is compact Hausdorff, π maps closed sets to closed sets, so $\pi(A \cap E)$ is closed. But this set is exactly $\{x \in X \mid \exists a \in A \text{ with } f(x) = f(a)\}$.

8.14i Notice that $x \sim x$ since $x - x = 0 \in Q$. If $x \sim y$, then $x - y \in Q$ and so $-(x - y) = y - x \in Q$, so $y \sim x$. Finally if $x \sim y$ and $y \sim z$, then $x - y \in Q$ and $y - z \in Q$, so $(x - y) + (y - z) = x - z \in Q$, so $x \sim z$.

Let $\pi : R \to R/\sim$ and give $R/\sim$ the quotient topology. Call the resulting space Y. Notice that Y is uncountable, since each equivalence class contains only countably many elements. In particular, Y has two distinct points. I will prove that Y has the concrete topology: the only open sets are $\emptyset$ and Y. If so, it will follow that Y is not Hausdorff because there exist distinct elements $y_1 \neq y_2$ and yet the only possible open neighborhoods of y_1 and y_2 are Y and Y and these are not disjoint.

Let $\mathcal{U} \subseteq Y$ be nonempty and open. We will prove $\mathcal{U} = Y$. Notice first that $\pi^{-1}(\mathcal{U})$ is nonempty and open; call this set $\mathcal{V}$. The set $\mathcal{V}$ contains a nontrivial interval (a, b). I claim that every real number is equivalent to an element in (a, b). Indeed, if $r \in R$ then the interval $(a - r, b - r)$ contains a rational q, so $a - r < q < b - r$ and then $a < r + q < b$ and $r + q \sim r$. Any $y \in Y$ is represented by some real number, which we can assume is in (a, b) and thus $y \in \pi(\pi^{-1}(\mathcal{U})) = \mathcal{U}$. So $\mathcal{U} = Y$.

8.14j To understand this problem, consider first the case when X is a large closed rectangle about the origin and $\mathcal{U}$ is a smaller open disk inside this rectangle.



The space $X/(X-\mathcal{U})$ is formed by gluing all points in $X-\mathcal{U}$ together. This means that all of the points which are darker gray become a single point. In particular, all the points on the boundary of $\mathcal{U}$ become a single point, so the light gray disk becomes a sphere. Notice that $\mathcal{U}^\infty$ is also a sphere, since $\mathcal{U}$ is homeomorphic to R^n and thus $\mathcal{U}^\infty$ is homeomorphic to $(R^n)^\infty$, which is a sphere.

Now we give the general argument. As a set, $\mathcal{U}^\infty = \mathcal{U} \cup \{\infty\}$. Next we analyze $X/(X-\mathcal{U})$ as a set. Notice that $X = \mathcal{U} \cup (X - \mathcal{U})$. Each point in $\mathcal{U}$ represents a unique point in $X/(X-\mathcal{U})$ and all of the points in $X-\mathcal{U}$ represent the same point, p. Therefore, as a set $X/(X-\mathcal{U})$ is $\mathcal{U} \cup \{p\}$ where p is the point represented by all elements of $X-\mathcal{U}$. Map $\mathcal{U}^\infty$ to $X/(X-\mathcal{U})$ by sending points in $\mathcal{U}$ to themselves and sending ∞ to p. This map is clearly one-to-one and onto. We must show that this map induces a one-to-one correspondence between open sets $\mathcal{V} \subseteq \mathcal{U}^\infty$ and open sets $\mathcal{W} \subseteq X/(X-\mathcal{U})$.

Incidentally, the previous argument assumes that $X - \mathcal{U}$ is nonempty. Otherwise there would be no p and we would be in trouble.

There are two types of open sets in $\mathcal{U}^\infty$. First there are open sets $\mathcal{V} \subseteq \mathcal{U}$. Second there are open sets of the form $\mathcal{V} = (\mathcal{U} - A) \cup \{\infty\}$ where $A \subseteq \mathcal{U}$ is compact and closed.

There are also two types of open sets in $X/(X-\mathcal{U})$, namely open sets which do not contain the special point p and open sets which contain this point. Each point of a subset $\mathcal{W}$ which does not contain p is represented by a unique point in $\mathcal{U}$, so we can identify such subsets of $X/(X-\mathcal{U})$ with subsets $\mathcal{W} \subseteq \mathcal{U}$, and such a set is open in $X/(X-\mathcal{U})$ exactly when its inverse image in X is open, i.e., exactly when $\mathcal{W} \subseteq \mathcal{U}$ is open.

The open sets of $X/(X-\mathcal{U})$ which contain p have the form $\mathcal{W} = \mathcal{V} \cup \{p\}$ where $\mathcal{V} \subseteq \mathcal{U}$. This set is open exactly when its inverse image in X is open. The inverse image is $\mathcal{V} \cup (X-\mathcal{U}) = X - (\mathcal{U} \cap \mathcal{V}^c)$ and is open just in case $\mathcal{U} \cap \mathcal{V}^c$ is closed in X. Since X is compact Hausdorff, this happens just in case $\mathcal{U} \cap \mathcal{V}^c$ is compact. Call this set A and notice that $\mathcal{V} = \mathcal{U} - A$.

To summarize, the open sets in $X/(X-\mathcal{U})$ have the form $\mathcal{W} \subset \mathcal{U}$ where $\mathcal{W}$ is open, or $\mathcal{W} = (\mathcal{U} - A) \cup \{p\}$ where $A \subseteq \mathcal{U}$ is closed and compact.

It is immediately clear that our map sets up a one-to-one correspondence between open sets in $\mathcal{U}^\infty$ and open sets in $X/(X-\mathcal{U})$.

Continuation of 8.14j Consider the special case $\mathcal{U} = X - \{p\}$. Then $\mathcal{U}^\infty = (X - \{p\})^\infty$ is homeomorphic to $X/(X - (X - \{p\})) = X/\{p\}$. This last space is X with the point p glued to itself, i.e., just X.