Assignment 5; Due Friday, October 28

6.6a Imagine that $X \times Y$ has some unspecified topology.

Suppose $X \times Y \to X$ is continuous and let $\mathcal{U} \subseteq X$ be open. The inverse image of this set is $\mathcal{U} \times Y$; by continuity this set is open.

Similarly if $X \times Y \to Y$ is continuous and $\mathcal{V} \subseteq Y$ is open, then $X \times \mathcal{V}$ is open. If both maps are continuous, then $\Big(\mathcal{U} \times Y\Big) \cap \Big(X \times \mathcal{V}\Big) = \mathcal{U} \times \mathcal{V}$ is open.

Consequently, if $\mathcal{U}_\alpha$ and $\mathcal{V}_\alpha$ are open in X and Y for each α, then $\mathcal{U}_\alpha \times \mathcal{V}_\alpha$ is open for each α and so $\cup\Big(\mathcal{U}_\alpha \times \mathcal{V}_\alpha\Big)$ is open. It follows that each open set in the product topology for $X \times Y$ is also open in the unspecified topology, which is what we were supposed to prove.

6.6d Let $I = [0,1]$. Think of the torus as I^2 modulo the equivalence relation $(0,v) \sim (1,v)$ and $(u,0) \sim (u,1)$ as in the picture in the book.

Consider the map $f : I \to R^2$ by $u \to e^{2\pi i u} = (\cos 2\pi u, \sin 2\pi u)$. This is a standard calculus map which is certainly continuous. The image is in S^1, so by theorem 4.5d (or just elementary reasoning), $I \to S^1$ is continuous.

Next consider $g = f \times f : I \times I \to S^1 \times S^1$. This map is continuous by a result in class. Since $g(0,v) = g(1,v)$ and $g(u,0) = g(u,1)$, g descends to a continuous map $I^2/\sim \to S^1 \times S^1$, that is, a continuous map $T^2 \to S^1 \times S^1$. This map is one-to-one and onto by easy reasoning.

It now suffices to prove that the inverse of this map is continuous. This follows because T^2 is compact and $S^1 \times S^1$ is Hausdorff. Indeed I^2 is compact and $\pi : I^2 \to T^2$ is onto, so T^2 is compact by a theorem in class. Also $S^1 \subseteq R^2$ is Hausdorff since any metric space is Hausdorff; the product of two Hausdorff spaces is Hausdorff.

6.6l Let $X = Z$ with the discrete topology. Then $X \times X = Z \times Z$ with the discrete topology, for if $m, n \in Z$ then $\{m\}$ and $\{n\}$ are open, so $\{m \times n\}$ is open in $Z \times Z$, and every subset of $Z \times Z$ is a union of such points and thus open.

Since Z and $Z \times Z$ are both countable, there is a one-to-one and onto map $f : Z \to Z \times Z$. This map and its inverse are continuous automatically since both spaces are discrete.

As a more interesting example, let X be the set of all infinite sequences $x = (x_1, x_2, x_3, \ldots)$ of real numbers such that all but finitely many x_i are zero. Define the distance between two such sequences x and y to be the maximum of $|x_i - y_i|$. This is easily shown to be a metric.

The space $X \times X$ consists of pairs $(x_1, x_2, \ldots) \times (y_1, y_2, \ldots)$. Map $f : X \times X \to X$ by mapping this pair to the single sequence $(x_1, y_1, x_2, y_2, \ldots)$, noticing that only finitely many terms of the resulting sequence are nonzero. This map is clearly one-to-one and onto.

Define a distance in $X \times X$ by letting $d\Big((x, y), (\tilde{x}, \tilde{y})\Big) = \max\Big(d(x, \tilde{x}), d(y, \tilde{y})\Big)$. By exercise 6.2b, this metric induces the usual product topology on $X \times X$. Notice that the map f preserves distance: $d(x \times y, \tilde{x} \times \tilde{y}) = d\Big(f(x \times y), f(\tilde{x} \times \tilde{y})\Big)$. Consequently f and its inverse are continuous.

7.13a D_n is closed and bounded, so compact. The second set is not closed, so not compact.The third set, which is a 1×4 closed rectangle, is compact.

The final set is compact because it is closed and bounded. Indeed, the set of all (s, t, u) with $s^2 + t^2 \le 1$ is closed, since if (s_n, t_n, u_n) is in the set and $(s_n, t_n, u_n) \to (s, t, u)$, then $s_n^2 + t_n^2 \le 1$ and so by taking limits, $s^2 + t^2 \le 1$. The set of all (s, t, u) with $t^2 + u^2 \le 1$ is also closed. Therefore the intersection of these sets is closed. This intersection is bounded, since if (s, t, u) is in the set then $|s| \le 1, |t| \le 1$, and $|u| \le 1$.

7.13b Let X be a compact subset of R^n. Define $\mathcal{U}_n = \{\ x \in X \mid ||x|| < n\ \}$. These $\mathcal{U}_n$ form an open cover of X. By compactness, a finite number of these sets already cover X, so X is bounded.

7.13c Suppose $f : I \to R$ is continuous. Define $id : I \to R$ by $id(x) = x$; clearly this map is continuous. Consequently by theorem 6.5, the map $id \times f : I \to R \times R$ is continuous; notice that this map sends I to the graph of f, which is a subset of $R \times R$. The continuous image of a compact space is compact, so this graph is compact.

Conversely, suppose the graph of f is compact. We will prove that $f^{-1}(A)$ is closed whenever A is closed; this is enough to prove that f is continuous. The natural maps $\pi_1 : I \times R \to I$ and $\pi_2 : I \times R \to R$ by projection on the first and second factors respectively are continuous. So $\pi_2^{-1}(A)$ is closed. The graph G of f is a compact subset of $I \times R$, so $\pi^{-1}(A) \cap G$ is compact. A continuous image of a compact set is compact, so $\pi_1(\pi_2^{-1}(A)$ is compact in the Hausdorff space I, so closed. However, this set is just $f^{-1}(A)$, so $f^{-1}(A)$ is compact. I'll let the reader verify that $f^{-1}(A) = \pi_1(\pi_2^{-1}(A) \cap G)$; the argument definitely requires that G be the graph of f.

Let $f(0) = 0$ and $f(x) = \frac{1}{x}$ for $x \in (0, 1]$. The graph of f is

$$(0, 0) \cup \{(x, 1/x) \mid x \in (0, 1]\}$$

This set is closed, for suppose (x_n, y_n) is in the graph and converges to (x, y). If infinitely many $x_n = 0$, then $(x_n, y_n) = (0, 0)$ for infinitely many n, so $(x, y) = (0, 0)$, which is in the graph. Otherwise $x_n \ne 0$ for infinitely many n and for these n we have $x_n \to x$ and $\frac{1}{x_n} \to y$. Then $x_n \cdot \frac{1}{x_n} \to x \cdot y$ and so $x \cdot y = 1$. In particular $x \ne 0$ and $y = \frac{1}{x}$ and therefore (x, y) is in the graph.

7.13g Suppose our compact X has an open cover $\{\mathcal{U}_\alpha\}$. If the result is false, then for each positive integer n we can find points x_n and y_n with $d(x_n, y_n) < \frac{1}{n}$ such that no $\mathcal{U}_\alpha$ contains both points. We are going to use the following

Lemma If X is a compact metric space, then every sequence x_n has a convergent subsequence x_{n_α}.

Accepting this lemma, choose a convergent subsequence x_{n_α} of x_n. Throw away all terms not in this subsequence and reindex; we can thus assume that $x_n \to x_0$. But then $y_n \to x_0$, for the y_n get arbitrarily close to the x_n, which get arbitrarily close to x_0 (details easily supplied). Find α such that $x_0 \in \mathcal{U}_\alpha$. fSince $x_n \to x_0$ and $y_n \to x_0$, we have $x_n \in \mathcal{U}_\alpha$ and $y_n \in \mathcal{U}_\alpha$ for all large n, a contradiction.

It remains to prove the lemma. Fix $p \in X$. If every neighborhood of p contains x_n for infinitely many n, then we can easily find a subsequence converging to p. So if the lemma is false, then for each p we can find an open $\mathcal{U}_p$ which contains only finitely many x_n. These $\mathcal{U}_p$ form an open cover of X, so there is a finite subcover $\mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_n}$, but then the union of these sets is everything and yet only contains a finite number of the x_n.

Remark: In class, a second proof was sketched. Since $\{U_\alpha\}$ is an open cover of our compact X, there is a finite subcover $\mathcal{U}_{\alpha_1} \cup \ldots \cup \mathcal{U}_{\alpha_k}$. Let C_i be the complement of $\mathcal{U}_{\alpha_i}$. The C_i are closed sets and $\cap C_i = \emptyset$. Let $f_i : X \to R$ be the function which assigns to x the minimum distance from x to a point of C_i. This function is continuous, so $g = \max(f_1, \ldots, f_k)$ is continuous. This function is strictly positive, for if $x \in X$ then x is not in some C_i, so $f_i(x) > 0$. Let δ be the minimum value of g on X.

Suppose A is a subset of X of diameter smaller than δ. Fix $a \in A$. Since $g(a) \geq \delta$, there is a C_i so $d(a, c) \geq \delta$ whenever $c \in C_i$. If $b \in A$, then $d(a, b) + d(b, c) \geq d(a, c)$, so $d(b, c) \geq d(a, c) - d(a, b) > \delta - \delta = 0$ for all $c \in C_i$, so $b \notin C_i$. Hence A does not intersect C_i and thus $A \subseteq \mathcal{U}_{\alpha_i}$.

7.13h The open sets in $X \cup \{\infty\}$ are $\mathcal{U}$ open in X together with $(X - A) \cup \{\infty\}$ where A is closed and compact in X. The entire set is open since it has this second form for $A = \emptyset$. The empty set is open since it is open in X.

Suppose $\mathcal{W}_1$ and $\mathcal{W}_2$ are open and consider $\mathcal{W}_1 \cap \mathcal{W}_2$. If both $\mathcal{W}_1$ and $\mathcal{W}_2$ contain infinity, then their intersections with X have the form $X - A_1$ and $X - A_2$ where A_1 and A_2 are closed and compact. So $\mathcal{W}_1 \cap \mathcal{W}_2 = (X - (A_1 \cup A_2)) \cup \{\infty\}$. Notice that $A_1 \cup A_2$ is closed and compact.

If one of the $\mathcal{W}_i$, say $\mathcal{W}_1$, contains infinity and the other does not, then their intersection is $\mathcal{W}_1 \cap (X - A_2)$, which is open in X and so open in X^∞. If both $\mathcal{W}_i$ omit infinity, then their intersection is $W_1 \cap W_2$, which is open in X.

We also need to show that a union of open sets $\{\mathcal{W}_\alpha\}$ is open in X^∞. This is obvious if all of the $\mathcal{W}_\alpha$ are contained in X. If at least one of the $\mathcal{W}_\alpha$ contains infinity, then the union contains infinity and we must show that the intersection of the union with X has the form $X - A$ for A closed and compact. But the intersection of each $\mathcal{W}_\alpha$ with X has the form $X - A_\alpha$ where A_α is closed, and in at least one case A_α is compact. The union of these sets is $X - \cap A_\alpha$. Notice that $\cap A_\alpha$ is closed, and it is compact since it is a closed subset of the compact A_α.

The set $X \subset X^\infty$ has the induced topology, for if $\mathcal{U}$ is open in X, then $\mathcal{U}$ is also open in X^∞ and its intersection with X is $\mathcal{U}$. Conversely if $\mathcal{W}$ is open in X^∞, then either $\mathcal{W}$ is open in X and so its intersection with X is open, or else $\mathcal{W} = (X - A) \cup \{\infty\}$ and its intersection with X is $X - A$, which is open since A is closed.

The space X^∞ is compact, for if $\mathcal{W}_\alpha$ is an open cover of X, then one of these open sets contains ∞ and thus has the form $(X - A) \cup \{\infty\}$. This set already covers everything except A. But A is compact and the $\mathcal{W}_\alpha \cap X$ form an open cover of A, so we can find a finite subcover.

Extra Problem 1 Notice that $R = \cup(-n, n)$, but no finite subset of this collection of open sets covers R. Similarly $[0, 1) = \cup[0, 1 - 1/n)$, and these sets are open in the relative topology, but no finite subset of this collection of open sets covers $[0, 1)$.

Extra Problem 2 $Q = \cup\Big((-n, n) \cap Q\Big)$, but this cover does not have a finite subcover. The sphere is an infinite union of points in the sphere with z coordinate less than $1 - 1/n$, and each of these sets is open, but there is no finite subcover.

The Klein bottle is compact. Indeed the square $[0, 1]^2$ is compact by the Heine-Borel theorem, and the Klein bottle is a continuous image of this compact set, and so compact by theorem 7.8

Extra Problem 3 By theorem 7.10, a closed subset of a compact space is compact, so A and B are compact.

Fix $b \in B$. For each $a \in A$, find open neighborhoods $\mathcal{U}_a$ of a and $\mathcal{V}_a$ of b which do not intersect. The $\mathcal{U}_a$ cover A; find a finite subcover $\mathcal{U}_{a_1}, \dots, \mathcal{U}_{a_m}$. Let $\mathcal{U} = \mathcal{U}_{a_1} \cup \dots \cup \mathcal{U}_{a_m}$ and let $\mathcal{V} = \mathcal{V}_{a_1} \cap \dots \cap \mathcal{V}_{a_m}$. Notice that $A \subseteq \mathcal{U}$ and $b \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Now repeat this argument varying b. For each $b \in B$, find open sets $\mathcal{U}_b$ and $\mathcal{V}_b$ such that $A \subseteq \mathcal{U}_b$ and $b \in \mathcal{V}_b$ and $\mathcal{U}_b \cap \mathcal{V}_b = \emptyset$. The $\mathcal{V}_b$ form an open cover of B. Find a finite subcover $\mathcal{V}_{b_1}, \dots \mathcal{V}_{b_n}$. Let $\mathcal{U} = \mathcal{U}_{b_1} \cap \dots \cap \mathcal{U}_{b_n}$ and $\mathcal{V} = \mathcal{V}_{b_1} \cup \dots \cup \mathcal{V}_{b_n}$. Notice that $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Extra Problem 4 Compactness is a topological property, so if two spaces are homeomorphic, they must both be compact or both be noncompact. But the open ball *is not* compact and the closed ball *is* compact.