Assignment 4; Due Friday, October 21

5.3b Note that $A \subseteq Y$ is closed if and only if $A^c \subseteq Y$ is open. By definition of the quotient topology, this happens if and only if $f^{-1}(A^c)$ is open, which happens if and only if the complement of this set, which equals $f^{-1}(A)$, is closed.

5.3c Consider $f : R \to S^1$ where $S^1 \subseteq R^2$. A subset $\mathcal{U} \subseteq S^1$ is open in the quotient topology if $f^{-1}(\mathcal{U})$ is open; this subset $\mathcal{U}$ is open in the induced topology if it has the form $\mathcal{V} \cap S^1$ for $\mathcal{V} \subseteq R^2$ open. So we must prove that $f^{-1}(\mathcal{U})$ is open if and only if $\mathcal{U} = \mathcal{V} \cap R^2$.

Let $g : R \to R^2$ be the map $i \circ f : R \to S^1 \to R^2$ obtained by following f by the natural inclusion map $i : S^1 \to R^2$. Notice that $g(t) = (\cos 2\pi t, \sin 2\pi t)$, which is continuous. Hence if $\mathcal{V} \subseteq R^2$ is open, $g^{-1}(\mathcal{V}) \subseteq R$ is open. This is half of what we are asked to prove.

To go the other way, suppose $f^{-1}(\mathcal{U})$ is open in R. Any open set in R is a union of open intervals $(a, b$, so $\mathcal{U}$ is a union of sets of the form $f((a, b))$ and it suffices to show that each of these sets is open in the induced topology. If $b - a > 2\pi$, then $f((a, b))$ equals the entire S^1, which is the intersection of R^2 with S^1. Otherwise the image of $f((a, b))$ is an open arc of the circle, which is the intersection of S^1 with an open wedge in R^2. This completes the proof.

5.3f Consider the map $f : S^2 \to R^4$ given by $(x_1, x_2, x_3) \to (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$. This map is clearly continuous. Notice that $f(-x) = f(x)$ because changing the sign of each of the x_i does not change the value of f. It follows that f induces a map $\tilde{f} : S^2/\sim \to R^4$ where $\sim$ is the equivalence relation $p \sim q$ if $p = \pm q$. The space $S^2/\sim$ is equal to projective space RP^2 by definition of projective space. The map $\tilde{f}$ is automatically continuous by definition of the quotient topology, and it suffices to show that this map is one-to-one.

Consider the map from the plane to the plane given by $(x_1, x_2) \to (x_1^2 - x_2^2, 2x_1x_2)$. Since $(x_1 + ix_2)^2 = x_1^2 - x_2^2 + 2ix_1x_2$, this map is just $z \to z^2$ in complex notation. But $z^2 = w^2$ if and only if $z^2 - w^2 = (z - w)(z + w) = 0$ if and only if $z = \pm w$.

Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) in S^2 map to the same point in R^4. By the previous paragraph, $(x_1, x_2) = \pm(y_1, y_2)$. If $x_1 = y_1 \neq 0$, then the equation $x_1x_3 = y_1y_3$ implies $x_3 = y_3$. If $x_1 = -y_1 \neq 0$, this same argument implies that $x_3 = -y_3$. Similar arguments work if $x_2 = y_2 \neq 0$ and if $x_2 = -y_2 \neq 0$ because $x_2x_3 = y_2y_3$. If $x_1 = x_2 = y_1 = y_2 = 0$, then $x_3 = \pm 1$ because $(x_1, x_2, x_3) \in S^2$, and $y_3 = \pm 1$ for similar reasons, and thus $x_3 = \pm y_3$.

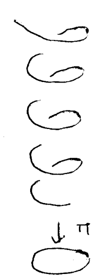
5.3h Consider the diagram below:

$$\begin{tikzcd} A \arrow[r, hook, "i_A"] \arrow[d, "\pi|_A"'] & X \arrow[d, "\pi"] \\ B \arrow[r, hook, "i_B"] & Y \end{tikzcd}$$

Suppose we give these four sets topologies as follows: give A the induced topology from $A \subseteq X$; give Y the quotient topology from $\pi : X \to Y$ and give B the induced topology from $B \subseteq Y$. The question is now: how is this topology on B related to the quotient topology from $\pi_A : A \to B$?

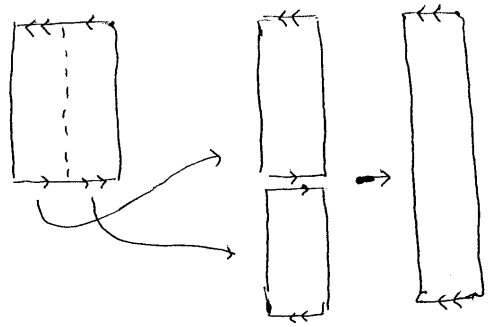
Suppose $\mathcal{U}$ is open in B. By definition, $\mathcal{U} = \mathcal{V} \cap B$ where $\mathcal{V}$ is open in Y. By definition of the quotient topology on Y, $\pi^{-1}(\mathcal{V})$ is open in X. Since A has the induced topology, $\pi^{-1}(\mathcal{V}) \cap A$ is open in A. Clearly this set equals $\pi_A^{-1}(\mathcal{U})$. It follows that open sets in the topology we have given B are also open in the quotient topology.

Now we show by example that the converse need not be true. Look at the special case when $X = R$ and $Y = S^1$ and $\pi : X \to Y$ is the map $t \to (\cos 2\pi t, \sin 2\pi t)$. Let $A \subseteq X$ be the set $(0, \frac{1}{2}] \cup (\frac{3}{2}, 1]$. Remember that this set has the induced topology from $A \subseteq R$. Notice that the image of this set in S^1 is all of S^1 since every point in the circle is represented by an angle "half way around" or an angle "the other half way around."

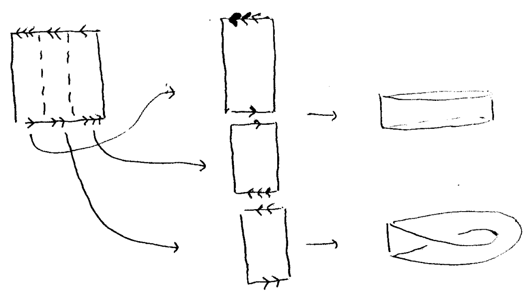


Now consider the set of points in the circle with angles $0 < \theta \leq \pi$. This set is not open in the induced topology $B \subseteq Y$ since $B = Y = S^1$. On the other hand, the inverse image in A is $(0, \frac{1}{2}]$ and this *is open* in the relative topology.

5.4b If you cut a Mobius band in the middle, you'll get one single band twice as long. This band has a double twist in it, but the band is homeomorphic to a single cylinder, as the picture below shows.



If you cut one third of the way across, you'll get two pieces, a cylinder and a Mobius band (up to homeomorphism). See the picture below.



6.2a By assumption there is a one-to-one and onto map $f : X_1 \to X_2$ such that when $\mathcal{U}$ is open in X_1 then $f(\mathcal{U})$ is open in X_2, and when $\mathcal{U}$ is open in X_2 then $f^{-1}(\mathcal{U})$ is open in X_1. There is a similar map $g : Y_1 \to Y_2$. Define $h = f \times g : X_1 \times Y_1 \to X_2 \times Y_2$. If $\mathcal{W}$ is open in $X_1 \times Y_1$, then $\mathcal{W} = \cup_\alpha (\mathcal{U}_\alpha \times \mathcal{V}_\alpha)$ where $\mathcal{U}_\alpha$ is open in X_1 and $\mathcal{V}_\alpha$ is open in Y_1. Then $(f \times g)(\mathcal{W}) = \cup (f(\mathcal{U}_\alpha) \times g(\mathcal{V}_\alpha))$ is open in $X_2 \times Y_2$. Similarly if $\mathcal{W}$ is open in $X_2 \times Y_2$, then $\mathcal{W} = \cup_\alpha (\mathcal{U}_\alpha \times \mathcal{V}_\alpha)$ where $\mathcal{U}_\alpha$ is open in X_2 and $\mathcal{V}_\alpha$ is open in Y_2 and then $(f \times g)^{-1}(\mathcal{W}) = (f \times g)^{-1}(\cup(\mathcal{U}_\alpha \times \mathcal{V}_\alpha)) = \cup((f^{-1}(\mathcal{U}_\alpha) \times g^{-1}(\mathcal{V}_\alpha))$ is open in $X_1 \times Y_1$.

6.2b I'll skip the straightforward check that d is a metric. Notice that an open ball of radius ϵ about $x \times y$ with respect to the d metric has the form $B_\epsilon(x) \times B_\epsilon(y)$, where we use the metric d_X to compute the first ball and d_Y to compute the second ball. This set has the form $\mathcal{U} \times \mathcal{V}$ for open $\mathcal{U} \subseteq X$ and $\mathcal{V} \subseteq Y$. Every open set in the *metric space* $X \times Y$ with respect to the metric d is a union of such epsilon balls, and thus a union of rectangles.

So open sets in $X \times Y$ with respect to the metric d are also open sets in $X \times Y$ with the product topology.

Conversely let $\mathcal{W}$ be open in $X \times Y$ with respect to the product topology. We claim that $\mathcal{W}$ is open with respect to the metric topology. Since $\mathcal{W}$ is a union of rectangles $\mathcal{U} \times \mathcal{V}$, it suffices to show that these rectangles are open in the metric topology. If $x \in \mathcal{U}$ and $y \in \mathcal{V}$, we can find $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{U}$ and $B_\epsilon(y) \subseteq \mathcal{V}$. Here the first ball is computed using the d_X metric and the second one is computed using the d_Y metric. The product of these balls is just $B_\epsilon(x \times y)$ with respect to the d metric. Clearly $\mathcal{W}$ is a union of such balls, and thus open in the d metric topology.

6.2d Map $R^2 - \{0\}$ to $R \times S^1$ by $f : x \to \ln ||x|| \times \frac{x}{||x||}$. Notice that $S^2 \subseteq R^2$; map $R \times S^1$ to $R^2 - \{0\}$ by $g : r \times p \to e^r p$. These maps are inverse to each other and both maps are continuous.

We will prove the first map continuous by using theorem 6.5. According to this result, if $f : A \to X$ and $g : A \to Y$ are continuous, then $f \times g : A \to X \times Y$ is continuous. So we must prove $R^2 - \{0\} \to R$ by $x \to ||x||$ and $R^2 - \{0\} \to S^1$ by $x \to \frac{x}{||x||}$ continuous.

If $A \subseteq X$ has the induced topology and $f : X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous. So to prove $R^2 - \{0\} \to R$ by $x \to ||x||$ continuous it is enough to prove that $R^2 \to R$ by $x \to ||x||$ continuous, and this is exercise 1.5a with $y = 0$.

If $B \subseteq Y$ has the induced topology and $f : X \to Y$ is continuous and the image is in B, then $f : X \to B$ is continuous (give the easy proof if you are unconvinced). So to show that $R^2 - \{0\} \to S^1$ by $x \to \frac{x}{||x||}$ is continuous it suffices to show that $R^2 - \{0\} \to R^2$ by the same map is continuous.

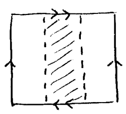
6.6h Consider the set $S_{p,q}$ of all $x \in R^n$ such that $x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 = 1$. Solving, we see that

$$x_1^2 + \ldots + x_p^2 = 1 + x_{p+1}^2 + \ldots + x_{p+q}^2$$

Think of this equation in the following way. The coordinates $x_{p+1}, \ldots, x_{p+q}$ and also the remaining "invisible" coordinates $x_{p+q+1}, \ldots, x_n$, can be chosen arbitrarily. Then $(x_1, \ldots, x_p)$ must be chosen to belong to the sphere of radius $\sqrt{1 + x_{p+1}^2 + \ldots + x_{p+q}^2}$. Another way to say this is to let $\lambda = \sqrt{1 + x_{p+1}^2 + \ldots + x_{p+q}^2}$, and require that $(x_1, \ldots, x_p) = \lambda y$ where y is a point on the unit sphere $S^{p-1} \subseteq R^p$.

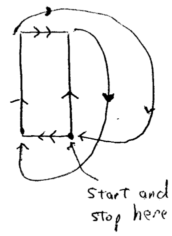
Thus we obtain a map $S_{p,q} \to S^{p-1} \times R^{n-p}$ by sending $(x_1, \ldots, x_n)$ to $\left(\frac{x_1}{\lambda}, \ldots, \frac{x_p}{\lambda}\right) \times (x_{p+1}, \ldots, x_n)$ where $\lambda = \sqrt{1 + x_{p+1}^2 + \ldots + x_{p+q}^2}$. The inverse map sends $(y_1, \ldots, y_p) \times (x_{p+1}, \ldots, x_n)$ to $(\lambda y_1, \ldots, \lambda y_p) \times (x_{p+1}, \ldots, x_n)$. These maps are clearly both continuous, and they are inverse to each other and thus both one-to-one and onto.

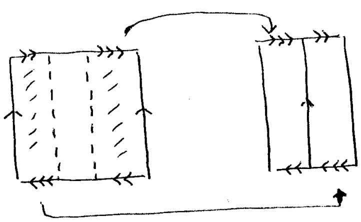
Extra Problem One



The middle piece is a Mobius band

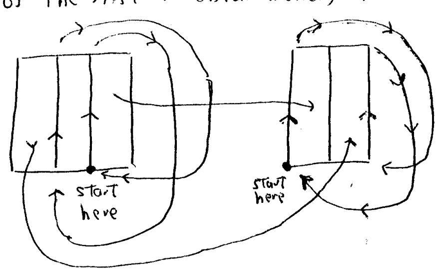
The boundary is a single circle



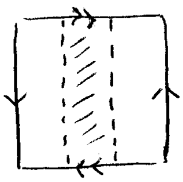


The outside is also a Mobius band

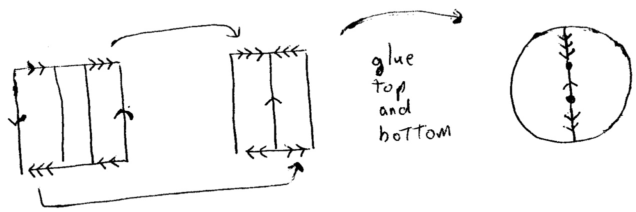
and the boundary, traced exactly as the boundary of the first Mobius band, is another circle



Extra Problem Two



The middle is a Mobius band as before



So the outside pieces form a disk with boundary a circle

Extra Problem Three Consider the diagram below:

$$\begin{tikzcd}
SO(3) \arrow[r, "f"] \arrow[d, "\pi"'] & S^2 \\
SO(3)/SO(2) \arrow[ur, dashed, "\tilde{f}"'] &
\end{tikzcd}$$

Define $f : SO(3) \to S^2 \subseteq R^3$ by sending the 3×3 rotation matrix A to $A(n)$ where n is the north pole $(0, 0, 1)$. This map is continuous since

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (a_{13}, a_{23}, a_{33})$$

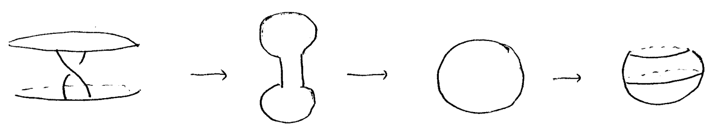
defines the map from $SO(3) \subseteq R^9$ to $S^2 \subseteq R^3$ given by very simple equations which are clearly continuous.

We claim that this map is constant on cosets $A\ SO(2)$. Indeed if $B \in SO(2)$ then $AB(n) = A(n)$ because B is rotation leaving the z-axis fixed, so $B(n) = n$.

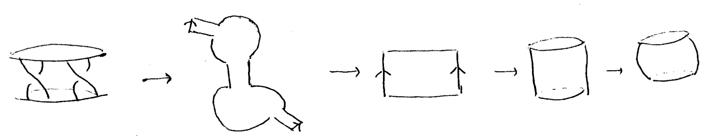
It follows that the map f induces a well-defined map $\tilde{f}$ from the set $SO(3)/SO(2)$ of cosets to S^2. This map is continuous by the general theory of the quotient topology. The map is onto because the north pole can be rotated to any point in the sphere by some rotation. The map is one-to-one, for if $A(n) = B(n)$ then $B^{-1}A(n) = n$, so $B^{-1}A \in SO(2)$. Call this element H. Then $A = BH$, so A and B define the same coset.

Finally $SO(3)$ is clearly closed and bounded, so it is compact as a subset of R^9. Since S^2 is Hausdorff, the map $SO(3)/SO(2) \to S^2$ has continuous inverse and so it is a homeomorphism.

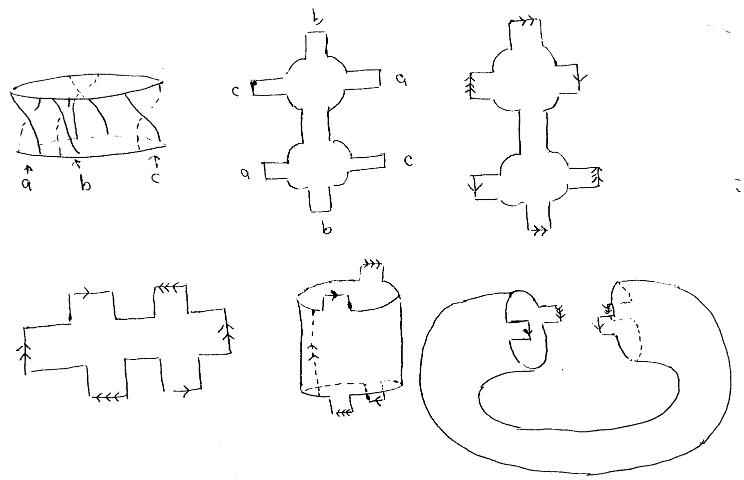
Extra Problem Four If we have one twisted strip, we can untwist the top and obtain an object homeomorphic to a disk. But a disk is homeomorphic to a sphere with a missing hole.



If we have two twisted strips, we can cut one strip, unfold, and glue the strip back together as we did last week. The result looks like a sphere with two holes removed.



Finally, if we have four twisted strips, we cut the three strips on the front, untwist the final strip, flip the top over and glue back together, obtaining a torus with two holes removed.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