Assignment 3; Due Friday, October 14

2.8a The closure of $\{1, 2, 3, \ldots\}$ is itself because the set is already closed. Indeed the complement is a union of open intervals. The closure of the set of rationals is all of R because every real number is a limit of a sequence of rationals. For example, $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ converges to π. The closure of the set of irrationals is also R.

2.8b The closure of $[a, b)$ is itself $[a, b)$ because this set is closed. Indeed its complement is $(-\infty, a) \cup [b, \infty)$ and this set is open because each point p in the set is contained in some interval $[p, s)$ in the set.

The closure of (a, b) is $[a, b)$. Indeed this later set is closed, so we need to check that (a, b) is not itself closed. But its complement is not open since the complement contains a but does not contain any interval $[a, s)$.

The closure of $[a, b]$ is itself because a brief check shows that the complement is open.

The closure of $(a, b]$ is $[a, b]$. Indeed this later set is closed, so we must check that (a, b) is not closed. But its complement is not open because a is in the complement and the complement does not contain any interval $[a, s)$.

2.9c The set $\overline{Y}$ is closed and contains $\overline{Y}$. So it is the smallest closed set which contains $\overline{Y}$ and thus must equal $\overline{\overline{Y}}$.

2.9d Notice that $A \cup B \subseteq \overline{A} \cup \overline{B}$. Since the set on the right is closed and $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Notice that $A \subseteq A \cup B \subseteq \overline{A \cup B}$. The set on the right is closed and $\overline{A}$ is the smallest closed set containing A, so $\overline{A} \subseteq \overline{A \cup B}$. A similar result holds for $\overline{B}$, so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Putting these two paragraphs together, we find that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

To prove the second result, notice that $A \cap B \subseteq \overline{A} \cap \overline{B}$ and the set on the right is closed. Since $\overline{A \cap B}$ is the smallest closed set containing $A \cap B$, we have $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

These two sets need not be equal. For example, let A be the set of rational numbers in R and let B be the set of irrational numbers in R. Then $A \cap B$ is empty and so is its closure. But $\overline{A}$ and $\overline{B}$ are both all of R and thus $\overline{A} \cap \overline{B}$ is R.

2.9e Notice that $X - Y \subseteq X - \overset{\circ}{Y}$. This last set is closed since it is the complement of the open set $\overset{\circ}{Y}$; here notice that X is the entire space. Since $\overline{X - Y}$ is the smallest closed set containing $X - Y$, $\overline{X - Y} \subseteq X - \overset{\circ}{Y}$.

Notice that $X - \overline{X - Y} \subseteq X - (X - Y) = Y$. Since the left side is open, it is contained in the largest open set inside Y, so $X - \overline{X - Y} \subseteq \overset{\circ}{Y}$. Taking complements, $\overline{X - Y} \supseteq X - \overset{\circ}{Y}$.

The exercise follows from the two previous paragraphs.

Extra Problem

I'll preserve my original solution of this problem. But many students in class gave a much easier solution, which goes like this: Notice that $\overline{A} - \overset{\circ}{A}$ is the intersection of $\overline{A}$ with the complement of $\overset{\circ}{A}$. By the previous exercise, this complement is $\overline{X - A}$, so $\overline{A} - \overset{\circ}{A} = \overline{A} \cap \overline{X - A}$.

OK, here is my original solution.

We first prove that $\overline{A} \cap \overline{X - A} \subseteq \overline{A} - \overset{\circ}{A}$. The set on the left is certainly in $\overline{A}$. It suffices to show that $\overline{X - A}$ is in the complement of $\overset{\circ}{A}$. Notice that $X - A \subseteq X - \overset{\circ}{A}$. This last set is closed, and $\overline{X - A}$ is the smallest closed set containing $X - A$, so $\overline{X - A} \subseteq X - \overset{\circ}{A}$ as desired.

Next we prove that $\overline{A} - \overset{\circ}{A} \subseteq \overline{A} \cap \overline{X - A}$. The set on the left is certainly in $\overline{A}$. Let p be a point in $\overline{A}$ which is not in the interior of A; we will show that $p \in \overline{X - A}$. According to a result in class, p is in the closure of $X - A$ is and only if every open neighborhood $\mathcal{U}$ of p intersects $X - A$. If this is false, then we can find an open neighborhood $\mathcal{U}$ of p which does not intersect $X - A$. So $\mathcal{U} \subseteq A$. Since $\overset{\circ}{A}$ is the largest open subset of A, $\mathcal{U} \subseteq \overset{\circ}{A}$ and in particular $p \in \overset{\circ}{A}$, a contradiction.

Consider $\mathring{A}$.

3.2b We are given that for any Y and for any $f : X \to Y$, f is continuous. Apply this result to the special case when Y is X with the discrete topology and f is the identity map. If $A \subset X$ is any set, it is open in Y by definition, so $f^{-1}(A) = A$ is also open in X by continuity. Thus any subset of X is open, so X has the discrete topology.

3.2c We are given that for any X and for any $f : X \to Y$, f is continuous. Apply this result to the special case when X is Y with the concrete topology and f is the identity map. If $\mathcal{U}$ is an open subset of Y, then $f^{-1}(\mathcal{U}) = \mathcal{U}$ is open in X. So $\mathcal{U}$ is either the empty set or the entire space. It follows that Y has the concrete topology.

4.2 Part One I claim the equivalence classes are the following. When you check my answer, notice carefully that TEX doesn't print the characters exactly as they appear in the book. So some characters in a given equivalence class aren't homeomorphic in the list below, but are homeomorphic in the book.

$\{C, G, I, J, L, M, N, S, U, V, W, Z, 1, 2, 5, 7\}$, $\{D, O, 0\}$, $\{E, F, T, Y, 3\}$, $\{A, R\}$, $\{H, K\}$,

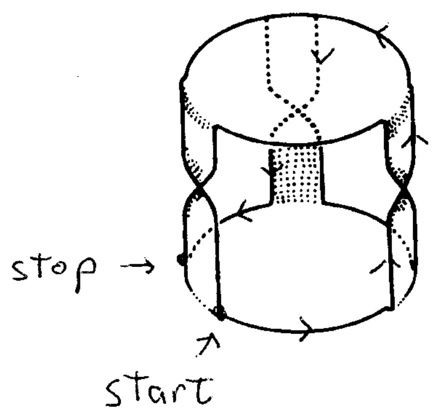
$\{4, X\}$, $\{P, 6, 9\}$, $\{B\}$, $\{8\}$, $\{Q\}$

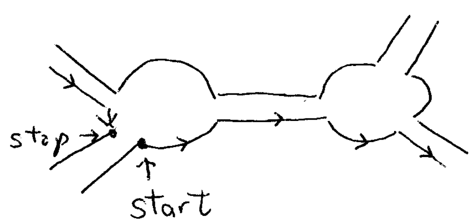
4.2 Part Two Number across the page and then down, the equivalence classes are

$\{1\}$, $\{2, 10\}$, $\{3, 6, 9\}$, $\{4, 7\}$, $\{5\}$ $\{8\}$, $\{11, 12\}$, $\{13, 14\}$, $\{15\}$, $\{16\}$

4.3 First cut the left two of the three twisted vertical strands. Then rotate the top to untwist the remaining strand on the right, and lay the resulting object down in the plane.

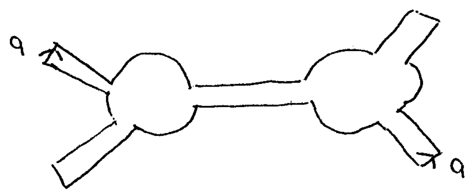
To keep track of where things came from, draw arrows along the boundary of the original object and notice that this boundary is a single circle. Let us just draw some of the arrows. Start at the bottom just after the left front vertical strip, continue along the bottom disk to the strip we did not cut, and up along its boundary to the top disk, continue to the left back vertical strip, down its boundary, and partly around the bottom disk. These arrows are shown on the top picture below and on the flattened picture just below it.



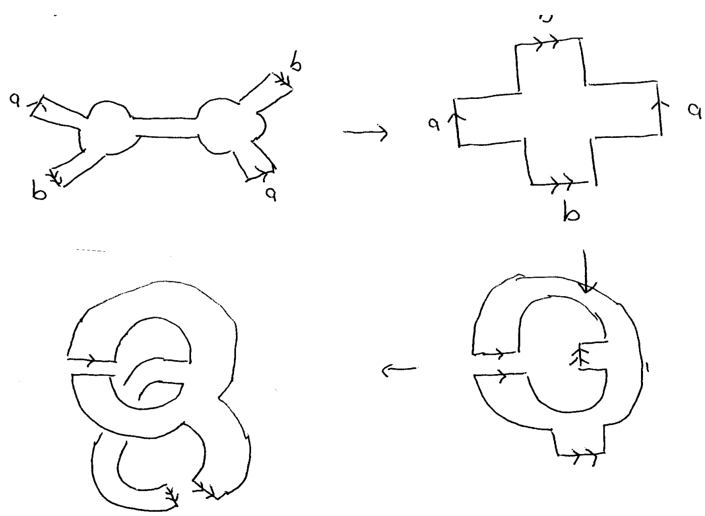


The arrows show that we must glue the top broken strip on the left to the bottom broken strip on the right, and they show the direction of this glue. See the sketch below.

By using the same technique, you can determine how to glue the remaining strip together.



The pictures below show this, and then the deformation which leads to the result in the book.



Notice that the second step of this deformation looks like our standard way of making a torus, with a large hole cut out near the four corners. Thus the objects in the book are homeomorphic to a torus with a small hole cut out. See the illustration below.