Assignment 2; Due Friday, October 7

1.5a For each fixed x, must prove that whenever $\epsilon > 0$, there is $\delta > 0$ such that if $d(p,x) < \delta$, then $|f(p) - f(x)| < \epsilon$. Since $f(x) = d(x,y)$, this last inequality can be written $|d(p,y) - d(x,y)| < \epsilon$.

The *idea* of the rest of the proof is easy. If p and x are really close, then they should both be about the same distance from y, so $|d(p,y) - d(x,y)|$ should be really small. We just have to quantify this.

Indeed $d(p,y) \le d(p,x) + d(x,y)$ and so $d(p,y) - d(x,y) \le d(p,x)$. Since p and x are arbitrary points, they can be interchanged to show that $d(x,y) - d(p,y) \le d(p,x)$. Putting these two results together, we get

$$\Big| d(p,y) - d(x,y) \Big| \le d(p,x)$$

Choose $\delta = \epsilon$. If $d(p,x) < \delta$, then $|d(p,y) - d(x,y)| \le d(p,x) < \delta = \epsilon$.

1.5b Suppose $f : M_0 \to M$ where M_0 is discrete. For fixed x, we must show that whenever $\epsilon > 0$ then there is a $\delta > 0$ such that if $d_0(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$. Choose $\delta = 1$. If $d_0(x,y) < 1$, then since d_0 is the discrete metric we have $d_0(x,y) = 0$ and so $x = y$. So $d(f(x), f(y)) = d(f(x), f(x)) = 0$ and so this distance is certainly less than ϵ.

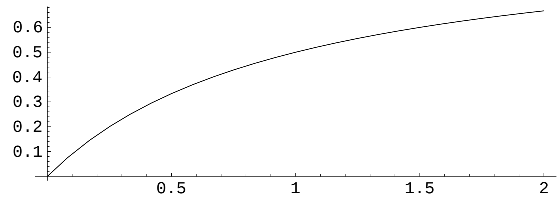
For the second part, let $f : M \to M_0$ be continuous. This time I will play the devil, and I choose $\epsilon = 1$. Since f is continuous, you can find $\delta > 0$ such that whenever $d(x,y) < \delta$ then $d_0(f(x), f(y)) < \epsilon = 1$. Since d_0 is the discrete metric, $d_0(f(x), f(y)) = 0$ and so $f(x) = f(y)$. So f is constant on the interval about x of radius δ and thus is certainly not injective.

1.6b We'll change notation slightly and write $\tilde{d}$ for the metric on B. We'll denote the two metrics on A by d_1 and d_2. If d is one of these two metrics, then by definition f is continuous if whenever $x \in A$ and $\epsilon > 0$, there is a $\delta > 0$ such that if $d(x,y) < \delta$ then $\tilde{d}(f(x), f(y)) < \epsilon$.

Suppose some other sucker knows how to prove this for d_1 and we want to prove it for d_2. So the devil picks ϵ and then that other sucker picks δ_1 that works for this ϵ and d_1. We then want to pick δ_2 that works the same way for d_2.

Our metrics are related by the function $g(t) = \frac{t}{1+t}$ plotted below.

In this picture, we plot d_1 along the horizontal axis and d_2 along the vertical axis. Since this function is increasing, $t < T$ if and only if $g(t) < g(T)$. So we can pick $\delta_2 = g(\delta_1)$. If $d_2(x,y) < \delta_2$, then $d_1(x,y) < \delta_1$, so if x and y satisfy our condition for d_2 and δ_2, then they satisfy the sucker's condition for d_1 and δ_1 and so $\tilde{d}(f(x), f(y)) < \epsilon$.



The proof the other direction works the same way. If the sucker picks δ_2, then we pick δ_1 so $g(\delta_1) = \delta_2$. But now there is a complication. If the sucker's δ_2 is greater than or equal to 1, there is no corresponding δ_1. However, in this special case the condition $d_2(x, y) < \delta_2$ is automatically true, so the sucker is claiming that any δ works for that particular ϵ. So in that case we can pick any δ_1 we like.

1.8a Let $y \in B_\epsilon(x)$. Then the distance from y to x is smaller than ϵ and so the distance from y on out to the boundary is $\epsilon - d(y, x)$, which is positive.

To show that $B_\epsilon(x)$ is open, we must find an open ball about any y in this set which is entirely inside $B_\epsilon(x)$. Let this ball be $B_\delta(y)$ where $\delta = \epsilon - d(y, x) > 0$. If z is in this ball, then $d(z, y) < \delta = (\epsilon - d(y, x))$ and so $d(z, x) \leq d(z, y) + d(y, x) < (\epsilon - d(y, x)) + d(y, x) = \epsilon$. So $z \in B_\epsilon(x)$.

1.8b I'm lazy and will give the answer without extensive reasons. Reading row by row, the first set is not open because the point $(1, 0)$ is on the right side boundary of the disk rather than inside it, so all neighborhoods of this point contain points outside the disk. The second example is a closed disk, and its boundary points do not have neighborhoods. The third example is an open strip and in particular open. The fourth example, is an open half plane, so open. The next example is a closed half plane, so not open. The final example is a straight line, which is not open in R^2. This would all be easier to see with pictures.

1.8d Let $\mathcal{U}_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$. As n increases, the lower end of the interval approaches 0 and the upper end of the interval approaches 1. On the other hand, 0 and 1 belong to all of these intervals. So the interection is $[0, 1]$. This set is closed, but not open.

2.2a Give M the discrete metric. We will show that this gives the right topology, and thus that in this metric space every subset of M is open.

So let $A \subseteq M$ be an arbitrary subset. To prove A open, we must show that for each $a \in A$ there is a positive ϵ such that $B_\epsilon(a) \subseteq A$. To prove this, choose $\epsilon = 1$. Then if $b \in B_\epsilon(a)$, $d(b, a) < 1$, so by definition of the discrete metric $d(b, a) = 0$ and $b = a$. So $B_\epsilon = \{a\} \subseteq A$.

2.2b Let $a \neq b$. Then $\epsilon = d(b, a) > 0$. But then $B_{\epsilon/2}(a)$ and $B_{\epsilon/2}(b)$ are open balls about a and b. These balls do not intersect, for if c were in both, then $d(c, a) < \frac{\epsilon}{2}$ and $d(c, b) < \frac{\epsilon}{2}$

and so $d(a,b) \le d(a,c) + d(c,b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, contradicting the definition of ϵ.

2.2c If X had the concrete topology, then the nonempty open sets $B_{\epsilon/2}(a)$ and $B_{\epsilon/2}(b)$ above would have to be X and thus would intersect.

Comment: We have seen that every metric space is a topological space. This example shows that the converse is false; there are topological spaces which do not come from metric spaces.

2.3a By direct definition, the empty set and the entire set are open. The intersection of open sets is open because $(-\infty, x) \cap (-\infty, y) = (-\infty, \min[x, y])$. The union of open sets is open because

$$\cup(-\infty, x_\alpha) = (-\infty, \sup(x_\alpha))$$

2.3b The empty set and N are open by definition. The intersection of $(m, m+1, \ldots)$ and $(n, n+1, \ldots)$ is $(\max(m,n), \max(m,n)+1, \ldots)$. The union of open sets is open, for if the open set $\mathcal{U}_\alpha$ starts at the integer n_α, then there is a smallest n_α since each of these integers is positive, and the union of the open sets starts at this smallest number.

2.3c The entire space is open by definition. A set is open if whenever it contains a it contains $[a, b)$ for some b. This condition is certainly true of the empty set, since there is nothing to test. If $\mathcal{U}$ and $\mathcal{V}$ are open and a is in their intersection, then $[a, b) \subseteq \mathcal{U}$ and $[a, c) \subseteq \mathcal{V}$ for some b and c. Then $[a, \min(b, c))$ is in the intersection.

The union of open sets is open, for if a is in the union and thus in some $\mathcal{U}_\alpha$, then b exists so $[a, b) \subseteq \mathcal{U}_\alpha$ and then $[a, b)$ is in the entire union.

2.3e The first is not a topology, because the union of the open sets $\left(-\infty, -\frac{1}{n}\right]$ is $(-\infty, 0)$, which is not open using the definition of this problem.

The second is not a topology, for $(0,1) \cup (2,3)$ is not an interval and so not open by this definition.

2.6b In the discrete topology, any set is open. A set is closed if its complement is open, but every such complement is open, so every set is closed. In particular, every set is simultaneously open and closed.

2.6c Finite unions of closed sets are closed. Since points are closed and since every subset is finite, every subset is closed. So the complement of any set is closed, and thus any set is open.

2.6d By definition, $[s, t)$ is open. To prove that this set is also closed, we show that its complement is open. The complement is $(-\infty, s) \cup [t, \infty)$. The first is open for is a is in this set, then so is $[a, s)$. The second is open because is a is in the set, then so is $[t, a+1)$.