Assignment 1; Due Friday, September 30

1.2: The triangle inequality must hold for every choice of a, b, and c. For instance, it must hold if $a = b$, so

$$d(a, b) + d(a, c) \geq d(b, c)$$

becomes

$$d(b, b) + d(b, c) \geq d(b, c)$$

Now $d(b, b) = 0$ by axiom one, so this gives $d(b, c) \geq d(b, c)$, which is obvious. You cannot win every time.

The remaining special cases give something interesting. Suppose we set b equal to c. Then the triangle inequality becomes

$$d(a, c) + d(a, c) \geq d(c, c)$$

Since $d(c, c) = 0$, this gives $2d(a, c) \geq 0$ and so $d(a, c) \geq 0$. Since a and c are arbitrary, we have proved that the distance between any two points is ≥ 0, as required.

Finally set $a = c$. The triangle inequality becomes

$$d(c, b) + d(c, c) \geq d(b, c)$$

Since $d(c, c) = 0$, we have $d(c, b) \geq d(b, c)$. This holds for any b and c, so interchanging b and c gives t $d(b, c) \geq d(c, b)$ and thus $d(b, c) = d(c, b)$ as required.

1.3a: Let $d(x, y) = ||x - y||$. Then $d(x, x) = \left(\sum (x_i - x_i)^2\right)^{1/2} = 0$, as required. Conversely, if $d(x, y) = 0$, then $\left(\sum (x_i - y_i)^2\right)^{1/2} = 0$. Squaring, $\sum (x_i - y_i)^2 = 0$. Each term in this expression is non-negative, so the expression can only be zero if each $x_i - y_i = 0$ and thus only if $x = y$.

We must prove the triangle inequality. Since $d(a, b) = d(b, a)$ by the first exercise, we can write the triangle inequality in its more standard form $d(x, z) \leq d(x, y) + d(y, z)$. Thus we must prove that

$$||x - z|| \leq ||x - y|| + ||y - z||$$

But it is known that $||a + b|| \leq ||a|| + ||b||$. Apply this when $a = x - y$ and $b = y - z$ to get $||x - z|| \leq ||x - y|| + ||y - z||$.

Finally, the graduate students need to prove that $||a + b|| \leq ||a|| + ||b||$. First we prove the Schwarz inequality $| < x, y > | \leq ||x||||y||$.

Indeed

$$\left\langle x - \frac{<x,y>}{||y||^2} y, x - \frac{<x,y>}{||y||^2} y \right\rangle \geq 0$$

because the length of any vector is non-negative. Expanding

$$\left\langle x, x \right\rangle - \frac{<x,y>^2}{||y||^2} \geq 0$$

and so

$$\left\langle x, x \right\rangle \left\langle y, y \right\rangle \geq \left\langle x, y \right\rangle^2$$

Since $<x,x>=||x||^2$, the Schwarz inequality follows by taking square roots.

But then

$$||a+b||^2 = <a+b, a+b> = <a,a> + 2<a,b> + <b,b>$$

and by the Schwarz inequality this is less than or equal to

$$<a,a> + 2||a||\ ||b|| + <b,b> = ||a||^2 + 2||a||\ ||b|| + ||b||^2 = \Big(||a|| + ||b||\Big)^2$$

The required inequality follows by taking square roots.

1.3a continued: Similar arguments hold for $d(x,y) = \sum |x_i - y_i|$. This expression is clearly zero if $x = y$. Conversely if it is zero, then each term of the sum must be zero, so $x_i = y_i$ for all i, so $x = y$.

According to the triangle inequality for ordinary real numbers, $|a+b| \leq |a| + |b|$. Set $a = x_i - y_i$ and $b = y_i - z_i$ to obtain $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. Summing

$$\sum |x_i - z_i| \leq \sum |x_i - y_i| + \sum |y_i - z_i|$$

so $d(x,z) \leq d(x,y) + d(y,z)$.

1.3b: If $d(x,y) = (x-y)^2$, the triangle inequality fails. For example, let $x = 0, y = 1$, $z = 2$. Then $d(x,y) + d(y,z) = 1^2 + 1^2 = 2$, but $d(x,z) = 2^2 = 4$, so it is not true that $d(x,z) \leq d(x,y) + d(y,z)$.

1.3e: In the exercise set for next week, we will show that the open sets using d and the open sets using d' are the same. So the metric spaces using d and using d' are homeomorphic. Notice that every subset in the metric space using d' is bounded because $d' < 1$. So the condition that a metric space is *bounded* is not *topologically* interesting; we can replace any metric space by a homeomorphic one in which every subset is bounded.

Clearly $d'(x,y) = 0$ if and only if $d(x,y) = 0$, and so if and only if $x = y$.

Thus we need only prove the triangle inequality. We know that

$$d(x,y) + d(y,z) \geq d(x,z)$$

and we would like to prove that

$$\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \geq \frac{d(x,z)}{1+d(x,z)}$$

To simplify the argument, let $d(x,y) = a, d(y,z) = b, d(x,z) = c$. Thus we know

$$a + b \geq c$$

and we want to prove

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c}$$

But

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{a}{1+a+b} + \frac{b}{1+a+b} = \frac{a+b}{1+a+b}$$

because dividing by a larger number gives a small number. So it suffices to prove that $a + b \geq c$ implies

$$\frac{a+b}{1+a+b} \geq \frac{c}{1+c}$$

and this follows because the function $f(x) = \frac{x}{1+x}$ is an increasing function, since its derivative is positive.